Planar $\mathcal{N} = 4$ at High Loops and Large Multiplicity

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Recent Developments in S-Matrix Theory
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JHEP 1908 (2019) 016 and JHEP 1909 (2019) 061, with
S. Caron-Huot, L. Dixon, F. Dulat, M. von Hippel, and G. Papathanasiou

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Motivation

- Computing scattering amplitude using traditional methods is hard, even in 'simpler' theories like planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and scalar ϕ^4 theory
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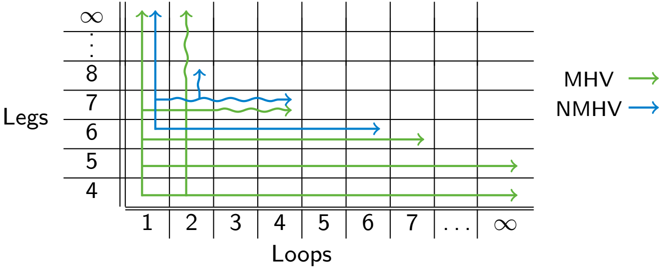
Can we bypass the evaluation of these integrals, and just look for a function that has all the expected properties of the amplitude?

- This is the 'bootstrap' philosophy (described also in Claude's lectures)
- Has proven highly successful in planar $\mathcal{N} = 4$

Status of Loops and Legs in Planar $\mathcal{N} = 4$



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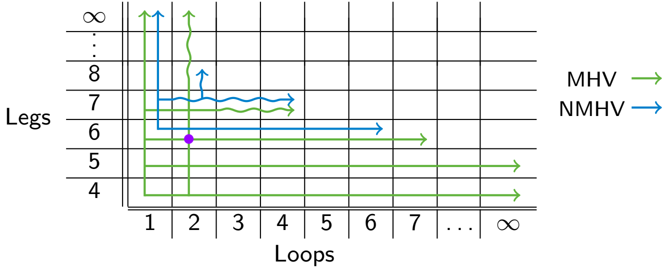

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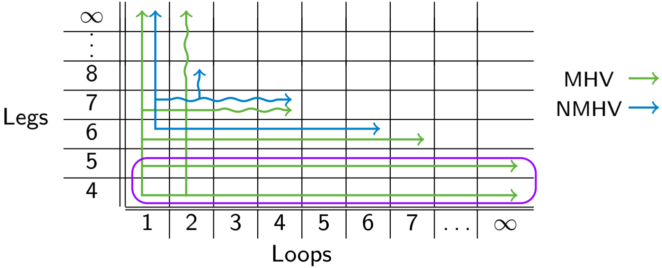

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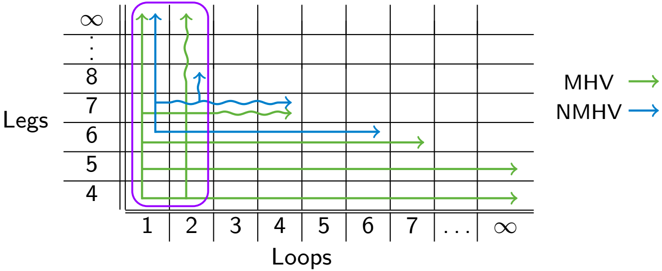

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- The last amplitude to be calculated by direct integration was at two loops and six points [Del Duca, Duhr, Smirnov]
- The four- and five-particle amplitudes are given by the BDS ansatz to all orders [Bern, Dixon, Smirnov]
- The one- and two-loop amplitudes can be computed using generalized unitarity and constraints from Dual Conformal Symmetry, respectively [Bern, Dixon, Dunbar, Kosower] [Caron-Huot, He]

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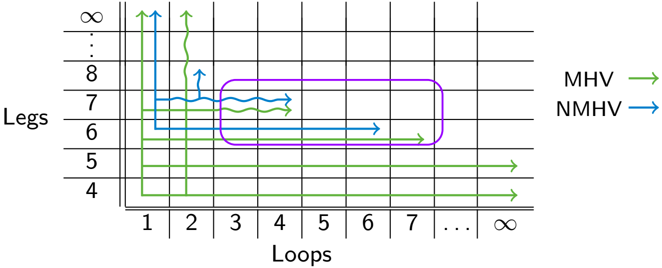

