Double soft theorems in generalized bi-adjoint scalars

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Introduction

One of the remarkable developments in S-matrix program is **Cachazo-He-Yuan** (CHY) formalism. Using this formalism several QFT amplitudes can be computed using integral representations over moduli space of punctured Riemann spheres.

CHY is important because:

- Applicable to large variety of non-supersymmetric theories including scalars, gluons and gravitons.
- Replaces conventional computation of Feynman diagrams.
- Several properties of S-matrix like color-kinematic duality, double copy relations are manifested easily which otherwise are difficult to study in Lagrangian descriptions.
- Studies of factorizations, soft limits, collinear limits of S-matrix are convenient.
- Hints at alternative description of scattering amplitudes without Lagrangian formulation.
- Extended beyond tree-level and massless amplitudes.

Scattering equations of CHY map kinematic space of Mandelstam invariants to moduli space of punctured Riemann spheres.

Recently **Cachazo, Early, Guevara, Mizera** (CEGM) introduced generalization of the *scattering equations*, which are defined on higher dimensional projective spaces, $\mathbb{CP}^{k-1}$.

$k = 2 \Rightarrow$ CHY.

Standard scattering equations defined on the space of n points on $\mathbb{CP}^1$ are also equations for n points on $\mathbb{CP}^{n-3}$.

A well studied example with this generalization is *bi-adjoint scalar amplitudes*.

Studying soft limits help us to understand the factorization properties of amplitudes and corresponding boundaries of underlying moduli space. In this work we explore *simultaneous double soft theorem* for biadjoint scalar amplitudes for $k = 3$ and in the process present some general results for arbitrary k.

Outline

CHY

CEGM

Soft limits

Double soft theorems

CHY

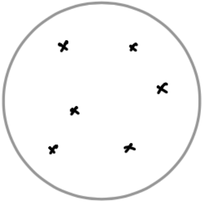
CEGM

Soft limits

Double soft theorems

Scattering equation

Scattering amplitudes are expressed in terms of Mandelstam variables, s_{ab}. These Mandelstam variables form a kinematic space.



The essential idea of this formalism is to map the kinematic space to the moduli space of all n-punctured Riemann spheres by the holomorphic mapping: $P(\sigma) = \sum_{j=1}^{n} \frac{k_j}{\sigma - \sigma_j}$. These are maps of Riemann sphere into complex momentum space.

Scattering equations:

$$\boxed{f_a := \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \qquad \forall a \in \{1, 2, \ldots n\}}$$

Properties:

- $SL(2,\mathbb{C})$ transformations: Scattering equations are invariant under $\sigma \to \frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}$ with $\alpha\delta - \beta\gamma = \mathbf{1}$.

  We can fix $\sigma_{\mathbf{1}} \to \mathbf{0}, \sigma_{\mathbf{2}} \to \mathbf{1}$ and $\sigma_{\mathbf{3}} \to \infty$.

- Dimension of the moduli space is $\mathbf{n} - \mathbf{3}$.

- Out of $\mathbf{n}$ only $\mathbf{n} - \mathbf{3}$ scattering equations are independent. Total number of solutions is $(\mathbf{n} - \mathbf{3})!$.

Scattering Amplitude

$$\boxed{M_n = \int \frac{\mathrm{d}^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{s_{ab}}{\sigma_a - \sigma_b}\right) I_n(\{k,\epsilon,\sigma\})}$$

$\mathrm{vol}\mathbb{SL}(2,\mathbb{C}) = \frac{\mathrm{d}\sigma_a \mathrm{d}\sigma_b \mathrm{d}\sigma_c}{(\sigma_a-\sigma_b)(\sigma_b-\sigma_c)(\sigma_c-\sigma_a)}$,

$\prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a . k_b}{\sigma_a-\sigma_b}\right) := (\sigma_i - \sigma_j)(\sigma_j - \sigma_k)(\sigma_k - \sigma_i) \prod_{a\neq i,j,k} \delta\left(\sum_{b\neq a} \frac{k_a . k_b}{\sigma_a-\sigma_b}\right)$.

- $\mathbb{SL}(2,\mathbb{C})$ **transformation**:

$$\mathrm{d}\mu_n \longrightarrow \mathrm{d}\mu_n \prod_{a=1}^{n} (\gamma\sigma_a + \delta)^{-4} \quad \Rightarrow \quad I_n \longrightarrow I_n \prod_{a=1}^{n} (\gamma\sigma_a + \delta)^{4}$$

- **Integration is localized to** $(n-3)!$ **points in** $\mathfrak{M}_{0,n}$:

$$M_n = \sum_{i=1}^{(n-3)!} \frac{I_n(\{k,\epsilon,\sigma\})}{\det{}'\Phi_n} \Big|_{i^{\mathrm{th}}\mathrm{solution}} \qquad (\Phi_n)_{ab} := \begin{cases} \frac{k_a \cdot k_b}{(\sigma_a-\sigma_b)^2}, & a \neq b \\ -\sum_{c\neq a} (\Phi_n)_{ac}, & a = b \end{cases}$$

Bi-adjoint scalar

Parke-Taylor factor:

$$\mathrm{PT}(\alpha) = \frac{1}{\left(\sigma_{\alpha(1)} - \sigma_{\alpha(2)}\right) \ldots \left(\sigma_{\alpha(n-1)} - \sigma_{\alpha(n)}\right) \left(\sigma_{\alpha(n)} - \sigma_{\alpha(1)}\right)}$$

Consider two copies of Parke -Taylor factors

$$I_n = \mathrm{PT}(\alpha)\,\mathrm{PT}(\beta)$$

Corresponds to amplitudes for $\mathcal{L}_{\mathrm{int}} = -f_{abc}\tilde{f}_{a'b'c'}\phi^{aa'}\phi^{bb'}\phi^{cc'}$.

$$\mathcal{M}_n = \sum_{\alpha,\beta} \mathrm{Tr}\left(T^{\alpha(1)} \ldots T^{\alpha(n)}\right) \mathrm{Tr}\left(\tilde{T}^{\beta(1)} \ldots \tilde{T}^{\beta(n)}\right) m_n(\alpha|\beta)$$

- $m_n(\alpha|\beta)$ is called **partial amplitude**.

