Soft Theorem and its Classical Limit

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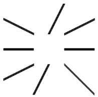
Bangalore, July 2020

Consider a violent explosion in space

D



A bound system breaks apart into fragments.

This process emits gravitational waves

Detector D placed far away detects $\mathbf{h}_{\mu\nu} \equiv (\mathbf{g}_{\mu\nu} - \eta_{\mu\nu})/2$

Examples: Explosion of supernova, binary black hole merger etc.

A more general situation: collision

A set of objects come together, interact strongly, and produce another set of objects.

This will also produce gravitational waves.

A simple example: Bullet cluster

A supercluster of galaxies passing through another supercluster of galaxies.

In general, computing gravitational wave-form $h_{\mu\nu}$ produced during such processes is complicated.

1. When the objects are close, they may undergo complicated, non-gravitational interactions, as in the case of explosion of supernova.

2. Gravity is non-linear

– even if the interactions were purely gravitational, e.g. in the case of black hole merger, the analysis is complicated.

However certain results involving S-matrix of quantum theory of gravity, known as soft graviton theorem, suggest some exact results for these classical problems.

The goal of these lectures is to explain the soft graviton theorem, its derivation, and the kind of results we can get by taking its classical limit.

We shall begin by giving a preview of some of the classical results that will come out of this analysis.

General case: Consider a scattering in space

A set of objects of four momenta $p'_1, \cdots p'_m$ come together, interact, and disperse as a set of other objects with four momenta $p_1, \cdots p_n$.

$$p_i^2 \equiv -(p_i^0)^2 + \vec{p}_i^2 = -m_i^2, \quad p_i'^2 = -m_i'^2, \quad i = 1, 2, \cdots ,$$

We shall choose the origin of space-time to be in the region where the scattering event takes place

Detector D placed at a far way point $\vec{x}$ detects $h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu})/2$ around time t_0:

$$t_0 = R/c + \text{correction}, \quad R \equiv |\vec{x}|$$

The correction is due to the gravitational drag on the gravitational radiation.

Define retarded time:

$$\mathbf{u} \equiv \mathbf{t} - \mathbf{t_0}$$

Our focus will be on the late and early time tail of the radiation – the value of $h_{\mu\nu}$ at D at large positive u and large negative u.

Define $e_{\mu\nu}$ via:

$$e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,h_{\rho\sigma} \quad \Leftrightarrow \quad h_{\mu\nu} = e_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,e_{\rho\sigma}$$

Up to gauge transformations and corrections of order R^{-2},

$$\boxed{\begin{aligned} e_{\mu\nu} &= A_{\mu\nu} + \frac{1}{u}\,B_{\mu\nu} + \mathcal{O}(u^{-2}\ln|u|), \quad \text{for large positive u} \\ e_{\mu\nu} &= \frac{1}{u}\,C_{\mu\nu} + \mathcal{O}(u^{-2}\ln|u|), \quad \text{for large negative u} \end{aligned}}$$

$A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}$ are given solely by the momenta of the ingoing and outgoing objects without requiring any knowledge of the details of the scattering process.

$$A^{\mu\nu} = \frac{2\,G}{R\,c^3}\left[-\sum_{i=1}^{n} p_i^{\mu}\,p_i^{\nu}\,\frac{1}{n.p_i} + \sum_{i=1}^{m} p_i'^{\mu}\,p_i'^{\nu}\,\frac{1}{n.p_i'}\right], \quad R \equiv |\vec{x}|, \quad n \equiv (1, \vec{x}/R)$$

$$\begin{aligned}
B^{\mu\nu} = -\frac{4\,G^2}{R\,c^7}\Bigg[&\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{p_i.p_j}{\{(p_i.p_j)^2 - m_i^2 m_j^2 c^4\}^{3/2}}\left\{\frac{3}{2}m_i^2 m_j^2 c^4 - (p_i.p_j)^2\right\} \\
&\times \frac{p_i^{\mu}}{n.p_i}\,(n.p_j\,p_i^{\nu} - n.p_i\,p_j^{\nu}) \\
&-\sum_{j=1}^{n} p_j.n\left\{\sum_{i=1}^{n}\frac{1}{p_i.n}\,p_i^{\mu}p_i^{\nu} - \sum_{i=1}^{m}\frac{1}{p_i'.n}\,p_i'^{\mu}p_i'^{\nu}\right\}\Bigg]
\end{aligned}$$

$$\begin{aligned}
C^{\mu\nu} = \frac{4\,G^2}{R\,c^7}\Bigg[&\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{p_i'.p_j'}{\{(p_i'.p_j')^2 - m_i'^2 m_j'^2 c^4\}^{3/2}}\left\{\frac{3}{2}m_i'^2 m_j'^2 c^4 - (p_i'.p_j')^2\right\} \\
&\times \frac{p_i'^{\mu}}{n.p_i'}\,(n.p_j'\,p_i'^{\nu} - n.p_i'\,p_j'^{\nu})\Bigg]\ .
\end{aligned}$$