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This talk will focus on the techniques used to calculate amplitudes for $n \geq 6$ and $L \geq 3$

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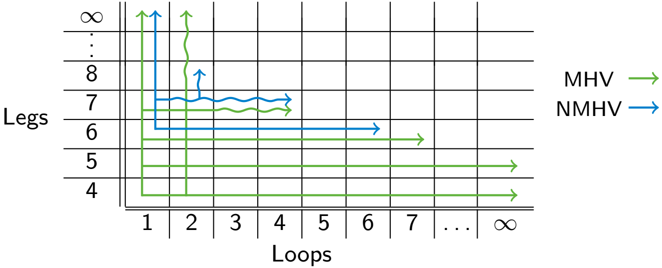

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This talk will focus on the techniques used to calculate amplitudes for $n \geq 6$ and $L \geq 3$

- Unexpected and striking structure exists in the the direction of both higher loops and legs
  - Extended Steinmann Relations
  - Cosmic Galois Coaction Principle
  - Cluster-Algebraic Structure

⋮ ⋮

Planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

Scattering Amplitudes

A number of nice simplifications occur in planar $\mathcal{N} = 4$ SYM

SUSY Ward identities	$\Rightarrow$	relate amplitudes with different helicity structure
Conformal symmetry	$\Rightarrow$	no running of the coupling or UV divergences
Planar limit	$\Rightarrow$	trivial color structure
AdS/CFT	$\Rightarrow$	dual to string theory on $AdS_5 \times S^5$

Much of what we learn here also augments our understanding of QCD

Planar Limit and Dual Conformal Symmetry

An additional simplification occurs in the planar limit, where $N_c \to \infty$ for fixed $g^2 = g^2_{\text{YM}} N_c/(16\pi^2)$

- The suppression of non-planar graphs allows us to endow the scattering particles with an ordering

$$\mathcal{A}_n^{\text{non-planar}} = \sum_{\sigma \in S_n} \text{Tr}\left[T^{a_{\sigma_1}} \cdots T^{a_{\sigma_n}}\right] \mathcal{A}_n\left(p^\mu_{\sigma_1}, \ldots, p^\nu_{\sigma_n}\right) + \mathcal{O}\left(\frac{1}{N_c^2}\right)$$

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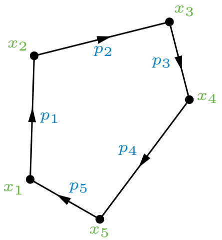
- This ordering gives rise to a natural set of dual coordinates (see also Mark's second lecture)

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu$$



- The coordinates x_i^μ can be thought of as labelling the cusps of a light-like polygonal Wilson loop in the dual theory, which respects a superconformal symmetry in this dual space

  [Alday, Maldacena] [Drummond, Korchemsky, Sokatchev]

- This strongly constrains the kinematic dependence of the amplitude

Helicity and Infrared Structure

- The infrared-divergent part of these amplitudes is accounted for at all particle multiplicity by the 'BDS ansatz' [Bern, Dixon, Smirnov]
- In the dual theory, the BDS ansatz solves an anomalous conformal Ward identity that determines the Wilson loop up to a function of dual conformal invariants [Drummond, Henn, Korchemsky, Sokatchev]
- Dual conformal invariants can first be formed in six-particle kinematics, so the four- and five-particle amplitudes are entirely described by the BDS ansatz

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- Dual conformal invariants can first be formed in six-particle kinematics, so the four- and five-particle amplitudes are entirely described by the BDS ansatz

$$\mathcal{A}_4 = \mathcal{A}_4^{\text{BDS}} \qquad \mathcal{A}_5 = \mathcal{A}_5^{\text{BDS}}$$

$$\mathcal{A}_n = \underbrace{\mathcal{A}_n^{\text{BDS}}}_{\text{IR structure}} \times \underbrace{\exp(R_n) \times \overbrace{\left(1 + \mathcal{P}_n^{\text{NMHV}} + \mathcal{P}_n^{\text{N}^2\text{MHV}} + \cdots + \mathcal{P}_n^{\overline{\text{MHV}}}\right)}^{\text{helicity structure}}}_{\text{finite function of dual conformal invariants}}$$