Building block with polarizations

$$\mathrm{Pf}(\mathbf{E_n}) := \sum_{\alpha \in \mathrm{P.M.}} \mathrm{sgn}(\alpha)(\mathbf{E_n})_{\alpha(\mathbf{1}),\alpha(\mathbf{2})}(\mathbf{E_n})_{\alpha(\mathbf{3}),\alpha(\mathbf{4})}\cdots(\mathbf{E_n})_{\alpha(\mathbf{n-1}),\alpha(\mathbf{n})}$$

If $\mathbf{E_n}$ is antisymmetric then $\mathrm{Pf}(\mathbf{E_n}) = \sqrt{\det \mathbf{E_n}}$.

$$\Psi_n = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

$$A_{ab} = \begin{cases} \frac{k_a.k_b}{\sigma_a-\sigma_b}, & a \neq b \\ 0, & a = b \end{cases}$$

$$B_{ab} = \begin{cases} \frac{\epsilon_a.\epsilon_b}{\sigma_a-\sigma_b}, & a \neq b \\ 0, & a = b \end{cases}$$

$$C_{ab} = \begin{cases} \frac{\epsilon_a.k_b}{\sigma_a-\sigma_b}, & a \neq b \\ -\sum_{c\neq a} \frac{\epsilon_a.k_c}{\sigma_a-\sigma_c}, & a = b \end{cases}$$

$\mathbf{\Psi_n}$ has a kernel of dimension two: $(1,1,\ldots,1;0,0,\ldots,0)^T$ **and** $(\sigma_1,\sigma_2,\ldots,\sigma_n;\mathbf{0},\mathbf{0},\ldots,\mathbf{0})^{\mathbf{T}}$**, hence** $\mathrm{Pf}\mathbf{\Psi_n}$ **vanishes.**

Therefore *reduced Pfaffian* is used

$$\mathrm{Pf}'\Psi_n = \frac{(-1)^{i+j}}{(\sigma_i-\sigma_j)}\mathrm{Pf}(\Psi_n)^{ij}_{ij}, \qquad \text{for any } i,j \in \{1,2,\ldots n\}.$$

Gluon and graviton amplitudes

- Color ordered Yang-Mills amplitude

$$A_n^{\mathsf{YM}}(\alpha) = \int d\mu_n \, \mathsf{PT}(\alpha) \, \mathrm{Pf}'\Psi_n$$

- Gravity amplitude

$$M_n^{\mathsf{GR}} = \int d\mu_n \, \left(\mathrm{Pf}'\Psi_n\right)^2$$

Examples:

$$\begin{aligned} m\,(123|123) &= 1 \\ m\,(1234|1243) &= -\frac{1}{s_{12}} \\ m\,(12345|13254) &= \frac{1}{s_{23}s_{45}} \end{aligned}$$

Kawai-Lewellen-Tye orthogonality: Double copy relations between gravity and Yang-Mills amplitudes

$$M_n^{GR} = A_n^{YM}(\beta)m^{-1}(\beta|\tilde{\beta})A_n^{YM}(\tilde{\beta})$$

Bern et al; Bjerrum-Bohr et al; Cachazo et al; Mizera

CHY

CEGM

Soft limits

Double soft theorems

Generalized scattering equations

Cachazo, Early, Guevara, Mizera

$\mathbf{k=2}$

Define **generalized potential function**

$$\mathcal{S} = \sum_{1\leq a<b\leq n} s_{ab} \log{(a,b)}$$

Cachazo, Mizera, Zhang

$$(a,b) = \begin{vmatrix} \sigma_a^1 & \sigma_b^1 \\ \sigma_a^2 & \sigma_b^2 \end{vmatrix} \quad \xrightarrow{x=\frac{\sigma^1}{\sigma^2}} \quad \begin{vmatrix} 1 & 1 \\ x_a & x_b \end{vmatrix}$$

$\sigma_{\mathbf{a}}$ are points on $\mathbb{CP}^1$.

Scattering equations:

$$\frac{\partial \mathcal{S}}{\partial x_a} = \sum_{b\neq a} \frac{s_{ab}}{x_a - x_b} = 0, \qquad \forall a$$

$\mathbb{CP}^{\mathbf{1}} \to \mathbb{CP}^{\mathbf{k-1}}$

$$|a_1\ a_2 \dots a_k| = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x^1_{a_1} & x^1_{a_2} & \dots & x^1_{a_k} \\ \vdots & \vdots & \vdots & \vdots \\ x^{k-1}_{a_1} & x^{k-1}_{a_2} & \dots & x^{k-1}_{a_k} \end{vmatrix}$$

Generalized potential:

$$\mathcal{S}_k = \sum_{1 \le a_1 < a_2 < \dots < a_k \le n} s_{a_1 a_2 \dots a_k} \log |a_1\ a_2 \dots a_k|$$

$\mathbf{s_{a_1 a_2 \dots a_k}}$ are **generalized Mandelstam variables** which satisfy

- $\displaystyle\sum_{\substack{a_2, a_3 \dots a_k \\ a_i \neq a_j}} s_{a_1 a_2 \dots a_k} = 0, \quad \forall a_1$ (momentum conservation)
- $s_{a_1 a_2 \dots a_k} = 0$ for any $a_i = a_j$ (massless condition)

Scattering equations:

$$\frac{\partial \mathcal{S}_k}{\partial x_a^i} = 0, \qquad \forall a, \quad i = 1, 2, \ldots k-1$$

As an example for $\mathbf{k} = \mathbf{3}$ scattering equations are

$$\sum_{\substack{1 \le b < c \le n \\ b,c \neq a}} \frac{s_{abc}\,(x_b - x_c)}{|abc|} = 0, \qquad \sum_{\substack{1 \le b < c \le n \\ b,c \neq a}} \frac{s_{abc}\,(y_b - y_c)}{|abc|} = 0, \qquad \forall a$$

Moduli space on which scattering equations are defined

$$X\,(k,n) := \mathrm{Gr}\,(k,n)\,/\,(C^*)^{n-1}\,, \qquad \mathrm{dim} X\,(k,n) = (k-1)\,(n-k-1)$$

- $\mathrm{Gr}\,(k,n)$ is the space of all k-planes in n-dimensional space
- $(C^*)^{n-1}$ is the $n-1$-torus action

Scattering equations transform covariantly under $\mathbf{SL}\left(\mathbf{k},\mathbb{C}\right)$ transformations. To remove the gauge redundancies positions of $(\mathbf{k}+\mathbf{1})$ points are fixed.

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-1} & x_2^{k-1} & \dots & x_n^{k-1} \end{pmatrix} \xrightarrow{\text{gauge fix}} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 & x_{k+2}^1 & \dots & x_n^1 \\ 0 & 0 & 1 & \dots & 0 & 1 & x_{k+2}^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 & x_{k+2}^{k-1} & \dots & x_n^{k-1} \end{pmatrix}$$

Number of undetermined coordinates

$$n\left(k-1\right)-\left(k+1\right)\left(k-1\right)=\left(\mathbf{k}-\mathbf{1}\right)\left(\mathbf{n}-\mathbf{k}-\mathbf{1}\right)$$

Generalized bi-adjoint amplitudes

For arbitrary **k** define Parke-Taylor factor

$$\mathrm{PT}^{(k)}(1,2,\ldots n) = \frac{1}{|12\ldots k||2\,3\ldots\ k+1|\ldots|n\,1\ldots k-1|}$$

Generalized bi-adjoint amplitudes:

$$m_n^{(k)}(\alpha|\beta) \ = \ \int\left(\frac{1}{\mathrm{Vol}\left[\mathbb{SL}\left(k,\mathbb{C}\right)\right]}\prod_{a=1}^{n}\prod_{i=1}^{k-1}dx_a^i\right)\prod_{a=1}^{n}{\prod_{i=1}^{k-1}}'\,\delta\left(\frac{\partial\mathcal{S}_k}{\partial x_a^i}\right) \times \ \mathrm{PT}^{(k)}(\alpha)\,\mathrm{PT}^{(k)}(\beta)$$

5-point example

3-point amplitude for $k = 2$ and 4-point amplitude for $k = 3$ are trivial. This is true for $(k+1)$-point amplitude for arbitrary k.