$$\mathbf{e}_{\mu\nu} = \mathbf{A}_{\mu\nu} + \frac{1}{\mathbf{u}}\,\mathbf{B}_{\mu\nu} + \mathcal{O}(\mathbf{u}^{-2}\ln|\mathbf{u}|), \quad \text{for large positive u}$$

$$\mathbf{e}_{\mu\nu} = \frac{1}{\mathbf{u}}\,\mathbf{C}_{\mu\nu} + \mathcal{O}(\mathbf{u}^{-2}\ln|\mathbf{u}|), \quad \text{for large negative u}$$

$\mathbf{A}_{\mu\nu}$: memory term

– a permanent change in the state of the detector after the passage of gravitational waves

Zeldovich, Polnarev; Braginsky, Grishchuk; Braginsky, Thorne; $\cdots$

– connected to the leading soft graviton theorem **Strominger; $\cdots$**

$\mathbf{B}_{\mu\nu}, \mathbf{C}_{\mu\nu}$: tail terms

– connected to logarithmic terms in the subleading soft graviton theorem **Laddha, A.S.; Sahoo, A.S.**

1. The result is a statement in classical GR, even though it was originally suggested by quantum soft graviton theorem.

Now we have a fully classical derivation. Saha, Sahoo, A.S.

2. $\mathbf{A}_{\mu\nu}, \mathbf{B}_{\mu\nu}, \mathbf{C}_{\mu\nu}$ can be expressed in terms of the momenta of incoming and outgoing objects without knowing what forces operated and how the objects moved during the scattering.

– consequence of soft graviton theorem

3. If a significant fraction of energy is carried away by radiation, then the sum over i,j includes integration over outgoing flux of radiation, regarded as a flux of massless particles.

4. The result matches explicit known results in special cases.

Peters; Ciafaloni, Colferai, Veneziano; Addazi, Bianchi, Veneziano

5. Explosion can be regarded as a special case of scattering when the initial state has just one object.

In this case $C_{\mu\nu}$ vanishes and $e_{\mu\nu}$ takes the form:

$$e_{\mu\nu} = A_{\mu\nu} + \frac{1}{u} B_{\mu\nu} + \mathcal{O}(u^{-2} \ln|u|), \quad \text{for large positive } u$$

$$e_{\mu\nu} = 0, \quad \text{for large negative } u$$

6. For explosion, the contribution to $B^{\mu\nu}$ vanishes unless there are at least two massive objects in the final state

– result of cancellation between different terms

$$B^{\mu\nu} = -\frac{4\,G^2}{R\,c^7}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{p_i.p_j}{\{(p_i.p_j)^2 - m_i^2 m_j^2 c^4\}^{3/2}} \left\{\frac{3}{2} m_i^2 m_j^2 c^4 - (p_i.p_j)^2\right\} \right.$$
$$\times \frac{p_i^\mu}{n.p_i}\left(n.p_j\, p_i^\nu - n.p_i\, p_j^\nu\right)$$
$$\left. - \sum_{j=1}^{n} p_j.n \left\{\sum_{i=1}^{n} \frac{1}{p_i.n} p_i^\mu p_i^\nu - \sum_{i=1}^{m} \frac{1}{p_i'.n} p_i'^\mu p_i'^\nu\right\}\right]$$

⇒ the coefficient of the 1/u tail vanishes for binary black hole merger!

Final state contains one massive object (remnant black hole) and a flux of massless particles (gravitational radiation).

In contrast, the coefficient of the 1/u term is non-zero for supernova explosion, binary neutron star merger etc.

PLAN

1. Quantum soft graviton theorem and its derivation

2. Classical limit

3. Classical proof

4. Issues with infrared divergences in D=4

5. Resolution

Convention: $\hbar = 1$, $c = 1$, $8\pi G = 1$

Quantum soft graviton theorem

What is soft graviton theorem?

Take a general coordinate invariant quantum theory of gravity coupled to matter fields

Consider an S-matrix element involving

– arbitrary number N of external particles of finite momentum $p_1, \cdots p_N$

– M external gravitons carrying small momentum $k_1, \cdots k_M$.

Soft graviton theorem: Expansion of this amplitude in power series in $k_1, \cdots k_M$ in terms of the amplitude without the low energy (soft) gravitons.

There are many explicit results.

1. **General results at leading order in k**

Weinberg; . . .