- Thus, the problem of calculating the n-point amplitude is reduced to the problem of calculating the DCI functions R_n and $\mathcal{P}_n^{\text{N}^k\text{MHV}}$

Dual Conformal Invariants

- In general, we can construct dual conformally invariant cross ratios out of combinations of Mandelstam invariants

$$x_{ij}^2 \equiv (x_i - x_j)^2 = (p_i + p_{i+1} + \cdots + p_{j-1})^2 \equiv s_{i,\dots,j-1}$$

that remain invariant under the dual inversion generator

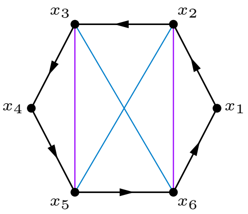
$$I(x_i^{\alpha\dot{\alpha}}) = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2} \qquad \Rightarrow \qquad I(x_{ij}^2) = \frac{x_{ij}^2}{x_i^2 x_j^2}$$

- For six particles, three dual conformal invariants can be formed

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2} = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



Bootstrapping Amplitudes in Planar $\mathcal{N} = 4$

The Coaction and Symbol

Recall from Claude's lectures that polylogarithms come equipped with a coaction and symbol:

- The coaction maps polylogarithms to a tensor product of lower-weight polylogarithms

$$\mathcal{H}_w \xrightarrow{\Delta} \bigoplus_{p+q=w} \mathcal{H}_p \otimes \mathcal{H}_q^\pi$$

Example: $\Delta \mathrm{Li}_m(z) = 1 \otimes \mathrm{Li}_m(z) + \sum_{k=0}^{m-1} \mathrm{Li}_{m-k}(z) \otimes \frac{\log^k z}{k!}$

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- If we iterate this map $w-1$ times we will arrive at a function's symbol, in terms of which all identities reduce to familiar logarithmic identities

Example: $\Delta_{1,\ldots,1} \mathrm{Li}_m(z) = -\log(1-z) \otimes \log z \otimes \cdots \otimes \log z$

(note that one still has to contend with algebraic identities between the symbol letters, which can be arbitrarily complicated)

The Coaction and Symbol

- The derivative of a polylogarithm is entirely determined by the $\Delta_{w-1,1}$ component of its coproduct

> Example: $$\Delta_{m-1,1}\mathrm{Li}_m(z) = \mathrm{Li}_{m-1}(z) \otimes \log z$$
> $$d\,\mathrm{Li}_m(z) = \frac{1}{z}\mathrm{Li}_{m-1}(z)dz$$

- The location of branch cuts is determined by the $\Delta_{1,w-1}$ component of its coproduct (up to terms involving powers of $i\pi$)

> Example: $$\Delta_{1,m-1}\mathrm{Li}_m(z) = -\log(1-z) \otimes \frac{\log^{m-1} z}{(m-1)!}$$
> $$\mathrm{Disc}_{\circlearrowleft \frac{1}{z}}\mathrm{Li}_m(z) = -2\pi i \frac{\log^{m-1} z}{(m-1)!}$$

Building the Function Space

We specialize to the six-particle MHV amplitude for concreteness

Our first task is to build the space of functions we expect the six-particle amplitude to be a member of

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Three (well-motivated) assumptions about this amplitude:

(1) It is polylogarithmic to all loop orders

(2) It has uniform transcendental weight $2L$ at L loops

(3) Its symbol alphabet to all loop orders is

$$\mathcal{S} \equiv \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

where

$$y_u = \frac{1+u-v-w-\sqrt{(1-u-v-w)^2-4uvw}}{1+u-v-w+\sqrt{(1-u-v-w)^2-4uvw}},$$

$$y_v = [y_u]_{u\to v\to w\to u}\,, \qquad y_w = [y_u]_{u\to w\to v\to u}$$

Building the Function Space

- Assumption (2) is consistent with all known polylogarithmic amplitudes in this theory
- The symbol alphabet $\mathcal{S}$ can be 'read off' of the two-loop amplitude, and is consistent with an all-loop analysis of the Landau equations [Prlina, Spradlin, Stanojevic]
- ...most importantly, in conjunction with the physical constraints described in a few slides, these assumptions give rise to a unique amplitude that passes a number of consistency checks (at least through seven loops)

So how do we construct this space of functions?