- **k = 2**

$$m_5^{(2)}\,(\mathrm{I}|\mathrm{I}) = \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{34}s_{51}} + \frac{1}{s_{45}s_{12}} + \frac{1}{s_{51}s_{23}}$$

- **k = 3**

$$m_5^{(3)}\,(\mathrm{I}|\mathrm{I}) = \frac{1}{s_{345}s_{512}} + \frac{1}{s_{451}s_{123}} + \frac{1}{s_{123}s_{345}} + \frac{1}{s_{234}s_{451}}$$

The two amplitudes are related by

$$m_5^{(2)}\,(\mathrm{I}|\mathrm{I}) = m_5^{(3)}\,(\mathrm{I}|\mathrm{I})\Big|_{s_{ab}\to s_{cde}} \qquad \{a,b,c,d,e\} = \{1,2,3,4,5\}$$

Equality in the example of 5-point amplitudes is a consequence of isomorphism between $\mathrm{Gr}\left(\mathbf{k},\mathbf{n}\right)$ and $\mathrm{Gr}\left(\mathbf{n}-\mathbf{k},\mathbf{n}\right)$.

As a result $\mathbf{X}\left(\mathbf{k},\mathbf{n}\right)$ and $\mathbf{X}\left(\mathbf{n}-\mathbf{k},\mathbf{n}\right)$ are dual to each other.

$$
\begin{aligned}
\mathcal{S}_k &= \sum_{1\le a_1<\ldots<a_k\le n} s_{a_1 a_2\ldots a_k} \log\left(a_1, a_2, \ldots a_k\right) \\
& \qquad\qquad \Big\downarrow {\scriptstyle \begin{array}{c} k\times k \text{ minor} \\ \downarrow \\ (n-k)\times(n-k) \text{ minor} \end{array}} \\
&= \sum_{1\le a_1<\ldots<a_k\le n} s_{a_1 a_2\ldots a_k} \log\left(\overline{a_1, a_2, \ldots a_k}\right)
\end{aligned}
$$

$\mathcal{S}_\mathbf{k}$ maps to generalized potential function on $\mathbb{CP}^{\mathbf{n}-\mathbf{k}-\mathbf{1}}$.

CHY

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Soft limits

Double soft theorems

Single soft limits

k = 2

Momentum of n-th external state is taken to be soft:

$$s_{na} = \tau \hat{s}_{na}, \qquad \lim \tau \to 0, \qquad a \in \{1, 2, \ldots, n-1\}$$

Scattering equations decouple

$$\begin{aligned} E_a &= \sum_{\substack{b=1 \\ b\neq a}}^{n-1} \frac{s_{ab}}{x_a - x_b} + \tau \frac{\hat{s}_{an}}{x_a - x_n} = 0 \\ E_n &= \tau \sum_{b=1}^{n-1} \frac{\hat{s}_{nb}}{x_n - x_b} = 0 \end{aligned}$$

For $k = 2$, $x_a - x_n$ is always $\mathcal{O}\left(\tau^0\right)$. Singular solution is absent.

- $\{\mathbf{E_a}\}$ gives solutions to $\mathbf{n-1}$ hard points. Total number of solutions is $(\mathbf{n-4})!$.
- $\mathbf{E_n}$ is a polynomial in $\mathbf{x_n}$ of degree $\mathbf{n-3}$. Hence there are $\mathbf{n-3}$ solutions to $\mathbf{x_n}$ for every set of $\{\mathbf{E_a}\}$. Therefore total number of solutions is $(\mathbf{n-3}) \times (\mathbf{n-4})! = (\mathbf{n-3})!$.

We consider soft limit in bi-adjoint scalar amplitude $\mathbf{m_n}(\mathrm{I}|\mathrm{I})$

$$
\begin{aligned}
m_n(\mathrm{I}|\mathrm{I}) &= \int dx_n\, \delta(E_n) \left[\frac{x_{n-1}-x_1}{(x_{n-1}-x_n)(x_n-x_1)}\right]^2 m_{n-1}(\mathrm{I}|\mathrm{I}) \\
&= \oint \frac{dx_n}{\underline{E_n}} \left[\frac{x_{n-1}-x_1}{(x_{n-1}-x_n)(x_n-x_1)}\right]^2 m_{n-1}(\mathrm{I}|\mathrm{I}) \\
&= -\frac{1}{\tau}\left[\frac{1}{\hat{s}_{n-1\,n}}+\frac{1}{\hat{s}_{n1}}\right] m_{n-1}(\mathrm{I}|\mathrm{I})
\end{aligned}
$$

$\mathbf{k = 3}$

Sepulvida, Guevara; Cachazo, Umbert, Zhang

Consider the soft limit: $\mathbf{s_{nab}} = \tau\, \hat{\mathbf{s}}_{\mathbf{nab}}, \quad \mathbf{a}, \mathbf{b} \in \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}-\mathbf{1}\}$.

Scattering equations decouple

$$
\begin{aligned}
E_a^{(i)} &= \sum_{b,c\neq a} \frac{s_{abc}}{|abc|} \frac{\partial}{\partial x_a^i} |abc| + \tau \sum_{b\neq a} \frac{\hat{s}_{abn}}{|abn|} \frac{\partial}{\partial x_a^i} |abn| = 0, \\
E_n^{(i)} &= \tau \sum_{1\leq a<b\leq n-1} \frac{\hat{s}_{abn}}{|abn|} \frac{\partial}{\partial x_n^i} |abn| = 0, \qquad i = 1, 2
\end{aligned}
$$

$|\mathbf{abn}| \to \mathcal{O}(\tau)$ **contribute to singular solutions. In soft factorization these solutions appear at sub-leading order.**

For convenience we will work with **homogeneous coordinates**.