2. **For <u>one</u> soft graviton, there are general subleading results in D=4 via BMS**

Strominger; Strominger, Zhiboedov; Campiglia, Laddha; . . .

3. **Results in specific theories in general dimensions**

White; Cachazo, Strominger; Bern, Davies, Di Vecchia, Nohle; Elvang, Jones, Naculich; . . .

Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha

Bianchi, Guerrieri; Di Vecchia, Marotta, Mojaza; . . .

Our goal: Study soft graviton amplitudes in generic quantum theory of gravity, in generic number of dimensions, for arbitrary mass and spin of external states

Validity

1. For tree amplitudes our analysis will be valid in all dimensions

2. For loop amplitudes the results will be valid if we assume that 1PI vertices do not generate soft factors in the denominator

True by power counting for

– subleading order for D > 5

– subsubleading order for D > 6

D: number of non-compact space-time dimensions

For single soft gravitons we can argue that the unwanted terms cancel in the sum over graphs and the results are also valid for D=5,6

We expect a similar result to hold for multiple soft gravitons, but this has not been proved.

In D=4 the S-matrix elements themselves are infrared divergent, introducing additional subtleties. Bern, Davies, Nohle

In $D = 4$, our analysis will apply to only tree amplitudes, but our results in D>4 will suggest how to modify the results in D=4.

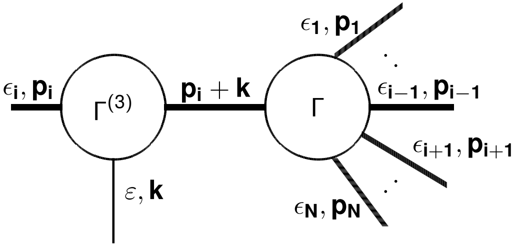
A.S. arXiv:1702.03934, 1703.00024: Subleading single soft

A. Laddha, A.S., arXiv:1706.00759: Sub-subleading single soft

Subhroneel Chakrabarti, Sitender Kashyap, Biswajit Sahoo, A.S., Mritunjay Verma, arXiv:1707.06803; subleading multiple soft

Single soft graviton

We divide the Feynman diagrams into two classes



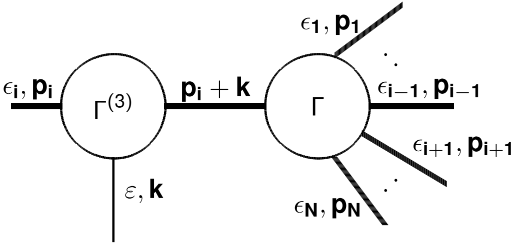
Γ: Full amputated Green's function

Internal lines: Full renormalized propagators

ε, k: polarization, momentum of soft graviton

ϵ_i, p_i: polarization, momentum of finite energy external particles.

p_i, k counted with + sign if outgoing and $-$ sign if ingoing



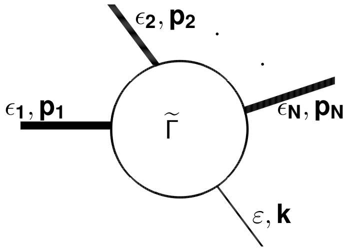
The internal line carrying momentum $p_i + k$ has denominator factor

$$\{(p_i + k)^2 + m^2\}^{-1} = (2p_i \cdot k)^{-1} \quad \text{if } m = m_i$$

using $p_i^2 + m_i^2 = 0$, $k^2 = 0$.

$\Rightarrow$ this starts contributing at the leading order.

Second class of diagrams



$\widetilde{\Gamma}$: **Amputated amplitudes in which the external soft graviton <u>does not</u> get attached to an external line**

– has no pole as k $\to$ 0

$\Rightarrow$ the contribution from this diagram begins at the subleading order.

Strategy for computation

1. Consider the gauge invariant one particle irreducible (1PI) effective action of the theory

2. Expand the action in powers of all fields, including the metric fluctuations, around the extremum of the action

3. Add manifestly Lorentz invariant gauge fixing terms.

4. This action is used to compute vertices and propagators of finite energy external states but <u>not</u> of <u>soft external gravitons</u>.

5. To calculate the coupling of the soft graviton $S_{\mu\nu}$ to the rest of the fields, we covariantize the gauge fixed action.

a. Replace the background metric $\eta_{\mu\nu}$ by $\eta_{\mu\nu} + 2\,S_{\mu\nu}$

b. Replace all derivatives by covariant derivatives computed with the metric $\eta_{\mu\nu} + 2\,S_{\mu\nu}$

This misses terms involving Riemann tensor computed from the metric $\eta_{\mu\nu} + 2\,S_{\mu\nu}$ but that contains two derivatives and hence is sub-subleading.