Building the Function Space

$\boxed{\text{Approach 1}}$

Use a basis of G functions, whose definition we recall:

$$G(a_1, \dots, a_k; z) \equiv \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_k; t), \qquad G(\underbrace{0, \dots, 0}_{k}; z) \equiv \frac{\log^k z}{k!}$$

- For instance, in the $\{y_u, y_v, y_w\}$ variables the functions

$$\left\{G_{\vec{a}}(y_u) \middle| a_i \in \{0,1\}\right\} \cup \left\{G_{\vec{a}}(y_v) \middle| a_i \in \left\{0, 1, \frac{1}{y_u}\right\}\right\}$$
$$\cup \left\{G_{\vec{a}}(y_w) \middle| a_i \in \left\{0, 1, \frac{1}{y_u}, \frac{1}{y_v}, \frac{1}{y_u y_v}\right\}\right\}$$

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  span the space of polylogarithms with the desired symbol letters

- Drawback: this space of functions grows very quickly; at four loops (weight eight) this gives rise to 1,675,553 functions!

Building the Function Space

$\boxed{\text{Approach 2}}$

'Build' a space of functions that has the right analytic structure out of coaction entries that make this structure manifest

- Constraints on the derivative of a function and the location of its discontinuities can be imposed by allowing only specific symbol letters to appear in the first and last entry

Building the Function Space

Approach 2

'Build' a space of functions that has the right analytic structure out of coaction entries that make this structure manifest

- Constraints on the derivative of a function and the location of its discontinuities can be imposed by allowing only specific symbol letters to appear in the first and last entry
- Drawback: a generic linear combinations of coaction terms does not necessarily correspond to a genuine function; in particular, partial derivatives must commute

Example:
$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}(\log x \otimes \log y) = \frac{1}{xy} \neq \frac{\partial}{\partial y}\frac{\partial}{\partial x}(\log x \otimes \log y) = 0$$

Building the Function Space

$\boxed{\text{Approach 2}}$

- We can construct the relevant space of functions iteratively, by considering an ansatz of $\Delta_{m-1,1}$ coproduct entries

$$\sum_{i,j} c_{ji} f_j^{(m-1)} \otimes \log s_i$$

where

  - c_{ji} is a free rational coefficient
  - $f_j^{(m-1)}$ is the jth weight $m-1$ function in our basis
  - s_i is the ith symbol letter in $\mathcal{S}$

and imposing the constraint that partial derivatives commute

- To satisfy these constraints we solve for some of the c_{ji} in terms of the others
- This generates the space of 'integrable' weight m functions

Imposing Physical Constraints

Once we have built a basis of weight $2L$ functions, we can now constrain a general ansatz of these functions to have the properties expected of the amplitude

- Analytic Constraints
- Symmetries
- Special kinematic limits

Analytic Constraints

$\boxed{\text{Physical Branch Cuts}}$

- Massless scattering amplitudes in the Euclidean region can only have branch cuts where one of the Mandelstam invariants vanishes
- In six-particle kinematics, this implies that the first symbol entries of the amplitude can only be u, v, and w

$$\Delta_{1,w-1} F = \log u \otimes {}^{u}F + \log v \otimes {}^{v}F + \log w \otimes {}^{w}F$$

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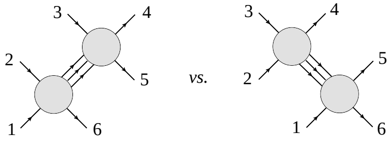
Steinmann Relations

- Additional restrictions come from the Steinmann relations, which tell us that amplitudes cannot have double discontinuities in partially overlapping channels [Steinmann] [Cahill, Stapp]



$$\text{Disc}_{s_{234}}(\text{Disc}_{s_{345}}(\mathcal{A}_n)) = 0$$

- However, to see this one must normalize the amplitude appropriately...