Notation: $(\mathbf{abc}) = \begin{vmatrix} \sigma_a^1 & \sigma_b^1 & \sigma_c^1 \\ \sigma_a^2 & \sigma_b^2 & \sigma_c^2 \\ \sigma_a^3 & \sigma_b^3 & \sigma_c^3 \end{vmatrix}$

Consider soft limit $\mathbf{m_n^{(3)}}\,(\mathrm{I}|\mathrm{I}) \to \mathrm{S}^{(3)}\mathbf{m_{n-1}^{(3)}}\,(\mathrm{I}|\mathrm{I})$

$$\mathrm{S}^{(3)} = \oint \frac{(\sigma\ d^2\sigma)\,(XY\sigma)}{\left(\sum_{b,c} \frac{s_{nbc}(Xbc)}{(\sigma bc)}\right)\left(\sum_{b,c} \frac{s_{nbc}(Ybc)}{(\sigma bc)}\right)} \left[\frac{(n-2\ n-1\ 1)\,(n-1\ 1\ 2)}{(n-2\ n-1\ \sigma)\,(n-1\ \sigma\ 1)\,(\sigma\ 1\ 2)}\right]^2$$

Deform the contour away from the original poles and apply Global Reside Theorem. Poles are encountered when *one* or *more than one* of the following terms vanish

$$(n-2\ n-1\ \sigma)\,, \qquad (n-1\ \sigma\ 1)\,, \qquad (\sigma\ 1\ 2)$$

Singularities are on **codimension 2** boundaries.

Collision singularities

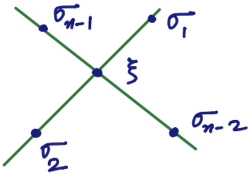
$\sigma \to \sigma_{n-1}$

$\sigma \to \sigma_1$

parametrize: $\sigma = \sigma_a + \epsilon A, \quad a = n-1, 1$

Collinear singularities

σ lies at the intersection of two straight lines, one joining $\sigma_{\mathbf{n-2}}$, $\sigma_{\mathbf{n-1}}$ and the other joining $\sigma_{\mathbf{1}}$, $\sigma_{\mathbf{2}}$.



parametrize: $\sigma = \alpha\sigma_1 + \beta\sigma_{n-1} + \xi$

Soft factor:

$$\mathrm{S}^{(3)} = \frac{1}{\tau^2}\left[\frac{1}{\hat{s}_{n-2\ n-1\ n}\ \hat{s}_{n12}} + \frac{1}{\sum_{b=2}^{n-1}\hat{s}_{1bn}}\left(\frac{1}{\hat{s}_{n-1\ n\ 1}} + \frac{1}{\hat{s}_{n12}}\right)\right.$$
$$\left. + \frac{1}{\sum_{b=1}^{n-2}\hat{s}_{b\ n-1\ n}}\left(\frac{1}{\hat{s}_{n-2\ n-1\ n}} + \frac{1}{\hat{s}_{n-1\ n\ 1}}\right)\right]$$

- For arbitrary **k** singularities lie at $\mathbf{k}-\mathbf{1}$ co-dimension boundaries.
- Scaling of soft factor: $\mathrm{S}^{(\mathbf{k})} \sim \frac{\mathbf{1}}{\tau^{\mathbf{k}-\mathbf{1}}}$.

CHY

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Soft limits

Double soft theorems

Double soft limits

$\mathbf{k = 2}$

Soft particles can be either **adjacent** or **non-adjacent**.

Adjacent soft limit

$$\mathbf{s_{na} = \tau \hat{s}_{na}, \quad s_{n-1\,a} = \tau \hat{s}_{n-1\,a}, \quad s_{n-1\,n} = \tau^2 \hat{s}_{n-1\,n}, \quad a \in \{1, \ldots, n-2\}}$$

Scattering equations:

$$\begin{aligned}
E_a &= \sum_{\substack{b=1\\ b\neq a}}^{n-2} \frac{s_{ab}}{x_a - x_b} + \tau \frac{\hat{s}_{a\,n-1}}{x_a - x_{n-1}} + \tau \frac{\hat{s}_{an}}{x_a - x_n} = 0 \\
E_{n-1} &= \tau \sum_{b=1}^{n-2} \frac{\hat{s}_{n-1\,b}}{x_{n-1} - x_b} + \tau^2 \frac{\hat{s}_{n-1\,n}}{x_{n-1} - x_n} = 0 \\
E_n &= \tau \sum_{b=1}^{n-2} \frac{\hat{s}_{nb}}{x_n - x_b} - \tau^2 \frac{\hat{s}_{n-1\,n}}{x_{n-1} - x_n} = 0
\end{aligned}$$

Soft factor:

$$S_{DS}^{(2)} = \int dx_n \, \delta(E_n) \int dx_{n-1} \, \delta(E_{n-1}) \left[\frac{x_{n-1} - x_1}{(x_{n-2} - x_{n-1})(x_{n-1} - x_n)(x_n - x_1)} \right]^2$$

The solutions to the scattering equations are of two types

1. **Degenerate solutions** - when $|\mathbf{x_{n-1}} - \mathbf{x_n}| \sim \mathcal{O}(\tau)$
2. **Non-degenerate solutions** - when $|\mathbf{x_{n-1}} - \mathbf{x_n}| \sim \mathcal{O}(\tau^{\mathbf{0}})$

- For gravity and gluon amplitudes leading contributions come from either non-degenerate solutions or both non-degenerate and degenerate ones.

Volovich, Wen, Zlotnikov; Chakrabarti, Kashyap, Sahoo, Sen, Verma; APS

- For scalars only degenerate solutions contribute to leading order in soft theorem.

Cachazo, He, Yuan

Yang-Mills

Color ordered partial amplitude for YM is given by

$$A_N^{\mathrm{YM}}(1,2,\ldots,N) = \int \mathrm{d}\mu_N \; C_N(1,2,\ldots,N)\,\mathrm{Pf}'\Psi_{n+2}\left(\{\sigma_i, p_i, \epsilon_i\}\right)$$

where

$$C_N(1,2,\ldots,N) = \frac{1}{\sigma_{12}\sigma_{23}\ldots\sigma_{N1}}, \qquad \text{with} \quad \sigma_{ij} \equiv \sigma_i - \sigma_j$$

is the color ordered Parke-Taylor factor.

Soft limit in the non-adjacent legs are straightforward and is given by product of single soft factors to leading order. We will present the analysis of soft limits for adjacent legs.