1. We choose

$$\mathbf{S}_{\mu\nu} = \varepsilon_{\mu\nu} \mathbf{e}^{\mathbf{ik\cdot x}}, \quad \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}, \quad \varepsilon^{\mu}{}_{\mu} = \mathbf{k}^{\mu}\varepsilon_{\mu\nu} = \mathbf{0}$$

<u>All indices raised and lowered by η.</u>

2. All fields representing finite energy external states are taken to carry tangent space Lorentz indices

– allows us to give uniform treatment to fermions and bosons.

3. To first order in $\mathbf{S}_{\mu\nu}$, we take the vielbeins to be

$$\mathbf{e}_{\mu}^{\ \mathbf{a}} = \delta_{\mu}^{\ \mathbf{a}} + \mathbf{S}_{\mu}^{\ \mathbf{a}}, \quad \mathbf{E}_{\mathbf{a}}^{\ \mu} = \delta_{\mathbf{a}}^{\ \mu} - \mathbf{S}_{\mathbf{a}}^{\ \mu}, \quad \mathbf{g}^{\mu\nu} = \eta^{\mu\nu} - 2\,\mathbf{S}^{\mu\nu}$$

$\{\phi_\alpha\}$: **set of all the fields in the theory with α's including Lorentz indices.**

$\mathbf{i}(\Sigma_{\mathbf{ab}})_\alpha{}^\beta$: **Infinitesimal Lorentz transformation matrix**

Covariantization: Acting on a field ϕ_α:

$$\partial_{\mathbf{a_1}} \cdots \partial_{\mathbf{a_n}} \Rightarrow \mathbf{E}_{\mathbf{a_1}}^{\mu_1} \cdots \mathbf{E}_{\mathbf{a_n}}^{\mu_n} \mathbf{D}_{\mu_1} \cdots \mathbf{D}_{\mu_n}, \quad \mathbf{E}_{\mathbf{a}}{}^{\mu} \equiv (\delta_{\mathbf{a}}{}^{\mu} - \mathbf{S}_{\mathbf{a}}{}^{\mu})$$

$$\mathbf{D}_\mu \phi_\alpha = \partial_\mu \phi_\alpha - \frac{\mathbf{i}}{\mathbf{2}} \omega_\mu^{\mathbf{ab}} (\Sigma_{\mathbf{ab}})_\alpha{}^\gamma \phi_\gamma$$

$$\mathbf{D}_{(\mu} \mathbf{D}_{\nu)} \phi_\alpha = \partial_\mu \partial_\nu \phi_\alpha - \mathbf{i}\, \omega_{(\mu}^{\mathbf{ab}} (\Sigma_{\mathbf{ab}})_\alpha{}^\gamma \partial_{\nu)} \phi_\gamma - \frac{\mathbf{i}}{\mathbf{2}} \partial_{(\mu} \omega_{\nu)}^{\mathbf{ab}} (\Sigma_{\mathbf{ab}})_\alpha{}^\gamma \phi_\gamma + \left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} \partial_\rho \phi_\alpha$$

etc.

$$\omega_\mu^{\mathbf{ab}} = \partial^{\mathbf{b}} \mathbf{S}_\mu{}^{\mathbf{a}} - \partial^{\mathbf{a}} \mathbf{S}_\mu{}^{\mathbf{b}}, \quad \mathbf{S}_{\mu\nu} = \varepsilon_{\mu\nu} \mathbf{e}^{\mathbf{ik \cdot x}}$$

$$\left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} = \frac{\mathbf{1}}{\mathbf{2}} \left[\partial_\mu \mathbf{S}_\nu{}^\rho + \partial_\nu \mathbf{S}_\mu{}^\rho - \partial^\rho \mathbf{S}_{\mu\nu} \right]$$

Consider a Lorentz invariant functional

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})\; \delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N})\, \mathbf{F}^{\alpha_1 \cdots \alpha_N}(\mathbf{p_1}, \ldots \mathbf{p_N})$$

Covariantization produces an additional term

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})\, \delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N} + \mathbf{k})$$
$$\sum_{i=1}^{N} \Big[-\delta_{\beta_i}^{\ \alpha_i} \varepsilon_{\mu}^{\ \nu} \mathbf{p}_{i\nu} \frac{\partial}{\partial \mathbf{p}_{i\mu}} - \frac{i}{2} (\mathbf{k^b} \varepsilon_{\mu}^{\ a} - \mathbf{k^a} \varepsilon_{\mu}^{\ b}) (\Sigma_{ab})_{\beta_i}^{\ \alpha_i} \frac{\partial}{\partial \mathbf{p}_{i\mu}}$$
$$- \frac{1}{2} \delta_{\beta_i}^{\ \alpha_i} \left\{ \mathbf{k}_\mu \varepsilon_{\nu}^{\ \rho} + \mathbf{k}_\nu \varepsilon_{\mu}^{\ \rho} - \mathbf{k}^\rho \varepsilon_{\mu\nu} \right\} \mathbf{p}_{i\rho} \frac{\partial^2}{\partial \mathbf{p}_{i\mu} \partial \mathbf{p}_{i\nu}} \Big]$$
$$\mathbf{F}^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N}(\mathbf{p_1}, \ldots, \mathbf{p_N}) + \mathcal{O}(\mathbf{k}^\mu \mathbf{k}^\nu) \,.$$