Shifted Infrared Subtraction

- The BDS ansatz exponentiates the one-loop amplitude, leading to products of Steinmann functions starting at two loops

$$\text{Disc}_{s_{i-1,i,i+1}}\left[\mathcal{A}_6^{(1)}\right] \neq 0$$
$$\Downarrow$$
$$\text{Disc}_{s_{234}}\left[\text{Disc}_{s_{345}}\left[\left(\mathcal{A}_6^{(1)}\right)^2\right]\right] \neq 0$$

- Therefore, we instead normalize by a 'BDS-like' ansatz that depends on only two-particle Mandelstam invariants

$$\mathcal{A}_n^{\text{MHV}} = \mathcal{A}_n^{\text{BDS}} \times \exp(R_n) \to \rho \times \mathcal{A}_n^{\text{BDS-like}} \times \mathcal{E}_n^{\text{MHV}}$$

where a transcendental constant ρ can also appear

- This only scrambles the Steinmann relations involving two-particle invariants, which are obfuscated in massless kinematics anyways

Analytic Constraints

- These analytic properties can be imposed directly on our space of functions in the first and second entry of the symbol, at the outset of our iterative construction
- This greatly decreases the number of functions that are generated by this procedure. For instance, at four loops:

Imposed Constraints	Number of Functions
Generalized polylogarithms with the correct kinematic dependence	1,675,553
That have branch cuts only in physical channels	6,916
That satisfy the Steinmann relations	839

Symmetries

Bose Symmetry

- The MHV amplitude is invariant under the dihedral group, representing permutations of the external particles that are consistent with the planar embedding
- In the $\{u, v, w\}$ variables, this corresponds to an S_3 permutation symmetry

$$\mathcal{A}_6^{\text{MHV}}(u, v, w) = \mathcal{A}_6^{\text{MHV}}(v, w, u)$$
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$\boxed{\bar{Q}\text{ Equation}}$

- The derivative of the amplitude is also constrained by the action of the dual superconformal generator $\bar{Q}$ [Caron-Huot, He]
- In the MHV sector, this constraint implies that the last entry of the symbol must be drawn from the set

$$\left\{ \frac{1-u}{u}, \frac{1-v}{v}, \frac{1-w}{w}, y_u, y_v, y_w \right\}$$

Kinematic Limits as Boundary Data

We can then match a general ansatz of such polylogarithms to the amplitude's known behavior in various kinematic limits

- Collinear Limits
- Multi-Regge Limits
- Near-Collinear Operator Product Expansion
- Multi-Particle Factorization
- Self-Crossing Limit

Collinear Limits

- The BDS ansatz smoothly limits to itself in collinear limits:

$$\mathcal{A}_n^{\text{BDS}} \xrightarrow{\text{collinear}} \mathcal{A}_{n-1}^{\text{BDS}}$$

- This implies that the same must be true of the remainder function:

$$R_n \xrightarrow{\text{collinear}} R_{n-1}$$

- In six-particle kinematics, the limit in which two external momenta become collinear corresponds to sending one of the cross ratios to zero and the sum of the remaining two cross ratios to one:

$$v \to 1 - u, \quad w \to 0$$

- Thus, we require that

$$R_6 \xrightarrow[]{\substack{v \to 1-u \\ w \to 0}} 0$$

which can be easily translated into a constraint on the BDS-like normalized amplitude

Multi-Regge Kinematics

- Describes $2 \to 4$ and $3 \to 3$ with large separations in rapidity

$$s_{12} \gg s_{345}, s_{456} \gg s_{34}, s_{45}, s_{56} \gg s_{23}, s_{61}, s_{234}$$

- In six-particle kinematics, this corresponds to the limit

$$v, w \to 0, \quad u \to 1 - \delta, \quad \delta \ll 1$$

  where we have fixed the ratios

$$\frac{v}{1-u} = \frac{1}{(1-z)(1-\bar{z})}, \qquad \frac{w}{1-u} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}$$

- To compare to physical predictions, we analytically continue to the Minkowski region $u \to e^{-2\pi i}|u|$ where we find

$$R_6^{(L)} \xrightarrow{\text{MRK}} 2\pi i \sum_{n=0}^{L-1} \log^n(1-u)\left[g_n^{(L)}(z,\bar{z}) + 2\pi i\; h_n^{(L)}(z,\bar{z})\right]$$

- These MRK limits can be compared to a pair of factorization relations in Fourier-Mellin transformed space [Fadin, Lipatov]

Current Results

These constraints have been used to uniquely determine the six-particle amplitude through seven (six) loops in the MHV (NMHV) sector