Non-degenerate solutions

Factorization of integrand at leading order is given by

$$\begin{aligned} C_n &= \frac{x_{n-2} - x_1}{(x_{n-2} - x_{n-1})(x_{n-1} - x_n)(x_n - x_1)} C_{n-2} \\ \mathrm{Pf}'\Psi_n &= -\left(\sum_{a=1}^{n-2} \frac{\epsilon_{n-1} \cdot p_a}{x_{n-1} - x_a}\right)\left(\sum_{b=1}^{n-2} \frac{\epsilon_n \cdot p_b}{x_n - x_b}\right) \mathrm{Pf}'\Psi_{n-2} \end{aligned}$$

Contributions from non-degenerate solutions are

$$\frac{1}{\tau^2}\left[\frac{\epsilon_{n-1} \cdot p_{n-2}}{k_{n-1} \cdot p_{n-2}} - \frac{\epsilon_{n-1} \cdot p_1}{k_{n-1} \cdot p_1}\right] \frac{\epsilon_n \cdot p_1}{k_n \cdot p_1} A_{n-2}^{\mathrm{YM}}$$

$$-\frac{1}{\tau^2} \oint_{\{A_i\}} \mathrm{d}x_{n-1} \left(\sum_{a=1}^{n-2} \frac{k_{n-1} \cdot p_a}{x_{n-1} - x_a}\right)^{-1} \left(\sum_{b=1}^{n-2} \frac{k_n \cdot p_b}{x_{n-1} - x_b}\right)^{-1}$$

$$\left(\sum_{c=1}^{n-2} \frac{\epsilon_n \cdot p_c}{x_{n-1} - x_c}\right)\left(\sum_{d=1}^{n-2} \frac{\epsilon_{n-1} \cdot p_d}{x_{n-1} - x_d}\right) \frac{(x_{n-2} - x_1)}{(x_{n-2} - x_{n-1})(x_{n-1} - x_1)} A_{n-2}^{\mathrm{YM}}$$

Degenerate solutions

Expanding $\boldsymbol{\Psi_n}$ at the leading order

$$
\Psi_n \approx \left(\begin{array}{c:c:c|c:c:c}
(A_{n-2})_{ab} & \frac{\tau\ k_a.p}{x_a-\rho} & \frac{\tau\ k_a.q}{x_a-\rho} & (-C^T_{n-2})_{ab} & \frac{-\epsilon_{n-1}.k_b}{\rho-x_b} & \frac{-\epsilon_n.k_b}{\rho-x_b} \\ \hdashline
\frac{\tau\ p.k_b}{\rho-x_b} & 0 & \frac{-\tau\ p.q}{\xi_1} & \frac{-\tau\ \epsilon_a.p}{x_a-\rho} & -C_{n-1,n-1} & \frac{-\epsilon_n.p}{\xi_1} \\ \hdashline
\frac{\tau\ q.k_b}{\rho-x_b} & \frac{\tau\ p.q}{\xi_1} & 0 & \frac{-\tau\ \epsilon_a.q}{x_a-\rho} & \frac{\epsilon_{n-1}.q}{\xi_1} & -C_{n,n} \\ \hline
(C_{n-2})_{ab} & \frac{\tau\ \epsilon_a.p}{x_a-\rho} & \frac{\tau\ \epsilon_a.q}{x_a-\rho} & (B_{n-2})_{ab} & \frac{\epsilon_a.\epsilon_{n-1}}{x_a-\rho} & \frac{\epsilon_a.\epsilon_n}{x_a-\rho} \\ \hdashline
\frac{\epsilon_{n-1}.k_b}{\rho-x_b} & C_{n-1,n-1} & \frac{-\epsilon_{n-1}.q}{\xi_1} & \frac{\epsilon_{n-1}.\epsilon_b}{\rho-x_b} & 0 & \frac{\epsilon_{n-1}.\epsilon_n}{-\tau\ \xi_1} \\ \hdashline
\frac{\epsilon_n.k_b}{\rho-x_b} & \frac{\epsilon_n.p}{\xi_1} & C_{n,n} & \frac{\epsilon_n.\epsilon_b}{\rho-x_b} & \frac{\epsilon_{n-1}.\epsilon_n}{\tau\ \xi_1} & 0
\end{array}\right)
$$

$$
\begin{aligned}
\mathrm{Pf}'\Psi_n \quad = \quad & \Bigg[-\left(\sum_{a=1}^{n-2}\frac{\epsilon_{n-1}\cdot p_a}{\rho-x_a}\right)\left(\sum_{b=1}^{n-2}\frac{\epsilon_n\cdot p_b}{\rho-x_b}\right)+\frac{1}{\xi_1}\Bigg\{\epsilon_{n-1}\cdot k_n\left(\sum_{a=1}^{n-2}\frac{\epsilon_n\cdot p_a}{\rho-x_a}\right) \\
& -\epsilon_n\cdot k_{n-1}\left(\sum_{b=1}^{n-2}\frac{\epsilon_{n-1}\cdot p_b}{\rho-x_b}\right)\Bigg\}+\frac{\epsilon_{n-1}\cdot\epsilon_n\ k_{n-1}\cdot k_n}{\xi_1^2}\Bigg]\mathrm{Pf}'\Psi_{n-2}
\end{aligned}
$$

Factorization of Parke-Taylor factor

$$
C_n = -\frac{(x_{n-2}-\sigma_1)}{\tau\xi_1\,(x_{n-2}-\rho)\,(\rho-\sigma_1)}C_{n-2}
$$

Extra terms are cancelled between degenerate and non-degenerate solutions.

Double soft theorem in YM

$$
\begin{aligned}
& A_n^{\mathrm{YM}}\left(p_1, \epsilon_1; p_2, \epsilon_2; \ldots; p_{n-2}, \epsilon_{n-2}; \tau k_{n-1}, \epsilon_{n-1}; \tau k_n, \epsilon_n\right) \\
= \quad & \frac{1}{\tau^2}\Bigg[\left(\frac{\epsilon_{n-1}\cdot p_{n-2}}{k_{n-1}\cdot p_{n-2}} - \frac{\epsilon_{n-1}\cdot p_1}{k_{n-1}\cdot p_1}\right)\frac{\epsilon_n\cdot p_1}{k_n\cdot p_1} \\
& +\frac{\epsilon_{n-1}\cdot p_1\ \epsilon_n\cdot p_1}{(k_{n-1}+k_n)\cdot p_1\ k_{n-1}\cdot p_1} - \frac{\epsilon_{n-1}\cdot p_{n-2}\ \epsilon_n\cdot p_{n-2}}{(k_{n-1}+k_n)\cdot p_{n-2}\ k_{n-1}\cdot p_{n-2}} \\
& +\frac{1}{k_{n-1}\cdot k_n}\Bigg\{\frac{\epsilon_{n-1}\cdot k_n\ \epsilon_n\cdot p_{n-2} - \epsilon_n\cdot k_{n-1}\ \epsilon_{n-1}\cdot p_{n-2}}{(k_{n-1}+k_n)\cdot p_{n-2}} \\
& -\frac{\epsilon_{n-1}\cdot k_n\ \epsilon_n\cdot p_1 - \epsilon_n\cdot k_{n-1}\ \epsilon_{n-1}\cdot p_1}{(k_{n-1}+k_n)\cdot p_1}\Bigg\} \\
& +\frac{1}{2}\frac{\epsilon_{n-1}\cdot\epsilon_n}{k_{n-1}\cdot k_n}\Bigg\{\frac{(k_{n-1}-k_n)\cdot p_{n-2}}{(k_{n-1}+k_n)\cdot p_{n-2}} - \frac{(k_{n-1}-k_n)\cdot p_1}{(k_{n-1}+k_n)\cdot p_1}\Bigg\}\Bigg] \\
& \qquad\qquad A_{n-2}^{\mathrm{YM}}\left(p_1, \epsilon_1; p_2, \epsilon_2; \ldots, p_{n-2}, \epsilon_{n-2}\right)
\end{aligned}
$$