After some algebra, this can be rewritten as

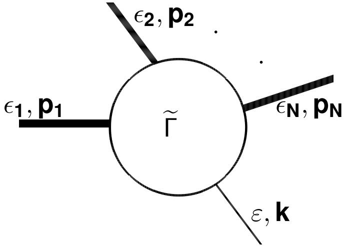
$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})$$

$$\sum_{i=1}^{N} \Bigg[-\delta_{\beta_i}^{\ \alpha_i} \varepsilon_{\mu}^{\ \nu} \mathbf{p}_{i\nu} \frac{\partial}{\partial \mathbf{p}_{i\mu}} - \frac{i}{2} (\mathbf{k^b} \varepsilon_{\mu}^{\ a} - \mathbf{k^a} \varepsilon_{\mu}^{\ b}) (\Sigma_{ab})_{\beta_i}^{\ \alpha_i} \frac{\partial}{\partial \mathbf{p}_{i\mu}}$$

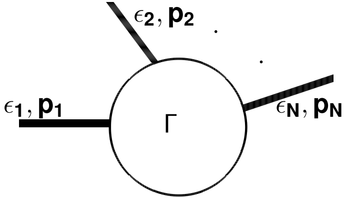
$$- \frac{1}{2} \delta_{\beta_i}^{\ \alpha_i} \left\{ \mathbf{k}_\mu \varepsilon_{\nu}^{\ \rho} + \mathbf{k}_\nu \varepsilon_{\mu}^{\ \rho} - \mathbf{k}^\rho \varepsilon_{\mu\nu} \right\} \mathbf{p}_{i\rho} \frac{\partial^2}{\partial \mathbf{p}_{i\mu} \partial \mathbf{p}_{i\nu}} \Bigg]$$

$$\left\{ \mathbf{F}^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N} (\mathbf{p_1}, \ldots, \mathbf{p_N})\, \delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N}) \right\} + \mathcal{O}(\mathbf{k}^\mu \mathbf{k}^\nu)\,.$$

Now consider



1. Take the amplitude without soft graviton.



2. Covariantize it to order k^μ

Denote the amplitude without the soft graviton by

$$\epsilon_{1,\alpha_1}(p_1)\ldots\epsilon_{N,\alpha_N}(p_N)\,\Gamma^{\alpha_1\ldots\alpha_N}(p_1,\ldots,p_N)$$

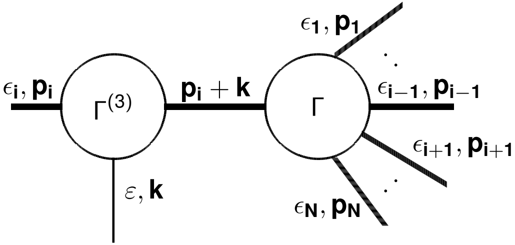
α_i**: tangent space tensor / spinor indices labelling all the fields of the theory**

$\Gamma^{\alpha_1\cdots\alpha_N}$ **includes the** $\delta^{(D)}(p_1+\cdots p_N)$ **factor.**

Then the result for $\widetilde{\Gamma}$ **is**

$$\sum_{i=1}^{N}\Big[-\delta_{\beta_i}^{\ \alpha_i}\varepsilon_\mu^{\ \nu}p_{i\nu}\frac{\partial}{\partial p_{i\mu}}-i\,k^b\varepsilon_\mu^{\ a}(\Sigma_{ab})_{\beta_i}^{\ \alpha_i}\frac{\partial}{\partial p_{i\mu}}$$
$$-\frac{1}{2}\delta_{\beta_i}^{\ \alpha_i}\left\{k_\mu\varepsilon_\nu^{\ \rho}+k_\nu\varepsilon_\mu^{\ \rho}-k^\rho\varepsilon_{\mu\nu}\right\}\,p_{i\rho}\,\frac{\partial^2}{\partial p_{i\mu}\partial p_{i\nu}}\Big]$$
$$\Gamma^{\alpha_1\cdots\alpha_{i-1}\beta_i\alpha_{i+1}\cdots\alpha_N}(p_1,\ldots,p_N)+\mathcal{O}(k^\mu k^\nu)\,.$$

Next consider



Need to focus on the three point coupling computed from the 1PI action.