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathcal{H}^{\text{hex}}$	6	27	105	372	1214	3692
2. Bose Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795)
3. $\bar{Q}$ Symmetry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0,0)$	$(0,2)$	(1,5)	(6,17)
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,2)
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Current Results

- Through five loops, collinear factorization and multi-Regge factorization are sufficient to determine the amplitude; all other limits provide cross-checks
- At six loops and seven loops, the near-collinear OPE is needed to fix a single residual ambiguity in the MHV sector
- The barrier to going to higher loops is "just" computational
- A similar bootstrap program has also been carried out in seven-particle kinematics through four loops

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- A similar bootstrap program has also been carried out in seven-particle kinematics through four loops

- ...however, note that the space of 'Hexagon functions' $\mathcal{H}^{\text{hex}}$ that was used as a starting point in this table has a smaller dimension at four loops than reported on an earlier slide
- This more restricted space of functions was built using novel constraints, which we now describe

Novel Analytic Properties and Symmetries

The Extended Steinmann Relations

- The Steinmann relations correspond to the constraint that certain sequences of letters never appear in the first two entries

$$\cancel{\log\left(\tfrac{u}{vw}\right) \otimes \log\left(\tfrac{w}{uv}\right)} \otimes \cdots \qquad \cancel{\log\left(\tfrac{u}{vw}\right) \otimes \log\left(\tfrac{v}{uw}\right)} \otimes \cdots$$

$$\frac{u}{vw} \sim s_{234}^2, \qquad \frac{v}{wu} \sim s_{345}^2, \qquad \frac{w}{uv} \sim s_{123}^2$$

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- However, the symbols of BDS-like normalized amplitudes exhibit a more surprising property: they obey the Steinmann relations in *all* adjacent entries of the symbol

  [Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou]

$$\cdots \otimes \cancel{\log\left(\tfrac{u}{vw}\right) \otimes \log\left(\tfrac{w}{uv}\right)} \otimes \cdots \qquad \cdots \otimes \cancel{\log\left(\tfrac{u}{vw}\right) \otimes \log\left(\tfrac{v}{uw}\right)} \otimes \cdots$$

Cosmic Galois Theory

However, we know that the symbol only captures some of the information encoded by the coaction

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- The cosmic Galois group extends the classical Galois theory to the study of periods—integrals of rational functions over rational domains
- Thus, we can explore the stability of amplitudes and integrals under the action of this Galois group

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- Thus, we can explore the stability of amplitudes and integrals under the action of this Galois group

In science you sometimes have to find a word that strikes, such as "catastrophe", "fractal", or "noncommutative geometry". They are words which do not express a precise definition but a program worthy of being developed.

- Pierre Cartier

The Coaction Principle

Specifically, we ask: does the space of Steinmann hexagon functions $\mathcal{H}^{\text{hex}}$ satisfy a 'coaction principle'? [Schnetz] [Brown]

$$\Delta \mathcal{H}^{\text{hex}} \subset \mathcal{H}^{\text{hex}} \otimes \mathcal{H}^{\pi}$$

- This can be formulated in terms of the action of the cosmic Galois group C as

$$C \times \mathcal{H}^{\text{hex}} \xrightarrow{?} \mathcal{H}^{\text{hex}}$$

- Part of the content of this statement is that the coaction preserves the locations of branch cuts (which we already know is the case from general physical principles)
- However, more general transcendental constants also appear in this space (multiple zeta values, alternating sums,...)
- These constants exhibit nontrivial structure under the coaction, which is not *a priori* constrained by physical principles

The coaction on MZVs

- For instance, we can consider our function space at $(u, v, w) = (1, 1, 1)$, where everything evaluates to multiple zeta values
- There exist natural derivations ∂_{2m+1} in the Lie algebra of the cosmic Galois group that act on the zeta values as

  $$\partial_{2m+1}\zeta_{2n+1} = \delta_{m,n}$$

  and that satisfy the Leibniz rule—for example,

  $$\partial_3(\zeta_7\zeta_3^2) = 2\zeta_7\zeta_3$$

- These operators act nontrivially on the multiple zeta values, for instance:

  $$\partial_3(\zeta_{5,3}) = 0, \quad \partial_5(\zeta_{5,3}) = -5\zeta_3$$

- There is no ∂_2, as the even zeta values are semi-simple elements of the coaction

