Soft Theorem and its Classical Limit

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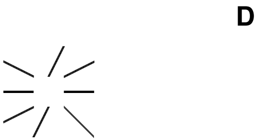
Bangalore, July 2020

Consider a violent explosion in space



A bound system breaks apart into fragments.

This process emits gravitational waves

Detector D placed far away detects $\mathbf{h}_{\mu\nu} \equiv (\mathbf{g}_{\mu\nu} - \eta_{\mu\nu})/2$

Examples: Explosion of supernova, binary black hole merger etc.

A more general situation: collision

A set of objects come together, interact strongly, and produce another set of objects.

This will also produce gravitational waves.

A simple example: Bullet cluster

A supercluster of galaxies passing through another supercluster of galaxies.

In general, computing gravitational wave-form h$_{\mu\nu}$ produced during such processes is complicated.

1. When the objects are close, they may undergo complicated, non-gravitational interactions, as in the case of explosion of supernova.

2. Gravity is non-linear

– even if the interactions were purely gravitational, e.g. in the case of black hole merger, the analysis is complicated.

However certain results involving S-matrix of quantum theory of gravity, known as soft graviton theorem, suggest some exact results for these classical problems.

The goal of these lectures is to explain the soft graviton theorem, its derivation, and the kind of results we can get by taking its classical limit.

We shall begin by giving a preview of some of the classical results that will come out of this analysis.

General case: Consider a scattering in space

A set of objects of four momenta $p'_1, \cdots p'_m$ come together, interact, and disperse as a set of other objects with four momenta $p_1, \cdots p_n$.

$$p_i^2 \equiv -(p_i^0)^2 + \vec{p}_i^2 = -m_i^2, \quad p_i'^2 = -m_i'^2, \quad i = 1, 2, \cdots ,$$

We shall choose the origin of space-time to be in the region where the scattering event takes place

Detector D placed at a far way point $\vec{x}$ detects $h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu})/2$ around time t_0:

$$t_0 = R/c + \text{correction}, \quad R \equiv |\vec{x}|$$

The correction is due to the gravitational drag on the gravitational radiation.

Define retarded time:

$$\mathbf{u} \equiv \mathbf{t} - \mathbf{t_0}$$

Our focus will be on the late and early time tail of the radiation – the value of $h_{\mu\nu}$ at D at large positive u and large negative u.

Define $e_{\mu\nu}$ via:

$$e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,h_{\rho\sigma} \quad \Leftrightarrow \quad h_{\mu\nu} = e_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,e_{\rho\sigma}$$

Up to gauge transformations and corrections of order R^{-2},

$$\boxed{\begin{aligned} e_{\mu\nu} &= A_{\mu\nu} + \frac{1}{u}\,B_{\mu\nu} + \mathcal{O}(u^{-2}\ln|u|), \quad \text{for large positive u} \\ e_{\mu\nu} &= \frac{1}{u}\,C_{\mu\nu} + \mathcal{O}(u^{-2}\ln|u|), \quad \text{for large negative u} \end{aligned}}$$

$A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}$ are given solely by the momenta of the ingoing and outgoing objects without requiring any knowledge of the details of the scattering process.

$$A^{\mu\nu} = \frac{2G}{Rc^3}\left[-\sum_{i=1}^{n} p_i^{\mu}\, p_i^{\nu}\, \frac{1}{n.p_i} + \sum_{i=1}^{m} p_i'^{\mu}\, p_i'^{\nu}\, \frac{1}{n.p_i'}\right], \quad R \equiv |\vec{x}|, \quad n \equiv (1, \vec{x}/R)$$

$$\begin{aligned} B^{\mu\nu} = -\frac{4G^2}{Rc^7}\Bigg[& \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{p_i.p_j}{\{(p_i.p_j)^2 - m_i^2 m_j^2 c^4\}^{3/2}} \left\{\frac{3}{2} m_i^2 m_j^2 c^4 - (p_i.p_j)^2\right\} \\ & \times \frac{p_i^{\mu}}{n.p_i}\,(n.p_j\, p_i^{\nu} - n.p_i\, p_j^{\nu}) \\ & - \sum_{j=1}^{n} p_j.n \left\{\sum_{i=1}^{n} \frac{1}{p_i.n}\, p_i^{\mu} p_i^{\nu} - \sum_{i=1}^{m} \frac{1}{p_i'.n}\, p_i'^{\mu} p_i'^{\nu}\right\}\Bigg] \end{aligned}$$

$$\begin{aligned} C^{\mu\nu} = \frac{4G^2}{Rc^7}\Bigg[& \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{p_i'.p_j'}{\{(p_i'.p_j')^2 - m_i'^2 m_j'^2 c^4\}^{3/2}} \left\{\frac{3}{2} m_i'^2 m_j'^2 c^4 - (p_i'.p_j')^2\right\} \\ & \times \frac{p_i'^{\mu}}{n.p_i'}\,(n.p_j'\, p_i'^{\nu} - n.p_i'\, p_j'^{\nu})\Bigg] . \end{aligned}$$

$$\mathbf{e}_{\mu\nu} = \mathbf{A}_{\mu\nu} + \frac{1}{\mathbf{u}}\,\mathbf{B}_{\mu\nu} + \mathcal{O}(\mathbf{u}^{-2}\ln|\mathbf{u}|), \quad \text{for large positive u}$$

$$\mathbf{e}_{\mu\nu} = \frac{1}{\mathbf{u}}\,\mathbf{C}_{\mu\nu} + \mathcal{O}(\mathbf{u}^{-2}\ln|\mathbf{u}|), \quad \text{for large negative u}$$

$A_{\mu\nu}$: memory term

– a permanent change in the state of the detector after the passage of gravitational waves

Zeldovich, Polnarev; Braginsky, Grishchuk; Braginsky, Thorne; $\cdots$

– connected to the leading soft graviton theorem

Strominger; $\cdots$

$B_{\mu\nu}, C_{\mu\nu}$: tail terms

– connected to logarithmic terms in the subleading soft graviton theorem

Laddha, A.S.; Sahoo, A.S.

1. The result is a statement in classical GR, even though it was originally suggested by quantum soft graviton theorem.

Now we have a fully classical derivation. Saha, Sahoo, A.S.

2. $\mathsf{A}_{\mu\nu}, \mathsf{B}_{\mu\nu}, \mathsf{C}_{\mu\nu}$ can be expressed in terms of the momenta of incoming and outgoing objects without knowing what forces operated and how the objects moved during the scattering.

– consequence of soft graviton theorem

3. If a significant fraction of energy is carried away by radiation, then the sum over i,j includes integration over outgoing flux of radiation, regarded as a flux of massless particles.

4. The result matches explicit known results in special cases.

Peters; Ciafaloni, Colferai, Veneziano; Addazi, Bianchi, Veneziano

5. Explosion can be regarded as a special case of scattering when the initial state has just one object.

In this case $\mathbf{C}_{\mu\nu}$ vanishes and $\mathbf{e}_{\mu\nu}$ takes the form:

$$\mathbf{e}_{\mu\nu} = \mathbf{A}_{\mu\nu} + \frac{\mathbf{1}}{\mathbf{u}}\mathbf{B}_{\mu\nu} + \mathcal{O}(\mathbf{u}^{-2}\ln|\mathbf{u}|), \quad \text{for large positive u}$$

$$\mathbf{e}_{\mu\nu} = \mathbf{0}, \quad \text{for large negative u}$$

6. For explosion, the contribution to $B^{\mu\nu}$ vanishes unless there are at least two massive objects in the final state

– result of cancellation between different terms

$$
\begin{aligned}
B^{\mu\nu} = -\frac{4\,G^2}{R\,c^7}\Bigg[& \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{p_i.p_j}{\{(p_i.p_j)^2 - m_i^2 m_j^2 c^4\}^{3/2}} \left\{\frac{3}{2} m_i^2 m_j^2 c^4 - (p_i.p_j)^2\right\} \\
& \times \frac{p_i^\mu}{n.p_i}\left(n.p_j\, p_i^\nu - n.p_i\, p_j^\nu\right) \\
& - \sum_{j=1}^{n} p_j.n \left\{\sum_{i=1}^{n} \frac{1}{p_i.n}\, p_i^\mu p_i^\nu - \sum_{i=1}^{m} \frac{1}{p_i'.n}\, p_i'^\mu p_i'^\nu\right\}\Bigg]
\end{aligned}
$$

$\Rightarrow$ the coefficient of the 1/u tail vanishes for binary black hole merger!

Final state contains one massive object (remnant black hole) and a flux of massless particles (gravitational radiation).

In contrast, the coefficient of the 1/u term is non-zero for supernova explosion, binary neutron star merger etc.

PLAN

1. Quantum soft graviton theorem and its derivation

2. Classical limit

3. Classical proof

4. Issues with infrared divergences in D=4

5. Resolution

Convention: $\hbar = 1, \quad c = 1, \quad 8\pi G = 1$

Quantum soft graviton theorem

What is soft graviton theorem?

Take a general coordinate invariant quantum theory of gravity coupled to matter fields

Consider an S-matrix element involving

– arbitrary number N of external particles of finite momentum $p_1, \cdots p_N$

– M external gravitons carrying small momentum $k_1, \cdots k_M$.

Soft graviton theorem: Expansion of this amplitude in power series in $k_1, \cdots k_M$ in terms of the amplitude without the low energy (soft) gravitons.

There are many explicit results.

1. General results at leading order in k — **Weinberg; . . .**

2. For <u>one</u> soft graviton, there are general subleading results in D=4 via BMS

Strominger; Strominger, Zhiboedov; Campiglia, Laddha; . . .

3. Results in specific theories in general dimensions

White; Cachazo, Strominger; Bern, Davies, Di Vecchia, Nohle; Elvang, Jones, Naculich; . . .

Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha

Bianchi, Guerrieri; Di Vecchia, Marotta, Mojaza; . . .

Our goal: Study soft graviton amplitudes in generic quantum theory of gravity, in generic number of dimensions, for arbitrary mass and spin of external states

Validity

1. For tree amplitudes our analysis will be valid in all dimensions

2. For loop amplitudes the results will be valid if we assume that 1PI vertices do not generate soft factors in the denominator

True by power counting for

– subleading order for D > 5

– subsubleading order for D > 6

D: number of non-compact space-time dimensions

For single soft gravitons we can argue that the unwanted terms cancel in the sum over graphs and the results are also valid for D=5,6

We expect a similar result to hold for multiple soft gravitons, but this has not been proved.

In D=4 the S-matrix elements themselves are infrared divergent, introducing additional subtleties. Bern, Davies, Nohle

In $D = 4$, our analysis will apply to only tree amplitudes, but our results in D>4 will suggest how to modify the results in D=4.

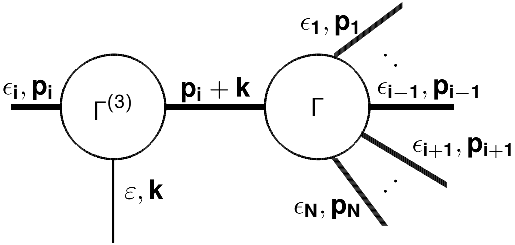
A.S. arXiv:1702.03934, 1703.00024: Subleading single soft

A. Laddha, A.S., arXiv:1706.00759: Sub-subleading single soft

Subhroneel Chakrabarti, Sitender Kashyap, Biswajit Sahoo, A.S., Mritunjay Verma, arXiv:1707.06803; subleading multiple soft

Single soft graviton

We divide the Feynman diagrams into two classes



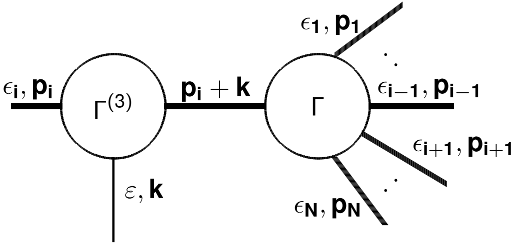
Γ: Full amputated Green's function

Internal lines: Full renormalized propagators

ε, k: polarization, momentum of soft graviton

ϵ_i, p_i: polarization, momentum of finite energy external particles.

p_i, k counted with + sign if outgoing and $-$ sign if ingoing



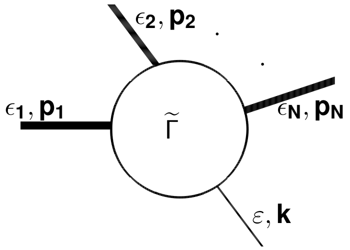
The internal line carrying momentum $p_i + k$ has denominator factor

$$\{(p_i + k)^2 + m^2\}^{-1} = (2p_i \cdot k)^{-1} \quad \text{if } m = m_i$$

using $p_i^2 + m_i^2 = 0$, $k^2 = 0$.

$\Rightarrow$ this starts contributing at the leading order.

Second class of diagrams



$\widetilde{\Gamma}$: Amputated amplitudes in which the external soft graviton <u>does not</u> get attached to an external line

– has no pole as k $\to$ 0

$\Rightarrow$ the contribution from this diagram begins at the subleading order.

Strategy for computation

1. Consider the gauge invariant one particle irreducible (1PI) effective action of the theory

2. Expand the action in powers of all fields, including the metric fluctuations, around the extremum of the action

3. Add manifestly Lorentz invariant gauge fixing terms.

4. This action is used to compute vertices and propagators of finite energy external states but <u>not</u> of <u>soft external gravitons</u>.

5. To calculate the coupling of the soft graviton $S_{\mu\nu}$ to the rest of the fields, we covariantize the gauge fixed action.

a. Replace the background metric $\eta_{\mu\nu}$ by $\eta_{\mu\nu} + 2\,S_{\mu\nu}$

b. Replace all derivatives by covariant derivatives computed with the metric $\eta_{\mu\nu} + 2\,S_{\mu\nu}$

This misses terms involving Riemann tensor computed from the metric $\eta_{\mu\nu} + 2\,S_{\mu\nu}$ but that contains two derivatives and hence is sub-subleading.

1. We choose

$$\mathbf{S}_{\mu\nu} = \varepsilon_{\mu\nu} \mathbf{e}^{\mathbf{ik\cdot x}}, \quad \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}, \quad \varepsilon^{\mu}{}_{\mu} = \mathbf{k}^{\mu} \varepsilon_{\mu\nu} = \mathbf{0}$$

All indices raised and lowered by η.

2. All fields representing finite energy external states are taken to carry tangent space Lorentz indices

– allows us to give uniform treatment to fermions and bosons.

3. To first order in $\mathbf{S}_{\mu\nu}$, we take the vielbeins to be

$$\mathbf{e}_{\mu}^{\ \mathbf{a}} = \delta_{\mu}^{\ \mathbf{a}} + \mathbf{S}_{\mu}^{\ \mathbf{a}}, \quad \mathbf{E}_{\mathbf{a}}^{\ \mu} = \delta_{\mathbf{a}}^{\ \mu} - \mathbf{S}_{\mathbf{a}}^{\ \mu}, \quad \mathbf{g}^{\mu\nu} = \eta^{\mu\nu} - 2\,\mathbf{S}^{\mu\nu}$$

$\{\phi_\alpha\}$: **set of all the fields in the theory with α's including Lorentz indices.**

$\mathbf{i}(\Sigma_{\mathbf{ab}})_\alpha{}^\beta$: **Infinitesimal Lorentz transformation matrix**

Covariantization: Acting on a field ϕ_α:

$$\partial_{\mathbf{a_1}} \cdots \partial_{\mathbf{a_n}} \Rightarrow \mathbf{E}_{\mathbf{a_1}}^{\ \mu_1} \cdots \mathbf{E}_{\mathbf{a_n}}^{\ \mu_n} \mathbf{D}_{\mu_1} \cdots \mathbf{D}_{\mu_n}, \quad \mathbf{E}_{\mathbf{a}}{}^{\mu} \equiv (\delta_{\mathbf{a}}{}^{\mu} - \mathbf{S}_{\mathbf{a}}{}^{\mu})$$

$$\mathbf{D}_\mu \phi_\alpha = \partial_\mu \phi_\alpha - \frac{\mathbf{i}}{\mathbf{2}} \omega_\mu^{\mathbf{ab}} (\Sigma_{\mathbf{ab}})_\alpha{}^\gamma \phi_\gamma$$

$$\mathbf{D}_{(\mu} \mathbf{D}_{\nu)} \phi_\alpha = \partial_\mu \partial_\nu \phi_\alpha - \mathbf{i}\, \omega_{(\mu}^{\mathbf{ab}} (\Sigma_{\mathbf{ab}})_\alpha{}^\gamma \partial_{\nu)} \phi_\gamma - \frac{\mathbf{i}}{\mathbf{2}} \partial_{(\mu} \omega_{\nu)}^{\mathbf{ab}} (\Sigma_{\mathbf{ab}})_\alpha{}^\gamma \phi_\gamma + \left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} \partial_\rho \phi_\alpha$$

etc.

$$\omega_\mu^{\mathbf{ab}} = \partial^{\mathbf{b}} \mathbf{S}_\mu{}^{\mathbf{a}} - \partial^{\mathbf{a}} \mathbf{S}_\mu{}^{\mathbf{b}}, \quad \mathbf{S}_{\mu\nu} = \varepsilon_{\mu\nu} \mathbf{e}^{\mathbf{ik \cdot x}}$$

$$\left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} = \frac{\mathbf{1}}{\mathbf{2}} \left[\partial_\mu \mathbf{S}_\nu{}^\rho + \partial_\nu \mathbf{S}_\mu{}^\rho - \partial^\rho \mathbf{S}_{\mu\nu} \right]$$

Consider a Lorentz invariant functional

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})$$
$$\delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N})\, \mathbf{F}^{\alpha_1 \cdots \alpha_N}(\mathbf{p_1}, \ldots \mathbf{p_N})$$

Covariantization produces an additional term

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})\, \delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N} + \mathbf{k})$$
$$\sum_{i=1}^{N} \Bigg[-\delta_{\beta_i}^{\ \alpha_i} \varepsilon_\mu^{\ \nu} \mathbf{p}_{i\nu} \frac{\partial}{\partial \mathbf{p}_{i\mu}} - \frac{i}{2} (\mathbf{k^b} \varepsilon_\mu^{\ a} - \mathbf{k^a} \varepsilon_\mu^{\ b}) (\Sigma_{ab})_{\beta_i}^{\ \alpha_i} \frac{\partial}{\partial \mathbf{p}_{i\mu}}$$
$$- \frac{1}{2} \delta_{\beta_i}^{\ \alpha_i} \left\{ \mathbf{k}_\mu \varepsilon_\nu^{\ \rho} + \mathbf{k}_\nu \varepsilon_\mu^{\ \rho} - \mathbf{k}^\rho \varepsilon_{\mu\nu} \right\} \mathbf{p}_{i\rho} \frac{\partial^2}{\partial \mathbf{p}_{i\mu} \partial \mathbf{p}_{i\nu}} \Bigg]$$
$$\mathbf{F}^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N}(\mathbf{p_1}, \ldots, \mathbf{p_N}) + \mathcal{O}(\mathbf{k}^\mu \mathbf{k}^\nu)\,.$$

After some algebra, this can be rewritten as

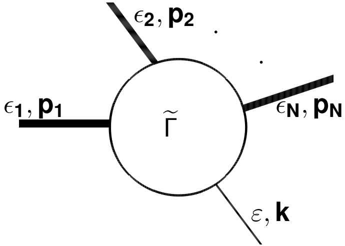
$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})$$

$$\sum_{\mathbf{i=1}}^{\mathbf{N}} \Bigg[-\delta_{\beta_i}^{\ \alpha_i} \varepsilon_\mu^{\ \nu} \mathbf{p_{i\nu}} \frac{\partial}{\partial \mathbf{p_{i\mu}}} - \frac{\mathbf{i}}{\mathbf{2}} (\mathbf{k^b} \varepsilon_\mu^{\ \mathbf{a}} - \mathbf{k^a} \varepsilon_\mu^{\ \mathbf{b}}) (\Sigma_{\mathbf{ab}})_{\beta_i}^{\ \alpha_i} \frac{\partial}{\partial \mathbf{p_{i\mu}}}$$

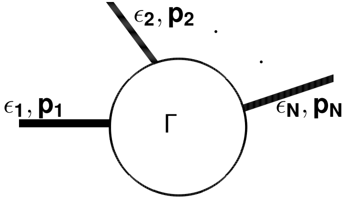
$$-\frac{\mathbf{1}}{\mathbf{2}} \delta_{\beta_i}^{\ \alpha_i} \left\{ \mathbf{k}_\mu \varepsilon_\nu^{\ \rho} + \mathbf{k}_\nu \varepsilon_\mu^{\ \rho} - \mathbf{k}^\rho \varepsilon_{\mu\nu} \right\} \mathbf{p_{i\rho}} \frac{\partial^2}{\partial \mathbf{p_{i\mu}} \partial \mathbf{p_{i\nu}}} \Bigg]$$

$$\left\{ \mathbf{F}^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N} (\mathbf{p_1}, \ldots, \mathbf{p_N})\, \delta^{(\mathbf{D})}(\mathbf{p_1} + \cdots \mathbf{p_N}) \right\} + \mathcal{O}(\mathbf{k}^\mu \mathbf{k}^\nu)\,.$$

Now consider



1. Take the amplitude without soft graviton.



2. Covariantize it to order $\mathbf{k}^\mu$

Denote the amplitude without the soft graviton by

$$\epsilon_{1,\alpha_1}(p_1)\ldots\epsilon_{N,\alpha_N}(p_N)\,\Gamma^{\alpha_1\ldots\alpha_N}(p_1,\ldots,p_N)$$

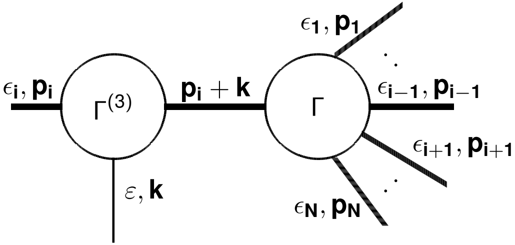
α_i: tangent space tensor / spinor indices labelling all the fields of the theory

$\Gamma^{\alpha_1\cdots\alpha_N}$ includes the $\delta^{(D)}(p_1+\cdots p_N)$ factor.

Then the result for $\widetilde{\Gamma}$ is

$$\sum_{i=1}^{N}\Bigg[-\delta_{\beta_i}^{\alpha_i}\varepsilon_\mu^{\ \nu}p_{i\nu}\frac{\partial}{\partial p_{i\mu}}-i\,k^b\varepsilon_\mu^{\ a}(\Sigma_{ab})_{\beta_i}^{\ \alpha_i}\frac{\partial}{\partial p_{i\mu}}$$
$$-\frac{1}{2}\delta_{\beta_i}^{\alpha_i}\left\{k_\mu\varepsilon_\nu^{\ \rho}+k_\nu\varepsilon_\mu^{\ \rho}-k^\rho\varepsilon_{\mu\nu}\right\}\,p_{i\rho}\,\frac{\partial^2}{\partial p_{i\mu}\partial p_{i\nu}}\Bigg]$$
$$\Gamma^{\alpha_1\cdots\alpha_{i-1}\beta_i\alpha_{i+1}\cdots\alpha_N}(p_1,\ldots,p_N)+\mathcal{O}(k^\mu k^\nu)\,.$$

Next consider



Need to focus on the three point coupling computed from the 1PI action.

Begin with two point function without the soft graviton and covariantize it to order $k^\mu k^\nu$.

$$\mathbf{S^{(2)}} = \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\frac{d^D q_2}{(2\pi)^D}\,\phi_\alpha(\mathbf{q_1})\mathcal{K}^{\alpha\beta}(\mathbf{q_2})\phi_\beta(\mathbf{q_2})\,(2\pi)^D\delta^{(D)}(\mathbf{q_1}+\mathbf{q_2})$$

$\{\phi_\alpha\}$: **set of all the fields**

$\mathcal{K}^{\alpha\beta}(\mathbf{q})$: **Kinetic operator, chosen to satisfy**

$$\mathcal{K}^{\alpha\beta}(\mathbf{q}) = \mathcal{K}^{\beta\alpha}(-\mathbf{q})$$

For grassmann odd fields the sign is opposite but the final result is not affected.

Covariantization $\Rightarrow$ coupling of ϕ_α to soft graviton

$$\begin{aligned}\mathbf{S^{(3)}} \;&=\; \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(\mathbf{q_1}+\mathbf{q_2}+\mathbf{k})\\ &\times\phi_\alpha(\mathbf{q_1})\left[-\varepsilon_{\mu\nu}\mathbf{q}_2^\nu\frac{\partial}{\partial \mathbf{q}_{2\mu}}\mathcal{K}^{\alpha\beta}(\mathbf{q_2}) + i\,\mathbf{k_a}\,\varepsilon_{\mathbf{b}\mu}\frac{\partial}{\partial \mathbf{q}_{2\mu}}\mathcal{K}^{\alpha\gamma}(\mathbf{q_2})\left(\Sigma^{ab}\right)_\gamma^{\;\beta}\right.\\ &\left.-\frac{1}{2}\left\{\mathbf{k}_\mu\varepsilon_\nu^{\;\rho}+\mathbf{k}_\nu\varepsilon_\mu^{\;\rho}-\mathbf{k}^\rho\varepsilon_{\mu\nu}\right\}\,\mathbf{q}_{2\rho}\,\frac{\partial^2\mathcal{K}^{\alpha\beta}(\mathbf{q_2})}{\partial \mathbf{q}_{2\mu}\partial \mathbf{q}_{2\nu}}\right]\phi_\beta(\mathbf{q_2}) + \mathcal{O}(\mathbf{k}^\mu\mathbf{k}^\nu)\end{aligned}$$

– determines the coupling of the soft graviton to the finite energy particles

$$
\begin{aligned}
&\Gamma^{(3)\alpha\beta}(\varepsilon, \mathbf{k}; \mathbf{p}, -\mathbf{p}-\mathbf{k}) \\
= \quad & \frac{\mathbf{i}}{2}\Bigg[-\varepsilon_{\mu\nu}(\mathbf{p}+\mathbf{k})^{\nu}\frac{\partial}{\partial \mathbf{p}_{\mu}}\mathcal{K}^{\alpha\beta}(-\mathbf{p}-\mathbf{k}) - \varepsilon_{\mu\nu}\mathbf{p}^{\nu}\frac{\partial}{\partial \mathbf{p}_{\mu}}\mathcal{K}^{\beta\alpha}(\mathbf{p}) \\
& -\frac{\mathbf{i}}{2}(\mathbf{k_a}\,\varepsilon_{\mathbf{b}\mu} - \mathbf{k_b}\,\varepsilon_{\mathbf{a}\mu})\frac{\partial}{\partial \mathbf{p}_{\mu}}\mathcal{K}^{\alpha\gamma}(-\mathbf{p}-\mathbf{k})\left(\Sigma^{\mathbf{ab}}\right)_{\gamma}^{\;\;\beta} \\
& +\frac{\mathbf{i}}{2}(\mathbf{k_a}\,\varepsilon_{\mathbf{b}\mu} - \mathbf{k_b}\,\varepsilon_{\mathbf{a}\mu})\frac{\partial}{\partial \mathbf{p}_{\mu}}\mathcal{K}^{\beta\gamma}(\mathbf{p})\left(\Sigma^{\mathbf{ab}}\right)_{\gamma}^{\;\;\alpha} \\
& -\frac{1}{2}\frac{\partial^{2}\mathcal{K}^{\alpha\beta}(-\mathbf{p}-\mathbf{k})}{\partial \mathbf{p}_{\mu}\partial \mathbf{p}_{\nu}}(-\mathbf{p}_{\rho}-\mathbf{k}_{\rho})\left(\mathbf{k}_{\mu}\varepsilon_{\nu}^{\;\;\rho} + \mathbf{k}_{\nu}\varepsilon_{\mu}^{\;\;\rho} - \mathbf{k}^{\rho}\varepsilon_{\mu\nu}\right) \\
& -\frac{1}{2}\frac{\partial^{2}\mathcal{K}^{\beta\alpha}(\mathbf{p})}{\partial \mathbf{p}_{\mu}\partial \mathbf{p}_{\nu}}\mathbf{p}_{\rho}\left(\mathbf{k}_{\mu}\varepsilon_{\nu}^{\;\;\rho} + \mathbf{k}_{\nu}\varepsilon_{\mu}^{\;\;\rho} - \mathbf{k}^{\rho}\varepsilon_{\mu\nu}\right) + \mathbf{O}(\mathbf{k}_{\mu}\mathbf{k}_{\nu})\Bigg]
\end{aligned}
$$

Note: This is determined in terms of $\mathcal{K}^{\alpha\beta}$.

We also need the propagator of ϕ_α carrying momentum q

Define $\quad \mathcal{N}^{j}_{\alpha\beta}(q) \equiv i\,(q^2 + m_j^2)\,\mathcal{K}^{-1}_{\alpha\beta}(q) \quad$ (fixed j)

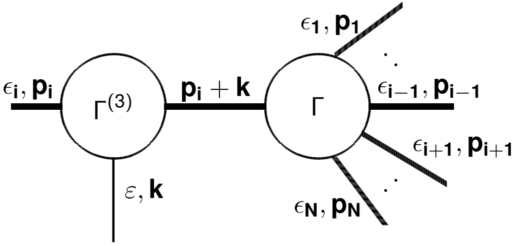
m_j: mass of the j-th external particle.

$N^j(q)$ has no pole at $q^2 + m_j^2 = 0$.

The propagator of the particle carrying momentum $p_j + k$

$$i\,\mathcal{K}^{-1}_{\alpha\beta}(p_j + k) \equiv (2p_j \cdot k)^{-1}\mathcal{N}^{j}_{\alpha\beta}(p_j + k)$$

Note: The same propagator is expressed differently for different external states to make the pole structure manifest.



Once we compute the Feynman diagram with these Feynman rules, we find that all dependence on $\mathcal{K}^{\alpha\beta}$ cancels, and we get a simple expression to subleading order in expansion in the soft momentum.

Need to use

– on-shell condition: $\epsilon_{\mathbf{i}\alpha}\mathcal{K}^{\alpha\beta} = 0$

– Lorentz invariance of $\mathcal{K}^{\alpha\beta}$ under simultaneous Lorentz transformation of α, β and momenta.

Final result for the soft graviton amplitude to subleading order:

$$\prod_{j=1}^{N} \epsilon_{j,\alpha_j}(p_j) \left[S^{(0)} \Gamma^{\alpha_1 \ldots \alpha_N} + \left\{ S^{(1)} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} \right]$$

$$S^{(0)} \equiv \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{\mu\nu} \, p_i^{\mu} \, p_i^{\nu}$$

$$\begin{aligned} \left\{ S^{(1)} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} &= \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{\mu\nu} \, p_i^{\mu} \, k_{\rho} \left(p_i^{\nu} \frac{\partial}{\partial p_{i\rho}} - p_i^{\rho} \frac{\partial}{\partial p_{i\nu}} \right) \Gamma^{\alpha_1 \cdots \alpha_N} \\ &- i \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{\mu b} \, p_i^{\mu} \, k_a \, (\Sigma^{ab})_{\gamma}^{\ \alpha_i} \, \Gamma^{\alpha_1 \cdots \alpha_{i-1} \gamma \alpha_{i+1} \cdots \alpha_N} \end{aligned}$$

This is the subleading soft graviton theorem

– agrees with all known results in field theory / string theory

This analysis can be extended to subsubleading order but we shall not discuss this

– has non-universal terms due to possible coupling of the soft graviton via Riemann tensor.

Multiple soft gravitons

Naive guess: To subleading order,

$$\prod_{j=1}^{N} \epsilon_{j,\alpha_j}(p_j) \left[\left\{ S^{(0)}(\varepsilon_1, k_1) + S^{(1)}(\varepsilon_1, k_1) \right\} \cdots \right.$$
$$\left. \cdots \left\{ S^{(0)}(\varepsilon_M, k_M) + S^{(1)}(\varepsilon_M, k_M) \right\} \Gamma \right]^{\alpha_1 \cdots \alpha_N}$$

Problem: $S^{(0)}$ and $S^{(1)}$ do not commute.

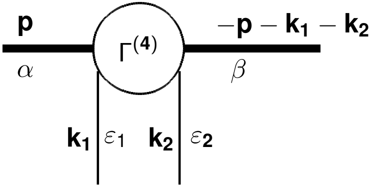
This expression is not symmetric under the exchange of the soft gravitons.

We have to do the analysis afresh.

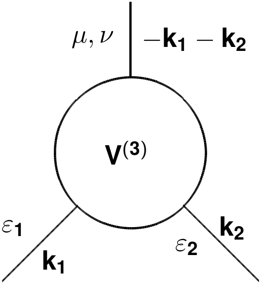
The procedure is identical, except that while constructing the soft graviton coupling via covariantization we need to include higher powers of the soft graviton field $S_{\mu\nu}$.

For computations up to subleading order, we need to keep terms up to two powers of $S_{\mu\nu}$ in the covariantization of the kinetic operator.

– leads to a four point interaction vertex



The other new vertex is three point coupling of soft gravitons.



We need this to leading order in soft momenta

– can be obtained by expanding the Einstein-Hilbert action

Result for M soft gravitons and N finite energy particles

Chakrabarti, Kashyap, Sahoo, A.S, Verma

$$\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}(p_i)\right\}\left[\left\{\prod_{r=1}^{M}S_r^{(0)}\right\}\Gamma^{\alpha_1\cdots\alpha_N}+\sum_{s=1}^{M}\left\{\prod_{\substack{r=1\\ r\neq s}}^{M}S_r^{(0)}\right\}\left[S_s^{(1)}\Gamma\right]^{\alpha_1\cdots\alpha_N}\right.$$

$$\left.+\sum_{\substack{r,u=1\\ r<u}}^{M}\left\{\prod_{\substack{s=1\\ s\neq r,u}}^{M}S_s^{(0)}\right\}\left\{\sum_{j=1}^{N}\{p_j\cdot(k_r+k_u)\}^{-1}\ \mathcal{M}(p_j;\varepsilon_r,k_r,\varepsilon_u,k_u)\right\}\Gamma^{\alpha_1\cdots\alpha_N}\right]$$

$$S^{(0)}\equiv\sum_{i=1}^{N}(p_i\cdot k)^{-1}\,\varepsilon_{\mu\nu}\,p_i^{\mu}\,p_i^{\nu}$$

$$\begin{aligned}\left\{S^{(1)}\Gamma\right\}^{\alpha_1\cdots\alpha_N} &= \sum_{i=1}^{N}(p_i\cdot k)^{-1}\,\varepsilon_{\mu\nu}\,p_i^{\mu}\,k_\rho\left(p_i^{\nu}\frac{\partial}{\partial p_{i\rho}}-p_i^{\rho}\frac{\partial}{\partial p_{i\nu}}\right)\Gamma^{\alpha_1\cdots\alpha_N}\\ &- i\sum_{i=1}^{N}(p_i\cdot k)^{-1}\,\varepsilon_{\mu b}\,p_i^{\mu}\,k_a\,(\Sigma^{ab})_{\gamma}^{\ \alpha_i}\,\Gamma^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_N}\end{aligned}$$

$\mathcal{M}$: ‘contact term’

$$
\begin{aligned}
&\mathcal{M}(\mathbf{p_i};\varepsilon_1,\mathbf{k_1},\varepsilon_2,\mathbf{k_2}) \\
= \;& (\mathbf{p_i}\cdot\mathbf{k_1})^{-1}(\mathbf{p_i}\cdot\mathbf{k_2})^{-1}\Big\{-\mathbf{k_1}\cdot\mathbf{k_2}\;\mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{p_i}\;\mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{p_i} \\
&+2\,\mathbf{p_i}\cdot\mathbf{k_2}\;\mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{p_i}\;\mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{k_1}+2\,\mathbf{p_i}\cdot\mathbf{k_1}\;\mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{p_i}\;\mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{k_2} \\
&-2\,\mathbf{p_i}\cdot\mathbf{k_1}\;\mathbf{p_i}\cdot\mathbf{k_2}\;\mathbf{p_i}\cdot\varepsilon_1\cdot\varepsilon_2\cdot\mathbf{p_i}\Big\} \\
&+(\mathbf{k_1}\cdot\mathbf{k_2})^{-1}\Big\{-(\mathbf{k_2}\cdot\varepsilon_1\cdot\varepsilon_2\cdot\mathbf{p_i})(\mathbf{k_2}\cdot\mathbf{p_i})-(\mathbf{k_1}\cdot\varepsilon_2\cdot\varepsilon_1\cdot\mathbf{p_i})(\mathbf{k_1}\cdot\mathbf{p_i}) \\
&+(\mathbf{k_2}\cdot\varepsilon_1\cdot\varepsilon_2\cdot\mathbf{p_i})(\mathbf{k_1}\cdot\mathbf{p_i})+(\mathbf{k_1}\cdot\varepsilon_2\cdot\varepsilon_1\cdot\mathbf{p_i})(\mathbf{k_2}\cdot\mathbf{p_i}) \\
&-\varepsilon_1^{\gamma\delta}\varepsilon_{2\gamma\delta}(\mathbf{k_1}\cdot\mathbf{p_i})(\mathbf{k_2}\cdot\mathbf{p_i})-2(\mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{k_2})(\mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{k_1}) \\
&+(\mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{p_i})(\mathbf{k_2}\cdot\varepsilon_1\cdot\mathbf{k_2})+(\mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{p_i})(\mathbf{k_1}\cdot\varepsilon_2\cdot\mathbf{k_1})\Big\},
\end{aligned}
$$

– agrees with results for two soft gravitons in specific theories

Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha

Classical limit

We take the external momenta $\{p_i\}$ to be large compared to the Planck mass.

$$S^{(0)} \equiv \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{\mu\nu} \, p_i^{\mu} \, p_i^{\nu}$$

is a large number.

Consider the leading amplitude for M soft gravitons

Amplitude: $\left\{ \prod_{r=1}^{M} S^{(0)}(\epsilon_r, k_r) \right\} \Gamma$

Γ: amplitude without soft graviton

Laddha, A.S., arXiv:1801.07719

Probability of producing M soft gravitons of

- **polarization ϵ,**

- **energy between ω and $\omega(1+\delta)$**

- **within a solid angle Ω around a unit vector $\hat{\mathbf{n}}$**

$$|\Gamma|^2\, \mathbf{A}^{\mathbf{M}}/\mathbf{M}!\,,$$

$$\begin{aligned} \mathbf{A} &\equiv |\mathbf{S}^{(0)}(\varepsilon,\mathbf{k})|^2 \frac{1}{(2\pi)^{D-1}}\frac{1}{2\omega}\,\omega^{D-2}\,\omega\,\delta\,\Omega \\ &\equiv \frac{1}{2^D\pi^{D-1}}|\mathbf{S}^{(0)}(\varepsilon,\mathbf{k})|^2\omega^{D-2}\,\Omega\,\delta\,. \end{aligned}$$

$$\mathbf{k}=\omega(1,\hat{\mathbf{n}})$$

Note: In the classical limit, A is large.

$$|\Gamma|^2\, A^M / M!\,,$$

is maximized at

$$\frac{\partial}{\partial M} \ln \left\{ |\Gamma|^2\, A^M / M! \right\} = 0$$

Assuming that M is large,

$$\Rightarrow \frac{\partial}{\partial M}(M \ln A - M \ln M + M) = 0$$

$$\Rightarrow M = A$$

Classical limit requires M to be large

$\Rightarrow$ A must be large (true since $S^{(0)}$ is large)

Probability distribution of M is sharply peaked

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{2^D}\pi^{\mathbf{D-1}}} |\mathbf{S^{(0)}}(\varepsilon, \mathbf{k})|^{\mathbf{2}} \omega^{\mathbf{D-2}}\, \Omega\, \delta$$

$$\mathbf{S^{(0)}}(\varepsilon, \mathbf{k}) \equiv \sum_{\mathbf{i=1}}^{\mathbf{N}} (\mathbf{p_i} \cdot \mathbf{k})^{-\mathbf{1}}\, \varepsilon_{\mu\nu}\, \mathbf{p_i^\mu}\, \mathbf{p_i^\nu}$$

The sum over i runs over the macroscopic objects taking part in the scattering.

Total energy radiated in this bin

$$\mathbf{A}\,\omega = \frac{\mathbf{1}}{\mathbf{2^D}\pi^{\mathbf{D-1}}} |\mathbf{S^{(0)}}(\varepsilon, \mathbf{k})|^{\mathbf{2}} \omega^{\mathbf{D-1}}\, \Omega\, \delta$$

This result can be related to the radiative part of the gravitational field $\mathbf{h}_{\mu\nu}$.

Define $\mathbf{e}_{\mu\nu}$ via

$$\mathbf{h}_{\mu\nu} = \mathbf{e}_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}\,\mathbf{e}_{\rho}^{\ \rho} \quad \Leftrightarrow \quad \mathbf{e}_{\mu\nu} = \mathbf{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathbf{h}_{\rho}^{\ \rho}$$

$\tilde{\mathbf{e}}_{\mu\nu}(\omega, \vec{\mathbf{x}})$: time Fourier transform of $\mathbf{e}_{\mu\nu}(\mathbf{t}, \vec{\mathbf{x}})$

Then the result for energy flux can be translated to the following result for the wave-form:

$$\varepsilon^{\mu\nu}\tilde{\mathbf{e}}_{\mu\nu}(\omega, \vec{\mathbf{x}}) = \left(\frac{\omega}{2\pi i R}\right)^{(D-2)/2} \frac{1}{2\omega}\,\mathbf{S}^{(0)}(\varepsilon, \mathbf{k}), \qquad \mathbf{R} = |\vec{\mathbf{x}}|, \quad \mathbf{k} = \omega(1, \vec{\mathbf{x}}/|\vec{\mathbf{x}}|)$$

up to an overall phase that does not affect the energy flux.

Subleading order:

One can check that when we take the classical limit by taking the energies to be large compared to the Planck mass, the effect of the contact term $\mathcal{M}$ gets suppressed.

Therefore the amplitude for M soft gravitons to subleading order is

$$\left[\prod_{r=1}^{M}\left\{S^{(0)}(\epsilon_r,k_r)+S^{(1)}(\epsilon_r,k_r)\right\}\right]\,\Gamma$$

Furthermore, $S^{(1)}$ becomes a multiplicative function:

$$S^{(1)}(\varepsilon,k)=-i\sum_{i=1}^{N}(p_i\cdot k)^{-1}\,\varepsilon_{\mu b}\,p_i^{\mu}\,k_a\,(J_i^{ab})$$

J_i^{ab} total angular momentum (orbital + spin) of the i-th particle

If the asymptotic trajectory of the i-th particle is $X_i^{\mu}(\tau)$: then

$$J_i^{\mu\nu}=X_i^{\mu}p_i^{\nu}-X_i^{\nu}p_i^{\mu}+\text{spin}$$

$$\left[\prod_{r=1}^{M}\left\{S^{(0)}(\epsilon_r, k_r) + S^{(1)}(\epsilon_r, k_r)\right\}\right]\Gamma$$

Since in the classical limit $S^{(0)}$ and $S^{(1)}$ commute, this is symmetric under the exchange of the soft gravitons.

We can now repeat previous analysis for computing the angular power spectrum in classical radiation to subleading order.

– replace $S^{(0)}$ by $S^{(0)} + S^{(1)}$.

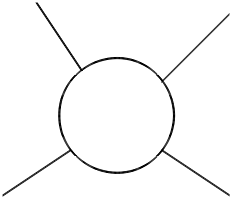
$$\varepsilon^{\mu\nu}\tilde{e}_{\mu\nu}(\omega, \vec{x}) = \left(\frac{\omega}{2\pi i R}\right)^{(D-2)/2}\frac{1}{2\omega}\left\{S^{(0)}(\varepsilon, k) + S^{(1)}(\varepsilon, k)\right\}$$

up to an overall phase that does not affect the energy flux.

If a significant amount of energy is carried away by finite wave-length radiation, its contribution must be included in the sum over i in $S^{(0)}$ and $S^{(1)}$.

A classical proof of the classical soft graviton theorem in D>4

We shall now describe a direct proof of the classical soft graviton theorem in D>5.



Strategy:

1. Consider a general classical scattering process in which we have a set of incoming particles and a set of outgoing particles.

2. Calculate the low frequency gravitational radiation during this process by solving Einstein's equation.

Recall:

$$\mathbf{e}_{\mu\nu} = \mathbf{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathbf{h}_\rho^{\ \rho}$$

In the de Donder gauge, the Einstein's equation takes the form:

$$\partial^\rho\partial_\rho\mathbf{e}_{\mu\nu} = -\mathbf{T}_{\mu\nu}$$

$\mathbf{T}_{\mu\nu}$ denotes the energy-momentum tensor, including that of the gravitational field obtained by taking the non-linear terms in the Einstein's equation to the right hand side.

All indices are raised and lowered by the flat metric $\eta_{\mu\nu}$.

Conservation law:

$$\partial^\mu\mathbf{T}_{\mu\nu} = 0$$

Formal 'solution':

$$\mathbf{e}_{\mu\nu}(\mathbf{x}) = -\int \mathbf{d}^\mathbf{D}\mathbf{x}'\,\mathbf{G}_\mathbf{r}(\mathbf{x},\mathbf{x}')\,\mathbf{T}_{\mu\nu}(\mathbf{x}')$$

$\mathbf{G}_\mathbf{r}$: retarded Green's function in flat space-time.

Note: $\mathbf{T}_{\mu\nu}$ depends on the metric and hence on $\mathbf{e}_{\mu\nu}$

Define:

$$\tilde{e}_{\mu\nu}(\omega,\vec{x}) \equiv \int dx^0\, e^{i\omega x^0}\, e_{\mu\nu}(x^0,\vec{x})$$

Then

$$\tilde{e}_{\mu\nu}(\omega,\vec{x}) = \int d^D x' \int \frac{d^{D-1}\ell}{(2\pi)^{D-1}}\, e^{i\omega x'^0 + i\vec{\ell}.(\vec{x}-\vec{x}')}\, \frac{1}{(\omega+i\epsilon)^2 - \vec{\ell}^{\,2}}\, T_{\mu\nu}(x')$$

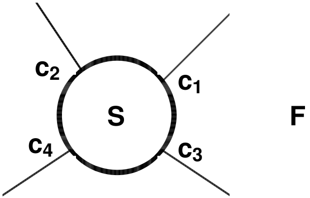
For large R=$|\vec{x}|$, one can do the integration over the longitudinal component of $\vec{\ell}$ by residue theorem and the transverse components by saddle point method:

$$\tilde{e}_{\mu\nu}(\omega,\vec{x}) \simeq i\,\mathcal{N}\, e^{i\,\omega\,|\vec{x}|} \int d^D x'\, e^{-ik\cdot x'}\, T_{\mu\nu}(x')$$

$$\mathcal{N} = \left(\frac{\omega}{2\pi i|\vec{x}|}\right)^{(D-2)/2} \frac{1}{2\omega}, \qquad k = \omega\left(1, \frac{\vec{x}}{|\vec{x}|}\right)$$

$$\tilde{\mathbf{e}}_{\mu\nu}(\omega, \vec{\mathbf{x}}) \simeq \mathbf{i}\,\mathcal{N}\,\mathbf{e}^{\mathbf{i}\,\omega\,|\vec{\mathbf{x}}|} \int \mathbf{d}^{\mathbf{D}}\mathbf{x}'\,\mathbf{e}^{-\mathbf{i}\mathbf{k}\cdot\mathbf{x}'}\,\mathbf{T}_{\mu\nu}(\mathbf{x}')$$

We shall evaluate the integral over $\mathbf{x}'$ by dividing the integration region into two parts.



1. The scattering region S has finite volume and therefore integration over this region will produce an analytic function of k.

2. Free region F, where we approximate $\mathbf{T}_{\mu\nu}$ by the energy momentum tensor of free incoming and outgoing particles.

Region F:

$$\mathbf{T}_{\mu\nu}(\mathbf{x}') = \sum_{\mathbf{i}} \int \mathbf{d}\tau \left[\mathbf{V}_{\mathbf{i}\mu} \mathbf{p}_{\mathbf{i}\nu} \delta^{(\mathbf{D})}(\mathbf{x}' - \mathbf{X}_{\mathbf{i}}(\tau)) \right.$$
$$\left. + \mathbf{V}_{\mathbf{i}(\mu} \Sigma_{\mathbf{i}\nu)\rho} \partial'^{\rho} \delta^{(\mathbf{D})}(\mathbf{x}' - \mathbf{X}_{\mathbf{i}}(\tau)) + \cdots \right]$$

$\mathbf{p}_\mathbf{i}$: **momentum of the i-th particle**

$\mathbf{V}_\mathbf{i} = \mathbf{p}_\mathbf{i}/\mathbf{m}_\mathbf{i}$: **velocity of the i-th particle**

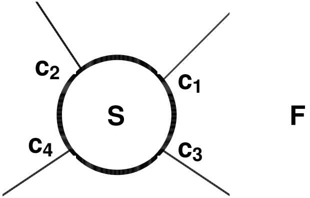
$\mathbf{X}_\mathbf{i}(\tau) = \mathbf{c}_\mathbf{i} + \mathbf{V}_\mathbf{i}\tau$: **trajectory of the i-th particle parametrized by proper time** τ

$\Sigma_\mathbf{i}$: **Spin angular momentum of the i-th particle**

This contribution depends only on $\mathbf{p}_\mathbf{i}^\mu$, $\Sigma_\mathbf{i}^{\mu\nu}$ **and** $\mathbf{c}_\mathbf{i}^\mu$.

The higher order terms encoded in $\cdots$ **are non-universal.**

Region S



Need to compute:

$$\mathbf{M}_{\mu\nu} \equiv \int_{\mathbf{S}} \mathbf{d^D x' e^{-ik.x'} T}_{\mu\nu}(\mathbf{x'})$$

– an analytic function of k since the integral is over a finite region

$$\mathbf{ik}^{\mu}\mathbf{M}_{\mu\nu} = -\int_{\mathbf{S}} \mathbf{d^D x'} \partial'^{\mu}(\mathbf{e^{-ik.x'}})\mathbf{T}_{\mu\nu}(\mathbf{x'}) = \text{boundary terms}$$

using conservation law $\partial'^{\mu}\mathbf{T}_{\mu\nu}(\mathbf{x'}) = 0$.

The boundary terms are computed from the trajectories in region F and depend on p_i^{μ}, $\Sigma_i^{\mu\nu}$ and c_i^{μ}.

$\mathbf{ik}^{\mu}\, \mathbf{M}_{\mu\nu}$ **is known**

Using analyticity of $M_{\mu\nu}$ this leads to a unique solution for $M_{\mu\nu}$ up to additive terms that are quadratic in k, e.g.

$$\mathbf{k}^{2}\eta_{\mu\nu} - \mathbf{k}_{\mu}\mathbf{k}_{\nu}$$

This gives the contribution to $e_{\mu\nu}$ from region S:

$$\mathbf{i}\,\mathcal{N}\,\mathbf{e}^{\mathbf{i}\,\omega\,|\vec{\mathbf{x}}|}\,\mathbf{M}_{\mu\nu}$$

The additive ambiguous terms do not contribute at the subleading order and we get a result that depends only on p_i^{μ}, $\Sigma_i^{\mu\nu}$ and c_i^{μ}.

Final result:

$$\tilde{e}_{\mu\nu} = \mathcal{N}\, e^{i\,\omega\,|\vec{x}|}\,\left[\sum_i \frac{p_{i\mu} p_{i\nu}}{k\cdot p_i} + i\sum_i \frac{1}{p_i\cdot k} p_{i(\mu} k^\rho J_{i\nu)\rho} + \mathcal{O}(k)\right],$$

where

$$J_{i\rho\mu} = \{c_{i\rho} p_{i\mu} - c_{i\mu} p_{i\rho} + \Sigma_{i\rho\mu}\}\ .$$

This is exactly the prediction of the classical limit of the subleading soft graviton theorem.

Some additional subtleties

1. We need to carefully check that in the region F the long range gravitational force between the particles can be neglected.

2. If the final state contains massless radiation, we have to analyze their contribution separately and show that they take the same form as above.

D=4

Quantum soft graviton theorem is a relation between S-matrices with and without external soft gravitons.

However the S-matrices themselves suffer from infrared (IR) divergences in D=4.

Nevertheless one could hope that with suitable regulator the IR divergences factor out from both sides leaving the soft theorem unaffected.

Unfortunately this is not true. Bern, Davies, Nohle

The model independent analysis described earlier breaks down due to the fact that the interaction terms of the 1PI effective action itself become singular in the soft limit.

Results of explicit one loop computation:

1.The leading soft graviton theorem remains unchanged.

2. The subleading soft graviton theorem gets modified.

Usual soft expansion is a power series expansion in powers of soft graviton energy ω.

Leading tern $\sim \omega^{-1}$, subleading term ω^0 etc.

Instead, in D=4 we find subleading terms $\propto \ln\omega$

Coefficients are free from IR divergences and depend on the momenta of the finite energy external particles.

Classical limit:

Unfortunately, the classical limit of this formula has not been fully understood.

For this reason there is no fully systematic procedure for deriving a classical soft theorem as limit of a quantum soft theorem in D=4.

Alternative approach:

1. Begin with the classical limit of the quantum soft theorem in D>4.

2. Set D=4 in this formula.

However this also runs into some hurdle.

Laddha, A.S., 1804.09193
Sahoo, A.S., arXiv:1808.03288

Problem: Due to long range force on the initial / final trajectories due to other particles, the trajectory of the i-th particle in D=4 takes the form:

$$X_i^\mu = c_i^\mu + m_i^{-1}\, p_i^\mu\, \tau + b_i^\mu \ln|\tau|$$

for some computable constants b_i^μ.

$$J_i^{\mu\nu} = (X_i^\mu p_i^\nu - X_i^\nu p_i^\mu) = (c_i^\mu p_i^\nu - c_i^\nu p_i^\mu) + (b_i^\mu p_i^\nu - b_i^\nu p_i^\mu) \ln|\tau|$$

Due to the $\ln|\tau|$ term, the soft factors do not have well defined $|\tau| \to \infty$ limit.

e.g. consider a non-relativistic particle in radially outward trajectory in an inverse square force:

$$m\, \frac{d^2 r}{dt^2} = -\frac{K}{r^2}$$

Solution

$$r = v\, t + c + b \ln t + \cdots, \qquad b = \frac{K}{m\, v^2}$$

Relativistic generalization is straightforward.

Guess: Soft radiation with frequency ω should not be sensitive to scales much larger than ω^{-1}.

In that case divergences in ln τ in the large τ limit is cut off at $\tau \sim \omega^{-1}$.

$$\begin{aligned} \mathbf{J}_i^{\mu\nu} &= (\mathbf{c}_i^\mu \mathbf{p}_i^\nu - \mathbf{c}_i^\nu \mathbf{p}_i^\mu) + (\mathbf{b}_i^\mu \mathbf{p}_i^\nu - \mathbf{b}_i^\nu \mathbf{p}_i^\mu) \ln|\tau| \\ &\Rightarrow (\mathbf{c}_i^\mu \mathbf{p}_i^\nu - \mathbf{c}_i^\nu \mathbf{p}_i^\mu) + (\mathbf{b}_i^\mu \mathbf{p}_i^\nu - \mathbf{b}_i^\nu \mathbf{p}_i^\mu) \ln \omega^{-1} \end{aligned}$$

This gives

$$\begin{aligned} \mathbf{S}^{(1)} &= i \sum_{i=1}^{N} (\mathbf{p}_i \cdot \mathbf{k})^{-1} \, \varepsilon_{\mu\nu} \, \mathbf{p}_i^\mu \, \mathbf{k}_\rho \, \mathbf{J}_i^{\rho\nu} \\ &= i \sum_{i=1}^{N} (\mathbf{p}_i \cdot \mathbf{k})^{-1} \, \varepsilon_{\mu\nu} \, \mathbf{p}_i^\mu \, \mathbf{k}_\rho \, (\mathbf{b}_i^\rho \mathbf{p}_i^\nu - \mathbf{b}_i^\nu \mathbf{p}_i^\rho) \ln \omega^{-1} \\ &\quad + \text{finite} \end{aligned}$$

We can now compute the $\mathbf{b}_i^\mu$'s and substitute.

$\ln|\tau| \Rightarrow \ln\omega^{-1}$ **rule gives:**

$$\tilde{e}^{\mu\nu} = \frac{2G}{iR}\sum_i \frac{p_i^\mu p_i^\nu}{p_i.n}\left\{\frac{1}{\omega} - 2iG\ln(\omega^{-1}R^{-1})\sum_{j,\eta_j=1} n.p_j\right\}$$

$$+2\frac{G^2}{R}\ln\omega^{-1}\sum_i \sum_{\substack{j\neq i\\ \eta_i\eta_j=1}} \frac{n_\rho p_i^{(\nu}}{p_i.n}\left(p_i^{\mu)}p_j^\rho - p_j^{\mu)}p_i^\rho\right)$$

$$\times \frac{p_j.p_i}{\{(p_j.p_i)^2 - m_i^2m_j^2\}^{3/2}}\left\{2(p_j.p_i)^2 - 3m_i^2m_j^2\right\} + \text{finite}\,.$$

η_i: **-1 if i is incoming, 1 if i is outgoing.**

$\mathbf{n} = \left(1, \vec{x}/|\vec{x}|\right)$, $\quad \mathbf{k} = \omega\,\mathbf{n}$ **Sahoo, A.S.**

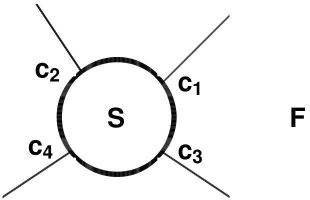
The second term in the first line is a pure phase and is not determined from soft theorem

– was originally determined by comparison with known results

Peters

Direct classical derivation

We proceed as in D>4 case, but there are two essential differences



1. We shall be looking for only the singular terms proportional to $1/\omega$ and $\ln\omega$

– allows us to ignore the contribution from integral over the finite region S which gives results that are analytic in ω

2. We can no longer treat the movement of the particles in region F as free since the effect of long range gravitational forces cannot be ignored.

Express Einstein's equation as

$$\sqrt{-\det \mathbf{g}}\left(\mathbf{R}^{\mu\nu} - \frac{1}{2}\mathbf{g}^{\rho\sigma}\mathbf{R}_{\rho\sigma}\,\mathbf{g}^{\mu\nu}\right) = 8\,\pi\,\mathbf{G}\,\mathbf{T}^{\mathbf{X}\mu\nu}$$

$\mathbf{T}^{\mathbf{X}\mu\nu}$: matter stress tensor

Note somewhat unusual definition of $\mathbf{T}^{\mathbf{X}\mu\nu}$ by including a factor of $\sqrt{-\det \mathbf{g}}$.

We shall write

$$\mathbf{g}_{\mu\nu} = \eta_{\mu\nu} + 2\,\mathbf{h}_{\mu\nu}, \quad \mathbf{e}_{\mu\nu} = \mathbf{h}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,\mathbf{h}_{\rho\sigma}, \quad \mathbf{e}^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}\mathbf{e}_{\alpha\beta}$$

and rewrite Einstein's equation in de Donder gauge $\partial^\mu \mathbf{e}_{\mu\nu} = 0$, as

$$\Box\,\mathbf{e}^{\mu\nu} = -8\,\pi\,\mathbf{G}\,\mathbf{T}^{\mu\nu}(\mathbf{x}), \qquad \Box \equiv \eta^{\rho\sigma}\partial_\rho\partial_\sigma \qquad \mathbf{T}^{\mu\nu} \equiv \mathbf{T}^{\mathbf{X}\mu\nu} + \mathbf{T}^{\mathbf{h}\mu\nu}$$

$\mathbf{T}^{\mathbf{h}\mu\nu}$ captures all terms quadratic and higher order in $\mathbf{h}_{\rho\sigma}$ on the left hand side of Einstein's equation.

From now on all indices will be raised and lowered by the flat metric η.

□ $\mathbf{e}^{\mu\nu} = -8\,\pi\,\mathbf{G}\,\mathbf{T}^{\mu\nu}$ **can be 'solved' as:**

$$\mathbf{e}^{\mu\nu}(\mathbf{x}) = -8\,\pi\,\mathbf{G}\int \mathbf{d}^4\mathbf{y}\,\mathbf{G_r}(\mathbf{x},\mathbf{y})\,\mathbf{T}^{\mu\nu}(\mathbf{y})$$

$\mathbf{G_r}(\mathbf{x},\mathbf{y})$**: retarded Green's function in flat space-time**

Using explicit form of $\mathbf{G_r}$ one finds that for large $\mathbf{R} \equiv |\vec{\mathbf{x}}|$,

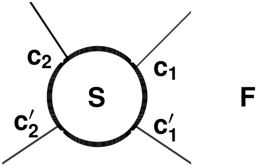
$$\tilde{\mathbf{e}}^{\mu\nu}(\omega,\vec{\mathbf{x}}) = \frac{2\,\mathbf{G}}{\mathbf{R}}\,\mathbf{e}^{\mathbf{i}\,\omega\,\mathbf{R}}\,\widehat{\mathbf{T}}^{\mu\nu}(\mathbf{k}), \quad \mathbf{k} = \omega(\mathbf{1},\hat{\mathbf{n}}), \quad \hat{\mathbf{n}} \equiv \vec{\mathbf{x}}/\mathbf{R}$$

$$\tilde{\mathbf{e}}^{\mu\nu}(\omega,\vec{\mathbf{x}}) = \int \mathbf{dt}\,\mathbf{e}^{\mathbf{i}\omega\mathbf{t}}\,\mathbf{e}^{\mu\nu}(\mathbf{t},\vec{\mathbf{x}}), \qquad \widehat{\mathbf{T}}^{\mu\nu}(\mathbf{k}) \equiv \int \mathbf{d}^4\mathbf{x}\,\mathbf{e}^{-\mathbf{ik.x}}\,\mathbf{T}^{\mu\nu}(\mathbf{x})$$

So far the analysis has been the same as in D>4 case.

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = \frac{2\,G}{R}\, e^{i\,\omega\,R}\, \widehat{T}^{\mu\nu}(k), \quad \widehat{T}^{\mu\nu}(k) \equiv \int d^4x\, e^{-ik.x}\, T^{\mu\nu}(x)$$

We shall divide the integration region over x into two parts:



1. Scattering region S: A region of large size L around x=0.

2. Asymptotic region F: Complement of S

Since our goal is to compute terms in $\widehat{T}_{\mu\nu}$ that are singular as $\omega \to 0$, we can ignore the contribution from the finite region S in $\int d^4x$.

In the asymptotic region, we can regard $T^{X\mu\nu}$ as due to the incoming and outgoing object trajectories, moving under each others' long range gravitational field.

$$\mathbf{T}^{\mathbf{X}\mu\nu}(\mathbf{x}) \equiv \sum_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{m_i} \int_0^\infty \mathbf{d}\tau \, \delta^{(4)}(\mathbf{x} - \mathbf{X_i}(\tau)) \frac{\mathbf{dX_i^\mu}}{\mathbf{d}\tau} \frac{\mathbf{dX_i^\nu}}{\mathbf{d}\tau}$$

$$+ \sum_{\mathbf{i}=1}^{\mathbf{m}} \mathbf{m'_i} \int_{-\infty}^0 \mathbf{d}\tau \, \delta^{(4)}(\mathbf{x} - \mathbf{X'_i}(\tau)) \frac{\mathbf{dX_i'^\mu}}{\mathbf{d}\tau} \frac{\mathbf{dX_i'^\nu}}{\mathbf{d}\tau} \, ,$$

$$\mathbf{T}^{\mu\nu}(\mathbf{x}) = \mathbf{T}^{\mathbf{X}\mu\nu}(\mathbf{x}) + \mathbf{T}^{\mathbf{h}\mu\nu}(\mathbf{x}),$$

$$\Box \, \mathbf{e}^{\mu\nu} = -8\,\pi\,\mathbf{G}\,\mathbf{T}^{\mu\nu} \, ,$$

$$\frac{\mathbf{d^2X_i^\mu}}{\mathbf{d}\tau^2} = -\Gamma^\mu_{\nu\rho}(\mathbf{X}(\tau)) \frac{\mathbf{dX_i^\nu}}{\mathbf{d}\tau} \frac{\mathbf{dX_i^\rho}}{\mathbf{d}\tau}, \qquad \frac{\mathbf{d^2X_i'^\mu}}{\mathbf{d}\tau^2} = -\Gamma^\mu_{\nu\rho}(\mathbf{X'}(\tau)) \frac{\mathbf{dX_i'^\nu}}{\mathbf{d}\tau} \frac{\mathbf{dX_i'^\rho}}{\mathbf{d}\tau} \, ,$$

Boundary conditions:

$$\mathbf{X_i^\mu}(\tau = \mathbf{0}) = \mathbf{c_i^\mu}, \quad \lim_{\tau\to\infty} \frac{\mathbf{dX_i^\mu}}{\mathbf{d}\tau} = \mathbf{V_i^\mu} = \frac{\mathbf{1}}{\mathbf{m_i}} \mathbf{p_i^\mu},$$

$$\mathbf{X_i'^\mu}(\tau = \mathbf{0}) = \mathbf{c_i'^\mu}, \quad \lim_{\tau\to-\infty} \frac{\mathbf{dX_i'^\mu}}{\mathbf{d}\tau} = \mathbf{V_i'^\mu} = \frac{\mathbf{1}}{\mathbf{m'_i}} \mathbf{p_i'^\mu} \, .$$

We solve these equations iteratively, starting with the solution:

$$\mathbf{e}_{\mu\nu} = \mathbf{0}, \quad \mathbf{X}_{\mathbf{i}}^{\mu}(\tau) = \mathbf{c}_{\mathbf{i}}^{\mu} + \mathbf{V}_{\mathbf{i}}^{\mu}\,\tau = \mathbf{c}_{\mathbf{i}}^{\mu} + \frac{\mathbf{1}}{\mathbf{m}_{\mathbf{i}}}\,\mathbf{p}_{\mathbf{i}}^{\mu}\,\tau,$$

$$\mathbf{X}_{\mathbf{i}}^{\prime\mu}(\tau) = \mathbf{c}_{\mathbf{i}}^{\prime\mu} + \mathbf{V}_{\mathbf{i}}^{\prime\mu}\,\tau = \mathbf{c}_{\mathbf{i}}^{\prime\mu} + \frac{\mathbf{1}}{\mathbf{m}_{\mathbf{i}}^{\prime}}\,\mathbf{p}_{\mathbf{i}}^{\prime\mu}\,\tau\,.$$

This generates a series expansion in $\mathbf{G}\,\mathbf{M}\,\omega$, possibly with corrections involving $\ln\omega$ factors.

In order to get ω^{-1} and $\ln\omega$ terms, it is enough to do one iteration.

Steps:

1. The starting solution has $T^{h\mu\nu} = 0$ and $T^{X\mu\nu}$ given by that of free particles.

From this compute $e^{\mu\nu}$ using retarded Green's function.

2. Use $e^{\mu\nu}$ to compute $T^{h\mu\nu}$.

3. From $e^{\mu\nu}$ compute the Christoffel symbol.

Calculate the Christoffel symbol at the position X_i^μ, $X_i'^\mu$ due to all objects other than the i-th object.

4. Use this to correct the trajectories $X_i^\mu(\tau)$, $X_i'^\mu(\tau)$.

5. Use the corrected trajectories to compute corrected $T^{X\mu\nu}$.

6. Take Fourier transform of $T^{X\mu\nu} + T^{h\mu\nu}$ to compute $\widehat{T}^{\mu\nu}$ and hence $\tilde{e}^{\mu\nu} = 2\,G\,R^{-1}\,e^{i\,\omega\,R}\,\widehat{T}^{\mu\nu}(k)$.

Result (of a Feynman diagram like expansion):

$$\tilde{e}^{\mu\nu} = \frac{2G}{R} e^{i\omega R} \widehat{T}^{\mu\nu}(k) = e^{i\omega R} \left[\frac{i}{\omega} A_{\mu\nu} - \ln(\omega + i\epsilon) B_{\mu\nu} + \ln(\omega - i\epsilon) C_{\mu\nu} + \mathcal{O}(1) \right]$$

$$A^{\mu\nu} = \frac{2\,G}{R} \left[-\sum_{i=1}^{n} p_i^{\mu}\, p_i^{\nu}\, \frac{1}{n.p_i} + \sum_{i=1}^{m} p_i'^{\mu}\, p_i'^{\nu}\, \frac{1}{n.p_i'} \right]$$

$$B^{\mu\nu} = -\frac{4\,G^2}{R} \left[\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{p_i.p_j}{\{(p_i.p_j)^2 - p_i^2 p_j^2\}^{3/2}} \left\{ \frac{3}{2} p_i^2 p_j^2 - (p_i.p_j)^2 \right\} \right.$$

$$\times \frac{n_\rho p_i^{\mu}}{n.p_i} (p_j^{\rho} p_i^{\nu} - p_j^{\nu} p_i^{\rho})$$

$$\left. - \sum_{j=1}^{n} p_j.n \left\{ \sum_{i=1}^{n} \frac{1}{p_i.n} p_i^{\mu} p_i^{\nu} - \sum_{i=1}^{m} \frac{1}{p_i'.n} p_i'^{\mu} p_i'^{\nu} \right\} \right]$$

$$C^{\mu\nu} = \frac{4\,G^2}{R} \sum_{i=1}^{m} \sum_{\substack{j=1 \\ i \neq j}}^{m} \frac{p_i'.p_j'}{\{(p_i'.p_j')^2 - p_i'^2 p_j'^2\}^{3/2}} \left\{ \frac{3}{2} p_i'^2 p_j'^2 - (p_i'.p_j')^2 \right\}$$

$$\times \frac{n_\rho p_i'^{\mu}}{n.p_i'} (p_j'^{\rho} p_i'^{\nu} - p_j'^{\nu} p_i'^{\rho}) \, .$$

$$n = k/\omega = (1, \vec{x}/R)$$

1. If we ignore the iϵ's, this agrees with the results obtained from classical limit of soft theorem with the ln$|\tau| \to$ ln ω^{-1} rule.

2. Taking Fourier transform in ω, we get the result for $e^{\mu\nu}$ stated at the beginning of these lectures.

$$\boxed{\begin{aligned} e_{\mu\nu} &= A_{\mu\nu} + \frac{1}{u} B_{\mu\nu} + \mathcal{O}(u^{-2}\ln|u|), \quad \text{for large positive } u, \\ e_{\mu\nu} &= \frac{1}{u} C_{\mu\nu} + \mathcal{O}(u^{-2}\ln|u|), \quad \text{for large negative } u, \end{aligned}}$$

Note: The iϵ prescriptions inside ln ω are needed in order to distinguish between the initial state contribution and the final state contribution.

Some details:

We represent the retarded Green's function in momentum space:

$$G_r(x, x') = \int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.(x-x')} \frac{1}{(\ell^0 + i\epsilon)^2 - \vec{\ell}^2}$$

– used to compute the gravitational field at a point x due to the sources.

Due to this, it is simpler to do the computation fully in the momentum space.

To first order in iteration, $\widehat{T}_{\mu\nu}(k)$ is expressed as a single momentum integral of the form

$$\int \frac{d^4\ell}{(2\pi)^4} F(\ell, k, \{p_i\}, \{p_i'\})$$

The $\ln\omega$ terms come from ℓ integration.

Two regions contribute to the $\ln\omega$ terms:

'UV': $\omega << |\ell^\mu| << L^{-1} \quad \Rightarrow \quad \ln(L\,\omega)$

'IR': $R^{-1} << |\ell^\mu| << \omega \quad \Rightarrow \quad \ln(R\,\omega)$

$C_{\mu\nu}$ gets contribution from only the UV region.

$$B^{\mu\nu} = -\frac{4\,G^2}{R}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} \frac{p_i.p_j}{\{(p_i.p_j)^2 - p_i^2 p_j^2\}^{3/2}}\left\{\frac{3}{2}p_i^2 p_j^2 - (p_i.p_j)^2\right\}\right.$$
$$\times\,\frac{n_\rho p_i^\mu}{n.p_i}\,(p_j^\rho p_i^\nu - p_j^\nu p_i^\rho)$$
$$\left. - \sum_{j=1}^{n} p_j.n\left\{\sum_{i=1}^{n}\frac{1}{p_i.n}\,p_i^\mu p_i^\nu - \sum_{i=1}^{m}\frac{1}{p'_i.n}\,p_i'^\mu p_i'^\nu\right\}\right]$$

The first term in $B_{\mu\nu}$ comes from the UV region but the second term comes from the IR region.

Nevertheless, for explosion, when the final state has at most one massive particle, the two terms cancel

Emboldened by the success of the soft theorem in predicting classical gravitational wave-form, we can attempt to arrive at new conjectures using subsubleading soft graviton theorem.

This has non-universal terms, but the term involving $J_a^{\mu\nu}$ is universal, and logarithmic term arises from this.

This leads to a prediction for terms of order $\omega\,(\ln\omega)^2$ in $\tilde{e}^{\mu\nu}(\omega,\vec{x})$

– translates to terms of order $\ln|u|/u^2$ in $e^{\mu\nu}(t,\vec{x})$. Saha, Sahoo, A.S.

$$\Delta_{\text{subsubleading}}\, e_{\mu\nu} \quad \to \quad u^{-2} \ln|u|\, F_{\mu\nu} \quad \text{as} \quad u \to \infty$$

$$\to \quad u^{-2} \ln|u|\, G_{\mu\nu} \quad \text{as} \quad u \to -\infty\,,$$

$$F^{\mu\nu} = 2\,\frac{G^3}{R}\Bigg[4 \sum_{j=1}^{n} p_b.n \sum_{\ell=1}^{n} p_\ell.n \left\{ \sum_{i=1}^{n} \frac{p_i^\mu p_i^\nu}{p_i.n} - \sum_{i=1}^{m} \frac{p_i'^\mu p_i'^\nu}{p_i'.n} \right\}$$

$$+4 \sum_{\ell=1}^{n} p_\ell.n \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{p_i.n} \frac{p_i.p_b}{\{(p_i.p_b)^2 - p_i^2 p_b^2\}^{3/2}} \{2(p_i.p_b)^2 - 3p_i^2 p_b^2\} \{n.p_b\, p_i^\mu\, p_i^\nu - n.p_i\, p_i^\mu\, p_b^\nu\}$$

$$+2 \sum_{\ell=1}^{n} p_\ell.n \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{1}{p_i'.n} \frac{p_i'.p_b'}{\{(p_i'.p_b')^2 - p_i'^2 p_b'^2\}^{3/2}} \{2(p_i'.p_b')^2 - 3p_i'^2 p_b'^2\} \{n.p_b'\, p_i'^\mu\, p_i'^\nu - n.p_i'\, p_i'^\mu\, p_b'^\nu\}$$

$$+ \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{\ell=1\\ \ell\neq i}}^{n} \frac{1}{p_i.n} \frac{p_i.p_b}{\{(p_i.p_b)^2 - p_i^2 p_b^2\}^{3/2}} \{2(p_i.p_b)^2 - 3p_i^2 p_b^2\} \frac{p_i.p_\ell}{\{(p_i.p_\ell)^2 - p_i^2 p_\ell^2\}^{3/2}}$$

$$\{2(p_i.p_\ell)^2 - 3p_i^2 p_\ell^2\} \{n.p_b\, p_i^\mu - n.p_i\, p_b^\mu\} \{n.p_\ell\, p_i^\nu - n.p_i\, p_\ell^\nu\} \Bigg]\,,$$

$$G^{\mu\nu} = -2\,\frac{G^3}{R}\Bigg[2 \sum_{\ell=1}^{n} p_\ell.n \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{1}{p_i'.n} \frac{p_i'.p_b'}{\{(p_i'.p_b')^2 - p_i'^2 p_b'^2\}^{3/2}} \{2(p_i'.p_b')^2 - 3p_i'^2 p_b'^2\}$$

$$\{n.p_b'\, p_i'^\mu\, p_i'^\nu - n.p_i'\, p_i'^\mu\, p_b'^\nu\}$$

$$- \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} \sum_{\substack{\ell=1\\ \ell\neq i}}^{m} \frac{1}{p_i'.n} \frac{p_i'.p_b'}{\{(p_i'.p_b')^2 - p_i'^2 p_b'^2\}^{3/2}} \{2(p_i'.p_b')^2 - 3p_i'^2 p_b'^2\} \frac{p_i'.p_\ell'}{\{(p_i'.p_\ell')^2 - p_i'^2 p_\ell'^2\}^{3/2}}$$

$$\{2(p_i'.p_\ell')^2 - 3p_i'^2 p_\ell'^2\} \{n.p_b'\, p_i'^\mu - n.p_i'\, p_b'^\mu\} \{n.p_\ell'\, p_i'^\nu - n.p_i'\, p_\ell'^\nu\} \Bigg]\,.$$

This is now under investigation by Biswajit Sahoo using direct classical analysis.

Stay tuned for further news!

Material not covered:

1. Soft photon theorem

– analysis proceeds similarly to soft graviton theorem

– non-universal terms appear at the subleading order due to coupling to the photon via $F_{\mu\nu}$.

Subleading soft theorem with multiple gravitons and photons is known. Bhatkar, Sahoo

Its classical limit understood at the same level as that of the soft graviton theorem

2. Relation to asymptotic symmetries

– see lectures next week

For the future:

Polynomials in ω in small ω expansion lead to delta function and their derivatives in the u-variable

– localized around u=0.

Power law decay in u should come from non-analytic functions of ω in frequency space

– arise from IR divergent terms and should therefore be determined by soft physics.

1. Can we develop a systematic procedure for computing all higher order terms in the large u expansion?

2. Do all such terms vanish for the binary black hole merger problem?