Begin with two point function without the soft graviton and covariantize it to order $k^\mu k^\nu$.

$$S^{(2)} = \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\,\frac{d^D q_2}{(2\pi)^D}\,\phi_\alpha(q_1)\mathcal{K}^{\alpha\beta}(q_2)\phi_\beta(q_2)\,(2\pi)^D\delta^{(D)}(q_1+q_2)$$

$\{\phi_\alpha\}$: set of all the fields

$\mathcal{K}^{\alpha\beta}(q)$: Kinetic operator, chosen to satisfy

$$\mathcal{K}^{\alpha\beta}(q) = \mathcal{K}^{\beta\alpha}(-q)$$

For grassmann odd fields the sign is opposite but the final result is not affected.

Covariantization $\Rightarrow$ coupling of ϕ_α to soft graviton

$$\begin{aligned} S^{(3)} \;=\; & \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\,\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k) \\ & \times\phi_\alpha(q_1)\left[-\varepsilon_{\mu\nu}q_2^\nu\frac{\partial}{\partial q_{2\mu}}\mathcal{K}^{\alpha\beta}(q_2) + i\,k_a\,\varepsilon_{b\mu}\frac{\partial}{\partial q_{2\mu}}\mathcal{K}^{\alpha\gamma}(q_2)\left(\Sigma^{ab}\right)_\gamma^{\;\beta}\right. \\ & \left. -\frac{1}{2}\left\{k_\mu\varepsilon_\nu^{\;\rho} + k_\nu\varepsilon_\mu^{\;\rho} - k^\rho\varepsilon_{\mu\nu}\right\}\,q_{2\rho}\,\frac{\partial^2\mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu}\partial q_{2\nu}}\right]\phi_\beta(q_2) + \mathcal{O}(k^\mu k^\nu) \end{aligned}$$

– determines the coupling of the soft graviton to the finite energy particles

$$
\begin{aligned}
&\Gamma^{(3)\alpha\beta}(\varepsilon, \mathbf{k}; \mathbf{p}, -\mathbf{p}-\mathbf{k}) \\
= \quad & \frac{\mathbf{i}}{\mathbf{2}}\Bigg[-\varepsilon_{\mu\nu}(\mathbf{p}+\mathbf{k})^{\nu}\frac{\partial}{\partial \mathbf{p}_\mu}\mathcal{K}^{\alpha\beta}(-\mathbf{p}-\mathbf{k}) - \varepsilon_{\mu\nu}\mathbf{p}^{\nu}\frac{\partial}{\partial \mathbf{p}_\mu}\mathcal{K}^{\beta\alpha}(\mathbf{p}) \\
& -\frac{\mathbf{i}}{\mathbf{2}}(\mathbf{k_a}\,\varepsilon_{\mathbf{b}\mu} - \mathbf{k_b}\,\varepsilon_{\mathbf{a}\mu})\frac{\partial}{\partial \mathbf{p}_\mu}\mathcal{K}^{\alpha\gamma}(-\mathbf{p}-\mathbf{k})\left(\Sigma^{\mathbf{ab}}\right)_{\gamma}^{\;\beta} \\
& +\frac{\mathbf{i}}{\mathbf{2}}(\mathbf{k_a}\,\varepsilon_{\mathbf{b}\mu} - \mathbf{k_b}\,\varepsilon_{\mathbf{a}\mu})\frac{\partial}{\partial \mathbf{p}_\mu}\mathcal{K}^{\beta\gamma}(\mathbf{p})\left(\Sigma^{\mathbf{ab}}\right)_{\gamma}^{\;\alpha} \\
& -\frac{\mathbf{1}}{\mathbf{2}}\frac{\partial^{\mathbf{2}}\mathcal{K}^{\alpha\beta}(-\mathbf{p}-\mathbf{k})}{\partial \mathbf{p}_\mu \partial \mathbf{p}_\nu}(-\mathbf{p}_\rho - \mathbf{k}_\rho)\left(\mathbf{k}_\mu \varepsilon_{\nu}^{\;\rho} + \mathbf{k}_\nu \varepsilon_{\mu}^{\;\rho} - \mathbf{k}^{\rho}\varepsilon_{\mu\nu}\right) \\
& -\frac{\mathbf{1}}{\mathbf{2}}\frac{\partial^{\mathbf{2}}\mathcal{K}^{\beta\alpha}(\mathbf{p})}{\partial \mathbf{p}_\mu \partial \mathbf{p}_\nu}\mathbf{p}_\rho\left(\mathbf{k}_\mu \varepsilon_{\nu}^{\;\rho} + \mathbf{k}_\nu \varepsilon_{\mu}^{\;\rho} - \mathbf{k}^{\rho}\varepsilon_{\mu\nu}\right) + \mathbf{O}(\mathbf{k}_\mu \mathbf{k}_\nu)\Bigg]
\end{aligned}
$$

Note: This is determined in terms of $\mathcal{K}^{\alpha\beta}$.