The coaction principle at (1,1,1)

Weight	Multiple Zeta Values	Appear in $\mathcal{H}^{\text{hex}}\big|_{u,v,w\to 1}$
0	1	1
1		
2	ζ_2	ζ_2
3	ζ_3	
4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$
6	$\zeta_3^2,\ \zeta_6$	ζ_6
7	$\zeta_7,\ \zeta_5\zeta_2,\ \zeta_3\zeta_4$	$\zeta_5\zeta_2 - 7\zeta_7 + 3\zeta_3\zeta_4$
8	$\zeta_5\zeta_3,\ \zeta_{5,3},\ \zeta_8,\ \zeta_3^2\zeta_2$	$\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2,\ \zeta_8$

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$$\Downarrow$$

$$\partial_3(\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2) = 5\zeta_5 - 2\zeta_3\zeta_2$$

$$\partial_5(\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2) = 0$$

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- Unexplained dropouts were required at low weights for the coaction principle to be nontrivial
- Each zeta value that drops out seeds an infinite tower of constraints at higher loop orders, which we find are satisfied

The Coaction Principle

Everywhere we have checked, the coaction principle is respected

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Coaction principles of this type have been observed in other settings

- Tree-level string theory amplitudes [Schlotterer, Stieberger]
- Feynman graphs in ϕ^4 theory [Panzer, Schnetz]
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It is tempting to believe these coaction principles point to some symmetry respected by quantum field theory more generally

- A coaction can also be defined on the more complicated types of functions that appear in scattering amplitudes [Brown] [Broedel, Duhr, Dulat, Penante, Tancredi]
- However, things become more complicated when one loses purity and uniform transcendental weight. . .

Cluster Algebras and Planar $\mathcal{N} = 4$

Recall from Mark's lectures that the kinematics of planar $\mathcal{N} = 4$ can be mapped to the Grassmannian Gr$(4, n)$, in the guise of n momentum twistors that are combined into a $4 \times n$ matrix

Gr$(4, n)$ has an associated 'cluster algebra' that appears in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in a number of striking ways

- This cluster algebra arises naturally in studying the positivity properties of Gr$_+(4, n)$
- Loosely, cluster algebras are collections of 'cluster coordinates' that come grouped into 'clusters', any of which can be used to parametrize n-particle kinematics
- Operationally, cluster algebras can be generated from an initial seed cluster via a process called mutation

Cluster-Algebraic Structure

- The symbol alphabets for $n \in \{6, 7\}$ and all two-loop MHV amplitudes in this theory are given by cluster coordinates on the Grassmannian $\mathrm{Gr}(4, n)$ [Golden, Goncharov, Spradlin, Vergu, Volovich]
- The extended Steinmann relations seem to be equivalent to requiring that all symbol entries in the amplitude are 'cluster adjacent' [Drummond, Foster, Gurdogan]
- The 'non-classical part' of all two-loop MHV amplitudes can be expressed in terms of functions defined on their A_2 and A_3 subalgebras [Golden, Paulos, Spradlin, Volovich]
- While cluster algebras cannot generate algebraic symbol letters, there are hints that this type of algebraic structure generalizes in the form of tropical Grassmannians [Drummond, Foster, Gurdogan, Kalousios] [Arkani-Hamed, Lam, Spradlin] [Henke, Papathanasiou]

Conclusions

- A large amount of information is encoded in the formal structure of amplitudes, much of which is still not well understood
- This structure can be leveraged to bootstrap amplitudes that are otherwise well beyond our technical reach
- A great deal of interesting work in planar $\mathcal{N}=4$ is still ongoing:
  - The two-loop NMHV eight-point amplitude recently calculated using the $\bar{Q}$ equation [Zhang, Li, He]
  - Recent progress has been made on methods for identifying algebraic symbol letters [Mago, Schreiber, Spradlin, Volovich] [He, Li]
  - We are starting to better understand the types of functions that appear in this theory beyond polylogarithms

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

- Bootstrap approaches are also being applied in more realistic theories such as QCD [Li, Zhu] [Almelid, Duhr, Gardi, McLeod, White]

Thanks!