Bi-adjoint scalars

Degenerate solutions

Change of variables

$$x_{n-1} = \rho + \xi, \qquad x_n = \rho - \xi, \qquad \xi \sim \mathcal{O}(\tau)$$

Evaluate contour integrations from two delta functions

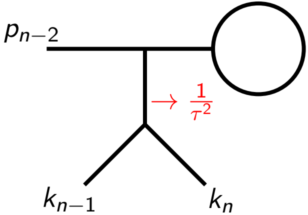
$$\mathrm{S}^{(2)}_{\mathrm{DS}} = \int d\rho \int d\xi\, \delta(E_{n-1}+E_n)\,\delta(E_{n-1}-E_n) \left[\frac{x_{n-2}-x_1}{\xi\,(x_{n-2}-\rho)\,(\rho-x_1)}\right]^2$$

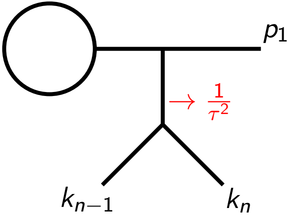
$$\boxed{\mathbf{E_{n-1}-E_n} = \sum_{\mathbf{b=1}}^{\mathbf{n-2}} \frac{\mathbf{s_{n-1\;b}-s_{n\;b}}}{\rho-\mathbf{x_b}} + \frac{\mathbf{s_{n-1\;n}}}{\xi}}$$

$$= -\int d\rho\, \delta(E_{n-1}+E_n) \oint_{\{\xi\to 0\}} \frac{d\xi}{\frac{s_{n-1\;n}}{\xi}} \left[\frac{x_{n-2}-x_1}{\xi\,(x_{n-2}-\rho)\,(\rho-x_1)}\right]^2$$

$$= \frac{\mathbf{1}}{\tau^{\mathbf{3}}}\, \frac{\mathbf{1}}{\mathbf{\hat{s}_{n-1\;n}}} \left[\frac{\mathbf{1}}{\mathbf{\hat{s}_{n-1\;n-2}+\hat{s}_{n\;n-2}}} + \frac{\mathbf{1}}{\mathbf{\hat{s}_{n-1\;1}+\hat{s}_{n\;1}}}\right]$$

$\frac{1}{\tau^3}$ scaling is easy to understand from Feynman diagrams





$$\frac{1}{s_{n-1\ n}} \times \frac{1}{s_{n-1\ n-2} + s_{n\ n-2}} \qquad \frac{1}{s_{n-1\ n}} \times \frac{1}{s_{n-1\ 1} + s_{n\ 1}}$$

Non-adjacent soft limit: Double soft factor is expressed as product of two single soft factors. For example, if soft particles are labelled by $\{\mathbf{n-2}, \mathbf{n}\}$ then

$$\mathrm{S}_{\mathrm{DS}}^{(2)} = \frac{1}{\tau^2}\left(\frac{1}{\hat{s}_{n\ n-1}} + \frac{1}{\hat{s}_{n\ 1}}\right)\left(\frac{1}{\hat{s}_{n-2\ n-3}} + \frac{1}{\hat{s}_{n-2\ n-1}}\right)$$

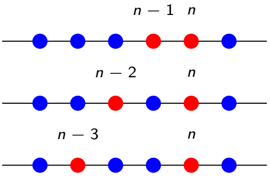
$\mathbf{k = 3}$

We consider **three** possible combinations:

Adjacent particles are soft

Next to adjacent particles are soft

Next to next to adjacent particles are soft



Abhishek, Hegde, Jatkar, APS

• Next to next to adjacent soft limits are trivial and are given by product of two single soft factors. Soft particles can be chosen to be $(\mathbf{n}, \mathbf{a})$, with $\mathbf{a} \in \{\mathbf{3}, \ldots, \mathbf{n-3}\}$. Soft factor scales as $\mathrm{S}^{(3)}_{\mathrm{DS}} \sim \frac{1}{\tau^4}$.

- **Adjacent soft limit**

$$s_{nab} = \tau \hat{s}_{nab}$$
$$s_{n-1\ a\ b} = \tau \hat{s}_{n-1\ a\ b} \qquad a, b \in \{1, \ldots, n-2\}$$
$$s_{n-1\ n\ a} = \tau^2 \hat{s}_{n-1\ n\ a}$$

Scattering equations for **regular solutions**

$$\left\{ |\mathbf{a\ b\ n-1}| \not\propto \tau, \quad |\mathbf{abn}| \not\propto \tau, \quad \forall \mathbf{a}, \mathbf{b} \right\}$$

$$E_a^{(i)} = \sum_{b,c \neq a} \frac{s_{abc}}{|abc|} \frac{\partial}{\partial x_a^{(i)}} |abc|$$

$$E_{n-1}^{(i)} = \tau \sum_{a,b} \frac{\hat{s}_{a\ b\ n-1}}{|a\ b\ n-1|} \frac{\partial}{\partial x_{n-1}^{(i)}} |a\ b\ n-1| + \tau^2 \sum_a \frac{\hat{s}_{a\ n-1\ n}}{|a\ n-1\ n|} \frac{\partial}{\partial x_{n-1}^{(i)}} |a\ n-1\ n|$$

$$E_n^{(i)} = \tau \sum_{a,b} \frac{\hat{s}_{abn}}{|abn|} \frac{\partial}{\partial x_n^{(i)}} |abn| + \tau^2 \sum_a \frac{\hat{s}_{a\ n-1\ n}}{|a\ n-1\ n|} \frac{\partial}{\partial x_n^{(i)}} |a\ n-1\ n|$$

Degenerate solutions contribute to leading order. There are **two** sources of degenerate solutions, $|\mathbf{a}\ \mathbf{n}-\mathbf{1}\ \mathbf{n}| \sim \mathcal{O}(\tau)$

- $\sigma_{\mathbf{n-1}}$ and $\sigma_{\mathbf{n}}$ collide with each other.
- $\sigma_{\mathbf{n-1}}$ and $\sigma_{\mathbf{n}}$ are collinear with *any one* hard puncture.

<u>**Soft punctures collide**</u>

$$\begin{aligned} x^i_{n-1} &= \rho^i + \xi^i, \\ x^i_n &= \rho^i - \xi^i, \end{aligned} \qquad \rho^i = \begin{pmatrix} 1 \\ \rho^1 \\ \rho^2 \end{pmatrix}, \quad \xi^i = \begin{pmatrix} 0 \\ \xi^1 \\ \xi^2 \end{pmatrix}, \qquad \xi^i \sim \tau$$

Soft factor

$$\begin{aligned} \mathrm{S}^{(3)}_{\mathrm{DS}} \;=\; & \int d^2\rho\, d^2\xi\, \delta^{(2)}\left(E_{n-1} + E_n\right) \delta^{(2)}\left(E_{n-1} - E_n\right) \\ & \times \left[\frac{|n-3\ n-2\ 1||n-2\ 1\ 2|}{|n-3\ n-2\ \rho||n-2\ \xi\ \rho||\rho\ \xi\ 1||\rho\ 1\ 2|}\right]^2 \end{aligned}$$

$$E^{(i)}_{n-1} + E^{(i)}_{n} = \sum_{a,b} \frac{s_{a\ b\ n-1} + s_{abn}}{|ab\rho|} \frac{\partial}{\partial \rho^i} |ab\rho| \quad \text{(fixes } \rho \text{ integration)}$$

$$E^{(i)}_{n-1} - E^{(i)}_{n} = \sum_{a,b} \frac{s_{a\ b\ n-1} - s_{abn}}{|ab\rho|} \frac{\partial}{\partial \rho^i} |ab\rho| + \sum_{a} \frac{s_{a\ n-1\ n}}{|a\rho\xi|} \frac{\partial}{\partial \xi^i} |a\rho\xi|$$

$$\Big\downarrow \text{dominates as } |a\rho\xi| \to 0$$

$$\frac{\partial}{\partial \xi^i} \underbrace{\sum_{a=1}^{n-2} s_{an-1n} \log |a\rho\xi|}_{\tilde{\mathcal{S}}}$$

$$\begin{aligned} \xi^1 &= \epsilon \\ \xi^2 &= \epsilon\alpha \end{aligned} \quad \Rightarrow \quad d\tilde{\mathcal{S}} = \sum_{a=1}^{n-2} s_{a\ n-1\ n} \left[\frac{d\epsilon}{\epsilon} + \frac{d\alpha}{\alpha - \alpha_a} \right], \qquad \alpha_a = \frac{x_a^2 - \rho^2}{x_a^1 - \rho^1}$$

Poles of ξ are at $\epsilon = \mathbf{0}$ and $\alpha = \alpha_{\mathbf{n-2}}, \alpha_{\mathbf{1}}$.

Adjacent double soft factor

$$\begin{aligned}
&\mathrm{S}_{\mathrm{DS}}^{(3)} \\
= \;& \frac{1}{\tau^6} \frac{1}{\sum_{a=1}^{n-2} \hat{s}_{a\; n-1\; n}} \left[\frac{1}{\hat{s}_{1\; n-1\; n}} + \frac{1}{\hat{s}_{n-2\; n-1\; n}} \right] \left\{ \frac{1}{\sum_{a=1}^{n-3} \left(\hat{s}_{a\; n-2\; n-1} + \hat{s}_{a\; n-2\; n} \right)} \right. \\
& \times \left(\frac{1}{\hat{s}_{n-3\; n-2\; n-1} + \hat{s}_{n-3\; n-2\; n}} + \frac{1}{\hat{s}_{n-2\; n-1\; 1} + \hat{s}_{n-2\; n\; 1}} \right) \\
& + \frac{1}{\sum_{a=2}^{n-2} \left(\hat{s}_{1\; a\; n-1} + \hat{s}_{1\; a\; n} \right)} \left(\frac{1}{\hat{s}_{n-2\; n-1\; 1} + \hat{s}_{n-2\; n\; 1}} + \frac{1}{\hat{s}_{n-1\; 1\; 2} + \hat{s}_{n\; 1\; 2}} \right) \\
& \left. + \frac{1}{\left(\hat{s}_{n-1\; 1\; 2} + \hat{s}_{n\; 1\; 2} \right) \left(\hat{s}_{n-3\; n-2\; n-1} + \hat{s}_{n-3\; n-2\; n} \right)} \right\}
\end{aligned}$$

When the soft punctures are collinear with any one hard puncture, at most one of the determinants in the integrand can scale as τ. Hence this configuration contributes to sub-leading soft theorem.

Consistency check: Reproduces consecutive soft factor
$(\mathbf{s_{nab}} = \tau_1 \hat{\mathbf{s}}_{\mathbf{nab}}, \quad \mathbf{s_{n-1\,a\,b}} = \tau_2 \hat{\mathbf{s}}_{\mathbf{n-1\,a\,b}}, \quad \tau_1 \ll \tau_2)$

$$
\begin{aligned}
& S^{(3)}_{CS} \\
= \quad & \frac{1}{\tau_1^2 \tau_2^4} \frac{1}{\sum_{a=1}^{n-2} \hat{s}_{a\;n-1\;n}} \left[\frac{1}{\hat{s}_{1\;n-1\;n}} + \frac{1}{\hat{s}_{n-2\;n-1\;n}} \right] \\
& \times \left\{ \frac{1}{\sum_{a=1}^{n-3} \hat{s}_{a\;n-2\;n-1}} \left(\frac{1}{\hat{s}_{n-3\;n-2\;n-1}} + \frac{1}{\hat{s}_{n-2\;n-1\;1}} \right) \right. \\
& \left. + \frac{1}{\sum_{a=2}^{n-2} \hat{s}_{1\;a\;n-1}} \left(\frac{1}{\hat{s}_{n-2\;n-1\;1}} + \frac{1}{\hat{s}_{n-1\;1\;2}} \right) + \frac{1}{\hat{s}_{n-1\;1\;2}\, \hat{s}_{n-3\;n-2\;n-1}} \right\}
\end{aligned}
$$

Generalisation to arbitrary k:

Soft limits in mandelstam variables:

$$\mathbf{s_{n-1\ a_1 \ldots a_{k-1}}},\ \mathbf{s_{n\ a_1 \ldots a_{k-1}}} \sim \mathcal{O}(\tau), \qquad \mathbf{s_{n-1\ n\ a_1 \ldots a_{k-2}}} \sim \mathcal{O}\left(\tau^{\mathbf{2}}\right).$$

Scattering equations are

$$\begin{aligned}
E_{a_1}^{(i)} &= \sum_{1 \le a_2 \ldots < a_k \le n-2} \frac{s_{a_1 \ldots a_k}}{|a_1 \ldots a_k|} \frac{\partial}{\partial x_{a_1}^i} |a_1 \ldots a_k|, \qquad \forall a_1 \\
E_{n-1}^{(i)} &= \sum_{1 \le a_1 \ldots < a_{k-1} \le n-2} \frac{s_{a_1 \ldots a_{k-1}\ n-1}}{|a_1 \ldots a_{k-1}\ n-1|} \frac{\partial}{\partial x_{n-1}^i} |a_1 \ldots a_{k-1}\ n-1|, \\
&+ \sum_{1 \le a_1 \ldots < a_{k-2} \le n-2} \frac{s_{a_1 \ldots a_{k-2}\ n-1\ n}}{|a_1 \ldots a_{k-2}\ n-1\ n|} \frac{\partial}{\partial x_{n-1}^i} |a_1 \ldots a_{k-2}\ n-1\ n|, \\
E_{n}^{(i)} &= \sum_{1 \le a_1 \ldots < a_{k-1} \le n-2} \frac{s_{a_1 \ldots a_{k-1}\ n}}{|a_1 \ldots a_{k-1}\ n|} \frac{\partial}{\partial x_{n}^i} |a_1 \ldots a_{k-1}\ n| \\
&+ \sum_{1 \le a_1 \ldots < a_{k-2} \le n-2} \frac{s_{a_1 \ldots a_{k-2}\ n-1\ n}}{|a_1 \ldots a_{k-2}\ n-1\ n|} \frac{\partial}{\partial x_{n}^i} |a_1 \ldots a_{k-2}\ n-1\ n|.
\end{aligned}$$

We look at the **degenerate solutions.**

Strategy:

- New variables:

$$
\begin{aligned}
x_{n-1}^i &= \rho^i + \xi^i, \\
x_n^i &= \rho^i - \xi^i, \qquad \xi^i \sim \mathcal{O}(\tau), \quad i = 1, \dots k-1.
\end{aligned}
$$

- First perform the ξ integration. $(\mathbf{k-1})$ dimensional integration is separated by choosing

$$
\begin{aligned}
\xi^1 &= \epsilon, \\
\xi^\alpha &= \epsilon\zeta^\alpha, \quad \alpha = 2, \dots k-1.
\end{aligned}
$$

  ϵ integration gives a residue $\frac{1}{\sum_{1 \le a_1 \dots < a_{k-2} \le n-2} s_{a_1 \dots a_{k-2}\ n-1\ n}}$.

  $(\mathbf{k-2})$ dimensional ζ integral produces $\mathbb{S}^{(\mathbf{k-1})}$.

- ρ integration produces another factor of $\mathbb{S}^{(\mathbf{k})}$.

Double soft factor for arbitrary **k**:

$$\boxed{\mathrm{S}^{(k)}_{\mathrm{DS}} = \frac{1}{\sum\limits_{1\le a_1\ldots<a_{k-2}\le n-2} s_{a_1\ldots a_{k-2}\; n-1\; n}} \mathrm{S}^{(k-1)}\left(s_{a_1\ldots a_{k-2}\; m} \to s_{a_1\ldots a_{k-2}\; n-1\; n}\right)\; \mathrm{S}^{(k)}}$$

Double soft factor for arbitrary **k** can be obtained from $\mathbf{k-1}$ and **k** single soft factors. This gives a recursive relation.

- Poles in $\mathrm{S}^{(\mathbf{k})}$ are shifted: $\mathbf{s_{a_1\ldots a_{k-1}m} \to s_{a_1\ldots a_{k-1}n-1} + s_{a_1\ldots a_{k-1}n}}$.
- Poles in $\mathrm{S}^{(\mathbf{k-1})}$ are obtained by the replacement: $\mathbf{s_{a_1\ldots a_{k-2}m} \to s_{a_1\ldots a_{k-2}\; n-1\; n}}$.
- Scaling of $\mathrm{S}^{(\mathbf{k})}_{\mathrm{DS}}$ is $\frac{\mathbf{1}}{\tau^{\mathbf{3(k-1)}}}$.

- **Next to adjacent soft limit** (soft particles: $\mathbf{n-2, n}$)

Integral expression of soft factor:

$$\int d^2\sigma_n \int d^2\sigma_{n-2}\, \delta^{(2)}(E_n)\, \delta^{(2)}(E_{n-2})$$

$$\times \quad \left[\frac{|n-4\ n-3\ n-1||n-3\ n-1\ 1||n-1\ 1\ 2|}{|n-4\ n-3\ n-2||n-3\ n-2\ n-1||n-2\ n-1\ n||n-1\ n\ 1||n12|}\right]^2$$

Degenerate solutions:

- In the case $\sigma_{\mathbf{n-2}} - \sigma_{\mathbf{n}} \sim \tau$, scaling of the integrals is $\frac{\mathbf{1}}{\tau^{\mathbf{4}}}$. In fact this contribution vanishes.
- Integrals scale as $\frac{\mathbf{1}}{\tau^{\mathbf{5}}}$ when $\sigma_{\mathbf{n-2}}$ and $\sigma_{\mathbf{n}}$ are nearly collinear with $\sigma_{\mathbf{n-1}}$

$$x^1_{n-2} = x^1_{n-1} + \alpha\left(x^1_n - x^1_{n-1}\right), \qquad x^2_{n-2} = x^2_{n-1} + \alpha\left(x^2_n - x^2_{n-1}\right) + \tau\xi.$$

$\frac{\mathbf{1}}{\tau^{\mathbf{5}}}$ scaling of soft factor is consistent with *consecutive soft limits*.

Non-degenerate solutions contribute to $\mathcal{O}\left(\frac{1}{\tau^4}\right)$ in soft factor.

Residues come from the following boundaries

$\sigma_{n-2} \to \sigma_{n-3}$ $\quad$ $\sigma_n \to \sigma_{n-1}$

$\sigma_n \to \sigma_1$

σ_n is collinear with $(\sigma_{n-3}, \sigma_{n-1})$ and (σ_1, σ_2)

$\sigma_{n-2} \to \sigma_{n-1}$ $\quad$ $\sigma_n \to \sigma_{n-1}$

$\sigma_n \to \sigma_1$

$\sigma_{n-2} \to \sigma_n$ $\quad$ $\sigma_n \to \sigma_{n-1}$

σ_{n-2} is collinear with $(\sigma_{n-3}, \sigma_{n-4})$ and (σ_{n-1}, σ_n) $\quad$ $\sigma_n \to \sigma_{n-1}$ $\quad$ $\sigma_n \to \sigma_1$ $\quad$ σ_n is collinear with (σ_1, σ_2) and $(\sigma_{n-3}, \sigma_{n-1})$

Summary

- We have presented adjacent double soft limits of generalized bi-adjoint scalar amplitudes for arbitrary k. Recursion relation relates this double soft factor to lower-k single soft factor.
- For arbitrary k scaling of double soft factors goes as

$$\text{adjacent} \sim \frac{\mathbf{1}}{\tau^{\mathbf{3(k-1)}}}$$

$$\mathbf{N^{k-1}}, \ldots, \mathbf{N^{n-k}} - \textit{adjacent} \sim \frac{\mathbf{1}}{\tau^{\mathbf{2(k-1)}}}$$

- $\frac{1}{\tau^{k-1}}$ poles are different from Weinberg poles. So the notions of locality and unitarity are not well understood. Studying soft theorems using planar collections of Feynman diagrams introduced by Borges and Cachazo may offer some insights.
- Connection to BCFW recursions and positive geometries?
- Generalizations to multiple soft theorems for arbitrary k?
- Total number of solutions to generalized scattering equations for any k is not known. Soft limits can be used to understand the solutions.

Thank you