We also need the propagator of ϕ_α carrying momentum $\mathbf{q}$

Define $\quad \mathcal{N}^{\mathbf{j}}_{\alpha\beta}(\mathbf{q}) \equiv \mathbf{i}\,(\mathbf{q}^2 + \mathbf{m}_\mathbf{j}^2)\,\mathcal{K}^{-1}_{\alpha\beta}(\mathbf{q}) \quad$ (fixed j)

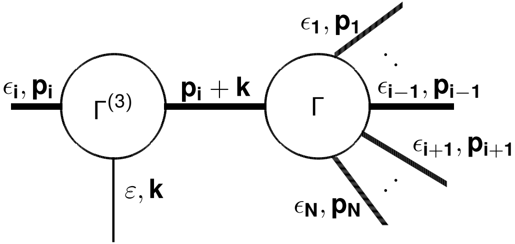
$\mathbf{m}_\mathbf{j}$: mass of the j-th external particle.

$\mathbf{N}^\mathbf{j}(\mathbf{q})$ has no pole at $\mathbf{q}^2 + \mathbf{m}_\mathbf{j}^2 = 0$.

The propagator of the particle carrying momentum $\mathbf{p}_\mathbf{j} + \mathbf{k}$

$$\mathbf{i}\,\mathcal{K}^{-1}_{\alpha\beta}(\mathbf{p}_\mathbf{j} + \mathbf{k}) \equiv (2\mathbf{p}_\mathbf{j} \cdot \mathbf{k})^{-1}\mathcal{N}^{\mathbf{j}}_{\alpha\beta}(\mathbf{p}_\mathbf{j} + \mathbf{k})$$

Note: The same propagator is expressed differently for different external states to make the pole structure manifest.



Once we compute the Feynman diagram with these Feynman rules, we find that all dependence on $\mathcal{K}^{\alpha\beta}$ cancels, and we get a simple expression to subleading order in expansion in the soft momentum.

Need to use

– on-shell condition: $\epsilon_{\mathbf{i}\alpha}\mathcal{K}^{\alpha\beta} = 0$

– Lorentz invariance of $\mathcal{K}^{\alpha\beta}$ under simultaneous Lorentz transformation of α, β and momenta.

Final result for the soft graviton amplitude to subleading order:

$$\prod_{j=1}^{N} \epsilon_{j,\alpha_j}(p_j) \left[S^{(0)} \Gamma^{\alpha_1 \ldots \alpha_N} + \left\{ S^{(1)} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} \right]$$

$$S^{(0)} \equiv \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{\mu\nu} \, p_i^\mu \, p_i^\nu$$

$$\begin{aligned} \left\{ S^{(1)} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} &= \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{\mu\nu} \, p_i^\mu \, k_\rho \left(p_i^\nu \frac{\partial}{\partial p_{i\rho}} - p_i^\rho \frac{\partial}{\partial p_{i\nu}} \right) \Gamma^{\alpha_1 \cdots \alpha_N} \\ &- i \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{\mu b} \, p_i^\mu \, k_a \, (\Sigma^{ab})_\gamma^{\ \alpha_i} \, \Gamma^{\alpha_1 \cdots \alpha_{i-1} \gamma \alpha_{i+1} \cdots \alpha_N} \end{aligned}$$

This is the subleading soft graviton theorem

– agrees with all known results in field theory / string theory

This analysis can be extended to subsubleading order but we shall not discuss this

– has non-universal terms due to possible coupling of the soft graviton via Riemann tensor.

Multiple soft gravitons

Naive guess: To subleading order,

$$\prod_{j=1}^{N} \epsilon_{j,\alpha_j}(p_j) \left[\left\{ S^{(0)}(\varepsilon_1, k_1) + S^{(1)}(\varepsilon_1, k_1) \right\} \cdots \right.$$
$$\left. \cdots \left\{ S^{(0)}(\varepsilon_M, k_M) + S^{(1)}(\varepsilon_M, k_M) \right\} \Gamma \right]^{\alpha_1 \cdots \alpha_N}$$

Problem: $S^{(0)}$ and $S^{(1)}$ do not commute.

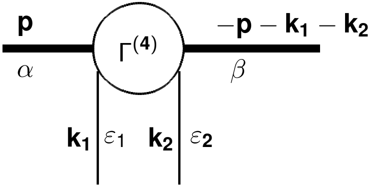
This expression is not symmetric under the exchange of the soft gravitons.

We have to do the analysis afresh.

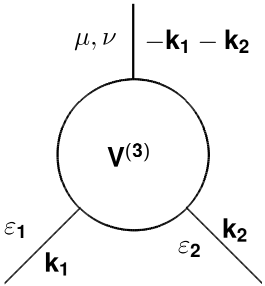
The procedure is identical, except that while constructing the soft graviton coupling via covariantization we need to include higher powers of the soft graviton field $S_{\mu\nu}$.

For computations up to subleading order, we need to keep terms up to two powers of $S_{\mu\nu}$ in the covariantization of the kinetic operator.

– leads to a four point interaction vertex



The other new vertex is three point coupling of soft gravitons.



We need this to leading order in soft momenta

– can be obtained by expanding the Einstein-Hilbert action

Result for M soft gravitons and N finite energy particles

Chakrabarti, Kashyap, Sahoo, A.S, Verma

$$\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}(p_i)\right\}\left[\left\{\prod_{r=1}^{M}S_r^{(0)}\right\}\Gamma^{\alpha_1\cdots\alpha_N}+\sum_{s=1}^{M}\left\{\prod_{\substack{r=1\\ r\neq s}}^{M}S_r^{(0)}\right\}\left[S_s^{(1)}\Gamma\right]^{\alpha_1\cdots\alpha_N}\right.$$

$$\left.+\sum_{\substack{r,u=1\\ r<u}}^{M}\left\{\prod_{\substack{s=1\\ s\neq r,u}}^{M}S_s^{(0)}\right\}\left\{\sum_{j=1}^{N}\{p_j\cdot(k_r+k_u)\}^{-1}\ \mathcal{M}(p_j;\varepsilon_r,k_r,\varepsilon_u,k_u)\right\}\Gamma^{\alpha_1\cdots\alpha_N}\right]$$

$S_r^{(0)}, S_r^{(1)}$: Soft factors defined earlier for r-th soft graviton

$\mathcal{M}$: ‘contact term’

$$
\begin{aligned}
&\mathcal{M}(\mathbf{p_i};\varepsilon_1,\mathbf{k_1},\varepsilon_2,\mathbf{k_2}) \\
= \quad &(\mathbf{p_i}\cdot\mathbf{k_1})^{-1}(\mathbf{p_i}\cdot\mathbf{k_2})^{-1}\Big\{-\mathbf{k_1}\cdot\mathbf{k_2}\ \mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{p_i}\ \mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{p_i} \\
&+2\,\mathbf{p_i}\cdot\mathbf{k_2}\ \mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{p_i}\ \mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{k_1}+2\,\mathbf{p_i}\cdot\mathbf{k_1}\ \mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{p_i}\ \mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{k_2} \\
&-2\,\mathbf{p_i}\cdot\mathbf{k_1}\ \mathbf{p_i}\cdot\mathbf{k_2}\ \mathbf{p_i}\cdot\varepsilon_1\cdot\varepsilon_2\cdot\mathbf{p_i}\Big\} \\
&+(\mathbf{k_1}\cdot\mathbf{k_2})^{-1}\Big\{-(\mathbf{k_2}\cdot\varepsilon_1\cdot\varepsilon_2\cdot\mathbf{p_i})(\mathbf{k_2}\cdot\mathbf{p_i})-(\mathbf{k_1}\cdot\varepsilon_2\cdot\varepsilon_1\cdot\mathbf{p_i})(\mathbf{k_1}\cdot\mathbf{p_i}) \\
&+(\mathbf{k_2}\cdot\varepsilon_1\cdot\varepsilon_2\cdot\mathbf{p_i})(\mathbf{k_1}\cdot\mathbf{p_i})+(\mathbf{k_1}\cdot\varepsilon_2\cdot\varepsilon_1\cdot\mathbf{p_i})(\mathbf{k_2}\cdot\mathbf{p_i}) \\
&-\varepsilon_1^{\gamma\delta}\varepsilon_{2\gamma\delta}(\mathbf{k_1}\cdot\mathbf{p_i})(\mathbf{k_2}\cdot\mathbf{p_i})-2(\mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{k_2})(\mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{k_1}) \\
&+(\mathbf{p_i}\cdot\varepsilon_2\cdot\mathbf{p_i})(\mathbf{k_2}\cdot\varepsilon_1\cdot\mathbf{k_2})+(\mathbf{p_i}\cdot\varepsilon_1\cdot\mathbf{p_i})(\mathbf{k_1}\cdot\varepsilon_2\cdot\mathbf{k_1})\Big\},
\end{aligned}
$$

– agrees with results for two soft gravitons in specific theories

Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha