Stability of many-body localization in two and higher dimension

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Indian Institute of Science

Correlation and Disorder in Classical and Quantum Systems
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Congratulations to HRK and Chandan for inspiring generations of students!!

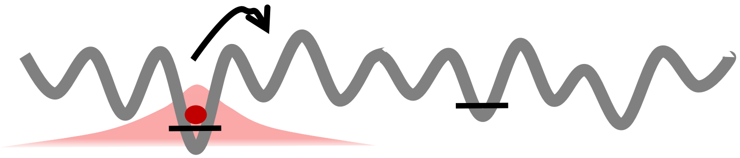
Stability of many-body localization in two and higher dimension

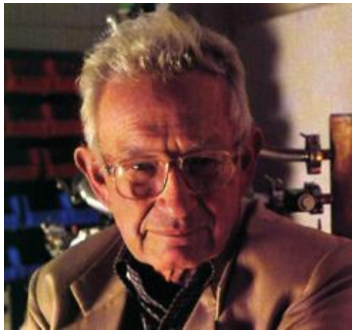
Ehud Altman
(UC Berkeley)

Ionut-Dragos Potirniche
(UC Berkeley)

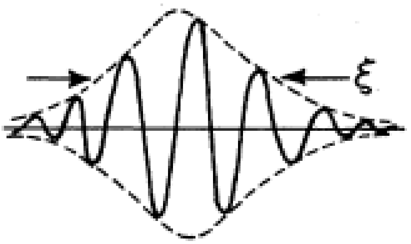
Anderson localization

Anderson (single-particle) localization (1958)





Localized | **Extended**



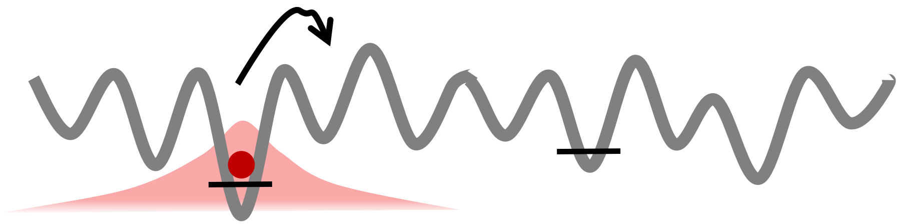
$$|\psi_\alpha(r)|^2 \sim e^{-\frac{|r-r_\alpha|}{\xi}}$$

$$\sim \frac{1}{L^{d/2}}$$

Abrahams et al. Scaling theory of localization (1979)

Lee & Ramakrishnan (1985), …

Many-body localization (MBL)



All states localized

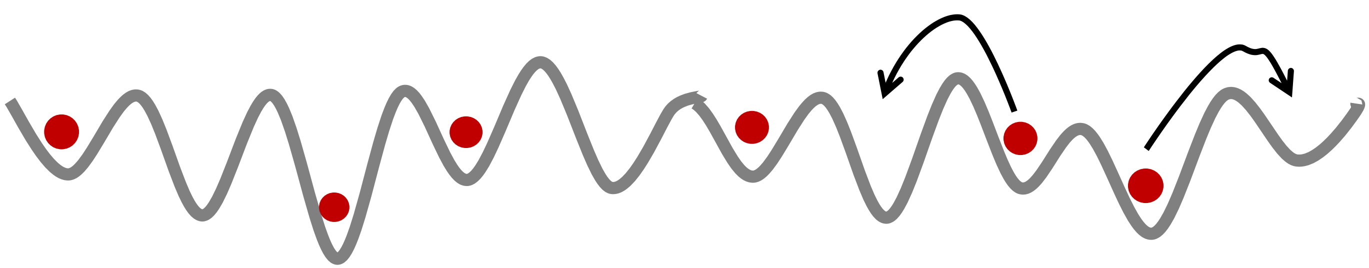
$$|\psi_\alpha(r)|^2 \sim \frac{e^{-\frac{|r-r_\alpha|}{\xi}}}{\xi^d}$$

$$\mathcal{H} = -t \sum_{<ij>} \left(c_i^\dagger c_j + h.c.\right) - \sum_i \varepsilon_i n_i \quad + V \sum_{\langle ij \rangle} n_i n_j$$

$\varepsilon_i \in [-W, W]$

$$= \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha \quad + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$

Add interaction

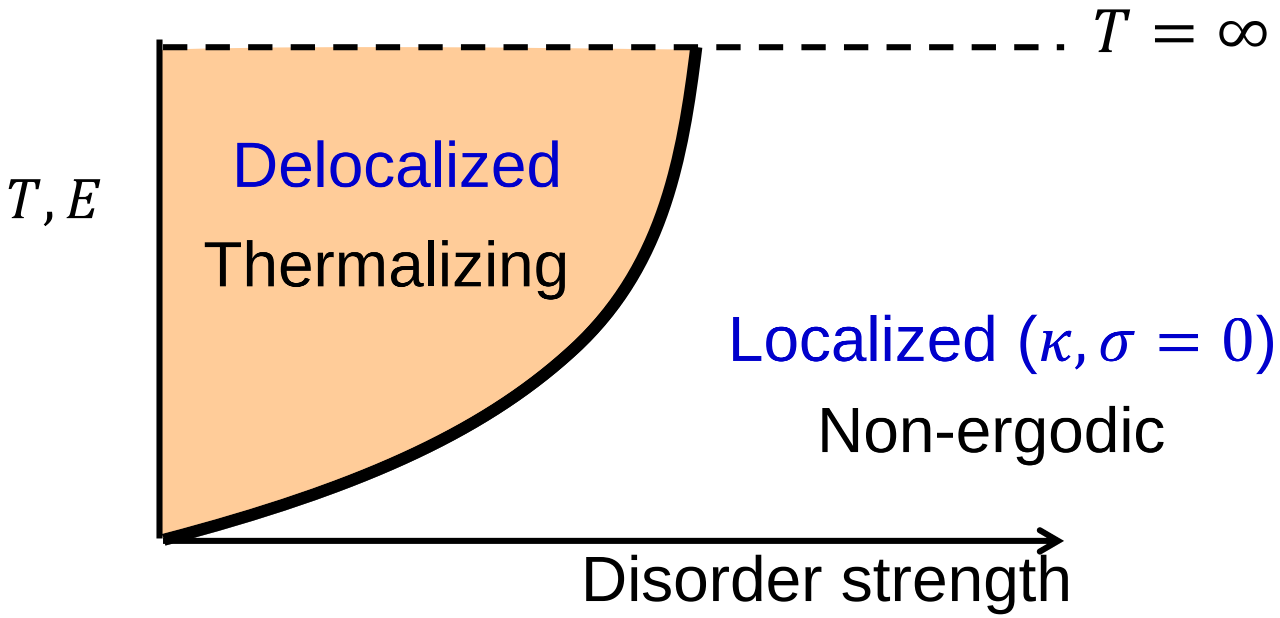


At high energies interaction connects between $\sim \exp(L^d)$ localized states ! Can localization survive?

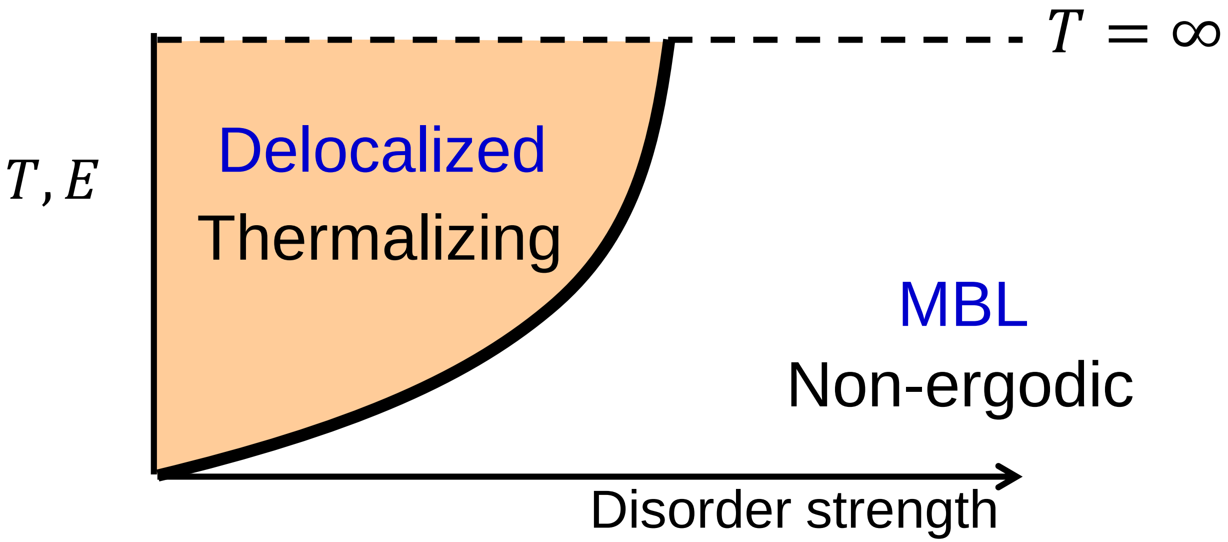
Many-body localization (MBL)

Yes!
For sufficiently strong disorder

Basko, Aleiner, Altshuler (2005); Gornyi, Mirlin, Polyakov (2005)



Oganesyan and Huse (2007), Pal and Huse (2010), ...



Existence of MBL →

○ 1D

-- Numerical evidence

Oganesyan and Huse (2007), Pal and Huse (2010), ……

-- Mathematical proof

Imbrie (2016)

Existence of MBL →

- General dimension, e.g. $d > 1$
  only perturbative proof

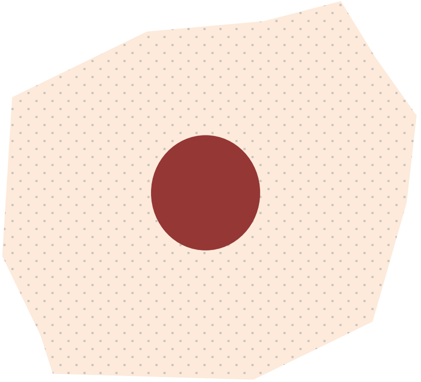
Basko, Aleiner, Altshuler (2005),
Gornyi, Mirlin, Polyakov (2005)

Perturbative treatment for weak interaction, $V/\Delta_\xi \ll 1$

V, interaction strength
Δ_ξ , level spacing within a localization volume

- Are there nonperturbative effects that can destabilize MBL in higher dimension?

Stability of MBL in the presence of ergodic grain?



De Roeck and Huveneers (2016)

Outline

- Introduction to MBL.
- Instability argument for $d > 1$.
- Solvable models and exact diagonalization studies to test the instability argument.
- Conclusions.

Models for MBL

- Disordered interacting fermions

$$\mathcal{H} = -t \sum_{<ij>} \left(c_i^\dagger c_j + h.c. \right) - \sum_i \varepsilon_i n_i + V \sum_{\langle ij \rangle} n_i n_j$$

$\varepsilon_i \in [-W, W]$

Hilbert space dimension
$D = 2^N$

- Disordered spin chains

$$\mathcal{H} = J \sum_i (S_i^+ S_{i+1}^- + h.c.) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

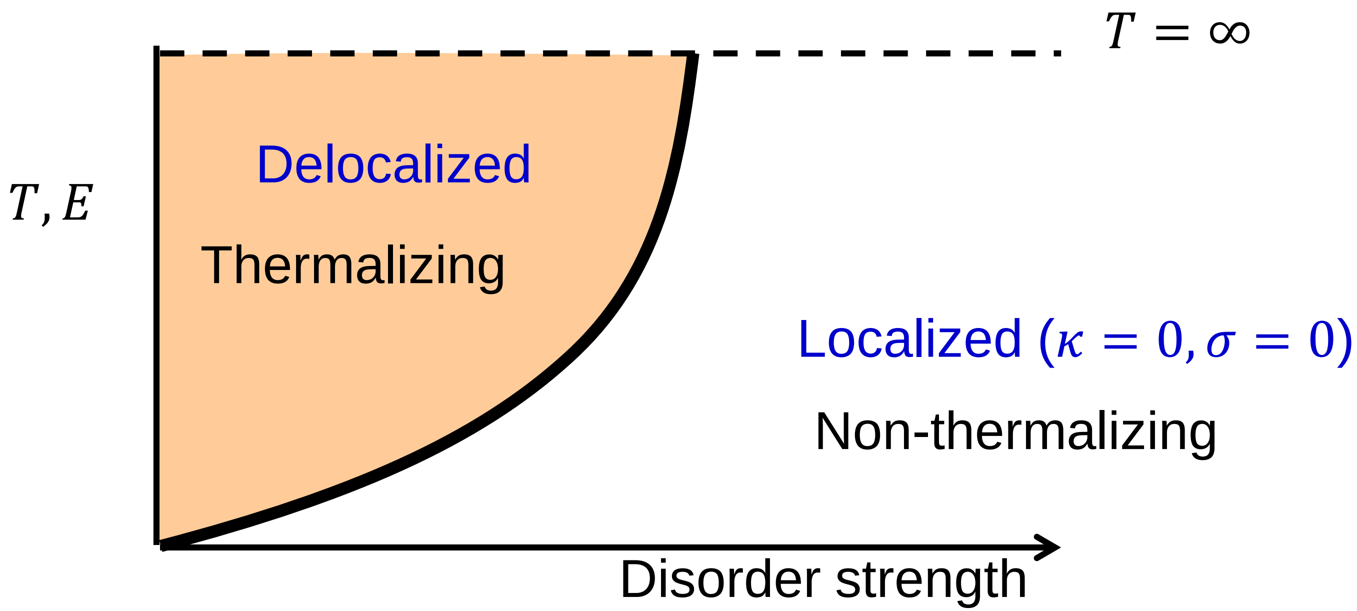
$h_i \in [-W, W]$

- Disordered Hubbard model, transverse field Ising model, ...

Many-body eigenstates $\mathcal{H} |\Psi_n\rangle = E_n |\Psi_n\rangle$ Bounded spectrum

Localization and thermalization in eigenstates

Diagnostic of MBL



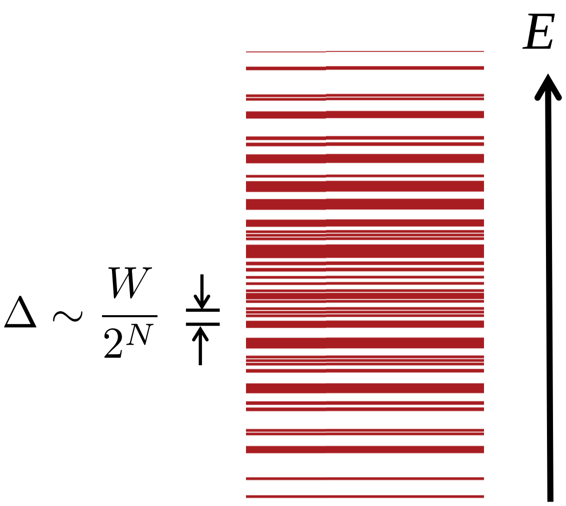
Many-body localization

$\Rightarrow$ Lack of thermalization in a generic system!

Localization and thermalization

Thermalization or ergodicity
⇒ “Eigenstate thermalization hypothesis” (ETH)

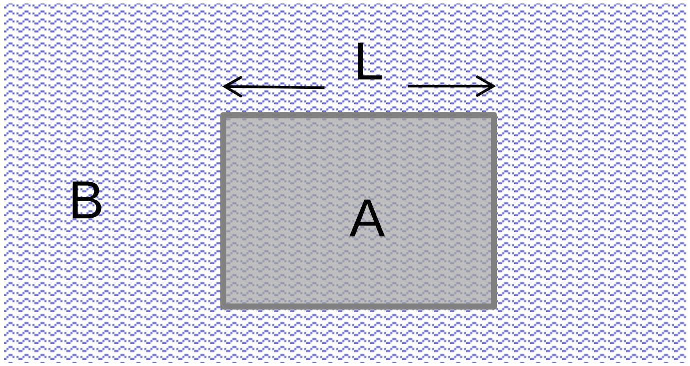
Deutsch 91, Srednicki 94



Generic high-energy eigenstates $|E\rangle$
(finite energy density above ground state)

→ Eigenstates of <u>thermalizing</u> system appear thermal to all local measurements

- $\rho_A = \mathrm{Tr_B}|E\rangle\langle E| \rightarrow \frac{1}{Z_A}\, e^{-\beta H_A}$
- $S_A = -\mathrm{Tr}[\rho_A \ln \rho_A] = s(E)L^d$ Volume law

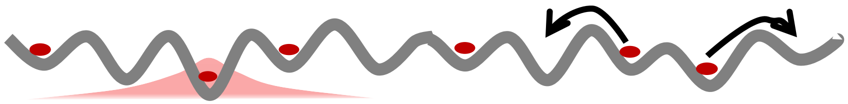


ETH fails for MBL states

Area law entanglement
$S_A \propto L^{d-1}$

Effective model of MBL

$$H = \sum_{\alpha} \epsilon_\alpha c_\alpha^\dagger c_\alpha + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$



$$c_\alpha^\dagger = \sum_i \psi_\alpha(i) c_i^\dagger$$

MBL fixed point → Emergent integrability

→ Complete set of quasi local integrals of motion (LIOMs)

Huse & Oganesyan (2013)
Serbyn et al. (2013)

$$\tilde{n}_\alpha = \tilde{c}_\alpha^\dagger \tilde{c}_\alpha$$

$$H_l = \sum_\alpha \epsilon_\alpha \tilde{n}_\alpha + \sum_{\alpha\beta} U_{\alpha\beta}\, \tilde{n}_\alpha \tilde{n}_\beta + \cdots \qquad U_{\alpha\beta} \sim e^{-|r_\alpha - r_\beta|/\xi}$$

$$\tilde{c}_\alpha^\dagger \simeq c_\alpha^\dagger + \sum_{\beta\gamma\delta} \frac{V_{\delta\gamma\beta\alpha}}{\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta} c_\delta^\dagger c_\gamma^\dagger c_\beta + \cdots$$

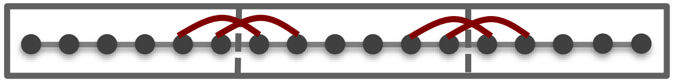
Basko, Aleiner, Altshuler (2005)

Effective Hamiltonian corresponding to real-space RG fixed point

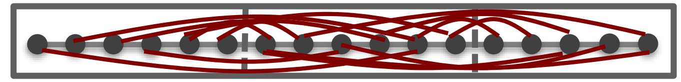
Dasgupta-Ma RG
Phys. Rev. B 22, 1305 (1980)

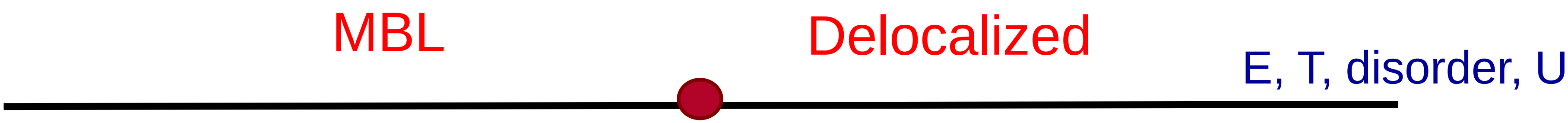
Vosk & Altman (2012,2014)

MBL

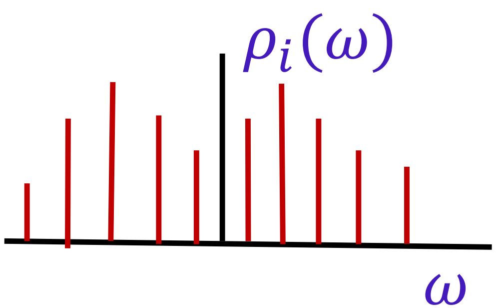


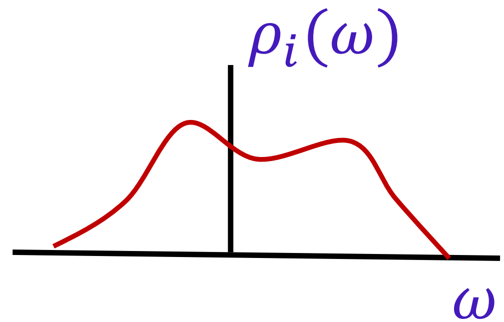
Dynamical transition





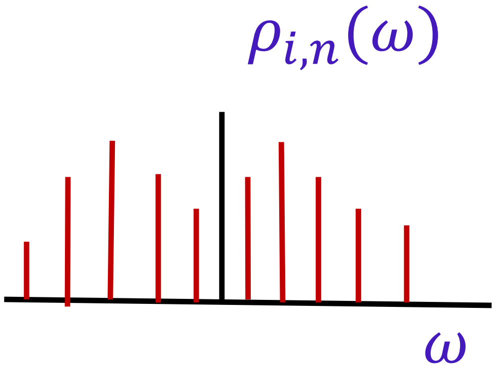
MBL	Delocalized
1. Memory of local initial Condition persists to $t \to \infty$	1. Memory of local initial condition is lost
2. Quantum dynamics	2. Classical hydrodynamics
3. Area law, $S_A \sim L^{d-1}$	3. Volume law, $S_A \sim L^{d}$
4. Discrete local spectra	4. Continuous local spectra





Spectral signatures of MBL

- Local spectral function of an eigenstate

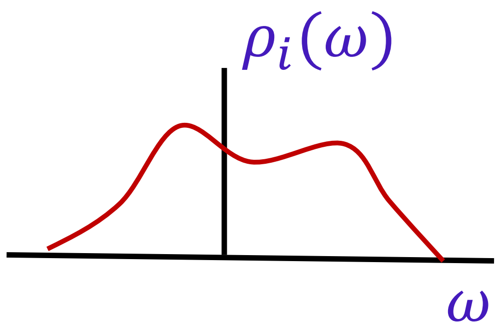


$\rho_{i,n}(\omega)$

← Single-particle Green's function

$$G^R_{i,n}(t) = -i\langle\Psi_n \left|\left\{c_i(t), c_i^\dagger(0)\right\}\right| \Psi_n\rangle\theta(t)$$

- Thermal spectral function



$$\rho_i^{th}(\omega) = \frac{1}{Z}\sum_n e^{-\beta E_n}\,\rho_{i,n}(\omega)$$

* Generically, $\rho_i^{th}(\omega)$ does not contain information about localization at $T \neq 0$

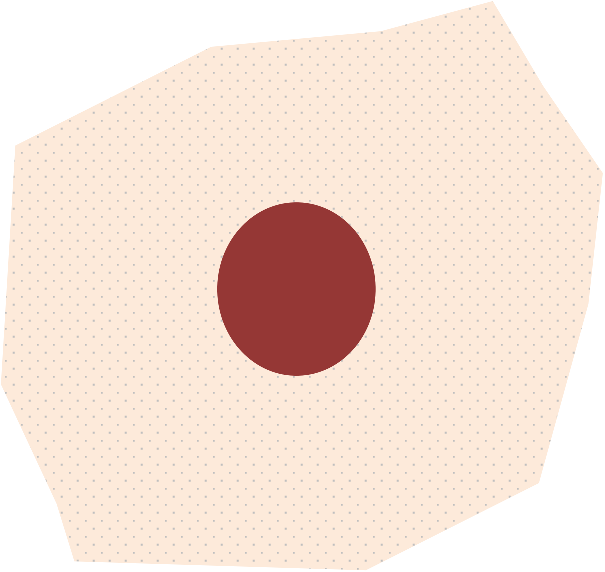
Infinite temperature ($\beta \to 0$), $\rho_i^{th}(\omega) = (1/D)\sum_n \rho_{i,n}(\omega)$

Hilbert space dimension D

- Delocalized state, $\rho_i^{th}(\omega) = \rho_{i,n}(\omega)$ for $T \to \langle E\rangle = E_n$

← ETH

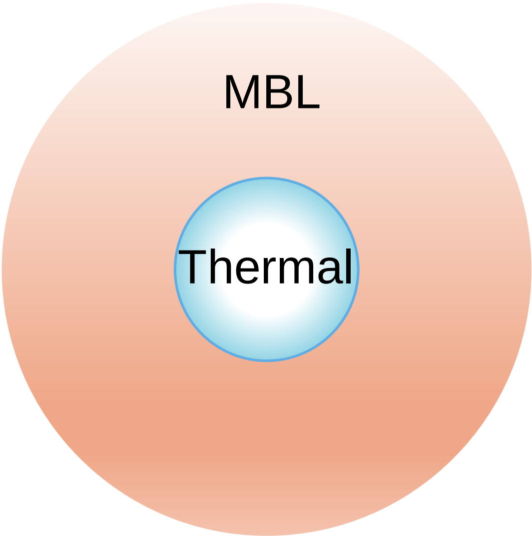
Is MBL stable in the presence of a thermal bubble?



No! for $d \geq 2$

De Roeck and Huveneers, PRB (2017)

Instability argument →



A finite ‘thermal bubble’ (disorder fluctuation) within a MBL

$$H = H_b + H_l + H_{bl}$$

- Ergodic bubble → RMT $H_b(\mathrm{N})$, fermions $c_i^\dagger$

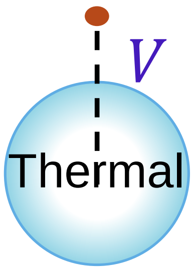
GOE random $2^N \times 2^N$ matrix
Level spacing $\delta_\epsilon \sim \mathcal{W}/2^N$

- MBL → $H_l = \sum_\alpha \epsilon_\alpha \tilde{n}_\alpha + \sum_{\alpha\beta} U_{\alpha\beta}\, \tilde{n}_\alpha \tilde{n}_\beta + \cdots$

$$\tilde{n}_\alpha = a_\alpha^\dagger a_\alpha \qquad a_\alpha^\dagger = \sum_i \psi_\alpha(i) a_i^\dagger + \cdots$$

- Coupling, $H_{bl} = \sum_{i\alpha}(V_{i\alpha}\, c_i^\dagger a_\alpha + h.c.)$ $\qquad V_{i\alpha} \sim V \exp(-r_\alpha/\xi)$

Can the single bubble destroy the entire MBL system?



- RMT Matrix element, ETH →

$$\langle \Psi_n | c_i^\dagger | \Psi_m \rangle \simeq \sqrt{\delta_\epsilon \, \rho(\omega)} \eta_{n,m} \qquad \omega = E_n - E_m$$

$\rho(\omega)$ → Single-particle spectral function of the bubble

The site is absorbed into the bubble if

$$\frac{V |\langle \Psi_\mathrm{n} | c_i^\dagger | \Psi_m \rangle|}{\delta_\epsilon} \gg 1$$

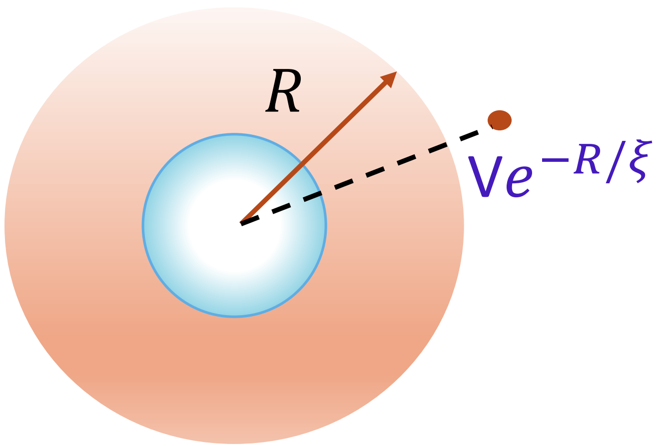
⇒ non-zero Fermi-Golden rule decay rate.

<u>Assumption:</u> The expanded bubble remains a ‘featureless’ RMT
→ the spectral function does not change

$$\rho(\omega) \to \tilde{\rho}(\omega) \qquad \delta_\epsilon \to \tilde{\delta}_\epsilon = \delta_\epsilon / 2$$

Thermal bubble grows! A better bath!

Instability of MBL in higher dimension



Level spacing $\delta_R \sim \delta_\epsilon \exp(-R^d)$

Matrix element

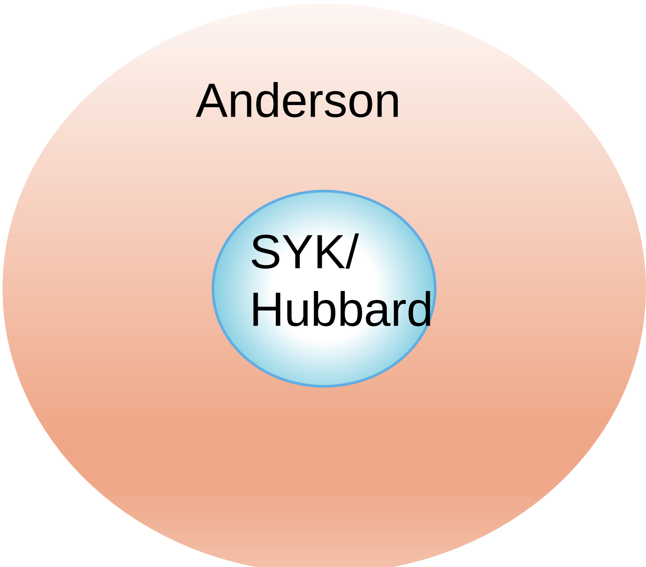
$$J_R \sim V \left| \langle \widetilde{\Psi}_b \left| c_i^\dagger \right| \widetilde{\Psi}_{b'} \rangle \right| \exp(-R/\xi)$$
$$\sim V \sqrt{\delta_\epsilon \rho(\omega)} \exp(-\mathrm{R}/\xi)$$

For $d > 1$, $\quad \frac{J_R}{\delta_R} \sim \exp\left(\frac{R^d}{2} - \frac{R}{\xi}\right) \gg 1$

→ No MBL in two and higher dimension ??!!!

Hard to prove or disprove, numerics in $d \geq 2$ is difficult

`Solvable' `Toy' models of the thermal bubble coupled to Anderson insulator



- Bubble - Sachdev-Ye-Kitaev (SYK) model with N sites

 --- Solvable in large-N

- Bubble - 'Large dimensional' Hubbard model

--- Solvable via DMFT

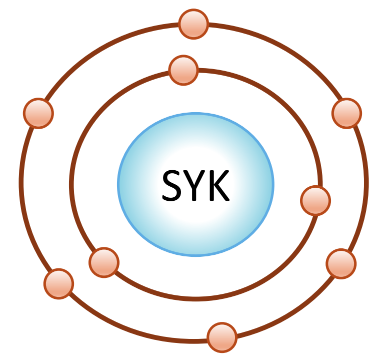
- Bubble – RMT

--- Exact diagonalization (ED) of small systems

- $\xi \gtrsim N^{\frac{1}{d}}$ Dynamical transition in spectral function
→ Instability argument breaks down
- $\xi < N^{\frac{1}{d}}$ ED → Instability argument breaks down

How does the spectral function change due to coupling to large number of localized sites in $d > 1$?

- Sachdev-Ye-Kitaev (SYK) model.
  -- Solvable model of thermalization.



- SYK model coupled to Anderson localized sites.

→(Large-N) controlled calculation of the spectral function.

$$\rho(\omega) \rightarrow \tilde{\rho}(\omega) \; ?$$

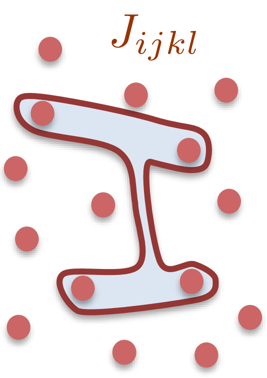
Transition in bath spectral function due to back reaction of the insulator.

Sachdev-Ye-Kitaev model

Sachdev & Ye, PRL (1993)
Kitaev, KITP (2015)
Sachdev, PRX (2015)

$$H_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l - \mu \sum_i^N c_i^\dagger c_i$$

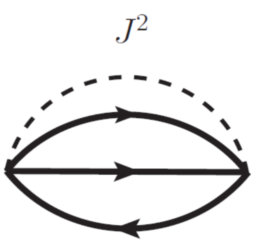
$$P(J_{ijkl}) \sim e^{-\frac{|J_{ijkl}|^2}{J^2}}$$



○ Self-consistency for Green's function ($N \to \infty$)

$$G^{-1}(\omega) = \omega + \mu - \Sigma_J(\omega)$$

$$\Sigma_J(\tau) = -J^2 G^2(\tau) G(-\tau)$$



→ Diverging DOS for $\omega \to 0$,

$$G(\omega) \sim 1/\sqrt{\omega}$$

Solvable model for thermalization and quantum chaos.

-- Lyapunov exponent, $\lambda_L = 2\pi T$

Maximally chaotic, like a black hole → model for holography

→ Use as a model for thermal bubble for large but finite N

Coupling to localized sites

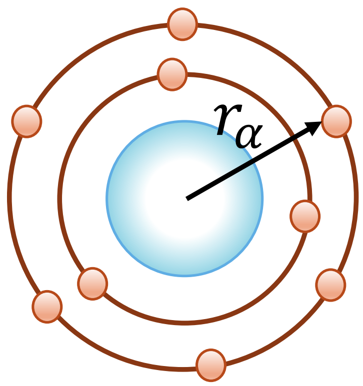
$$H_l = \sum_{\alpha} \epsilon_\alpha a_\alpha^\dagger a_\alpha$$

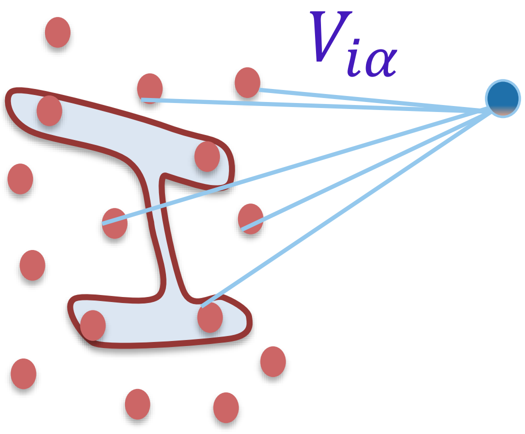
$\epsilon_\alpha \in [-W, W]$

Fixed $\{\epsilon_\alpha\}$,
no disorder averaging

$\alpha = 1, \dots, M$

$$H_{bl} = \frac{1}{\sqrt{N}} \sum_{i\alpha} (V_{i\alpha} c_i^\dagger a_\alpha + h.c.)$$





Random coupling

$$P(V_{i\alpha}) \sim \exp(-|V_{i\alpha}|^2/V_\alpha^2)$$

$$V_\alpha^2 = V^2 \exp(-r_\alpha/\xi)$$

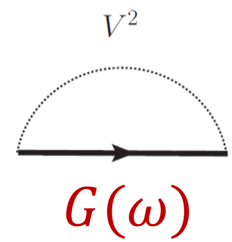
Self-consistency equations for Green's functions

- SYK model+ localized sites, large N

Disorder averaging over $J_{ijkl}, V_{i\alpha}$ →

$$G^{-1}(\omega) = \omega - \Sigma_J(\omega) - \frac{1}{N}\sum_{\alpha} V_\alpha^2\, \mathcal{G}_\alpha(\omega)$$

$$\mathcal{G}_\alpha^{-1}(\omega) = \omega + \epsilon_\alpha - V_\alpha^2 G(\omega)$$



Fixed $\{\epsilon_\alpha\}$ realization

→ Bubble spectral function $\rho(\omega) \sim -\mathrm{Im}G(\omega)$

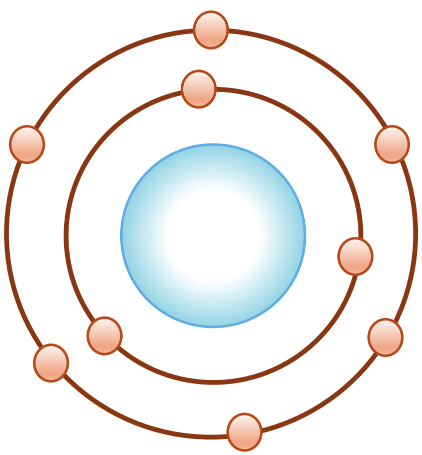
Coupling

V= 0.1

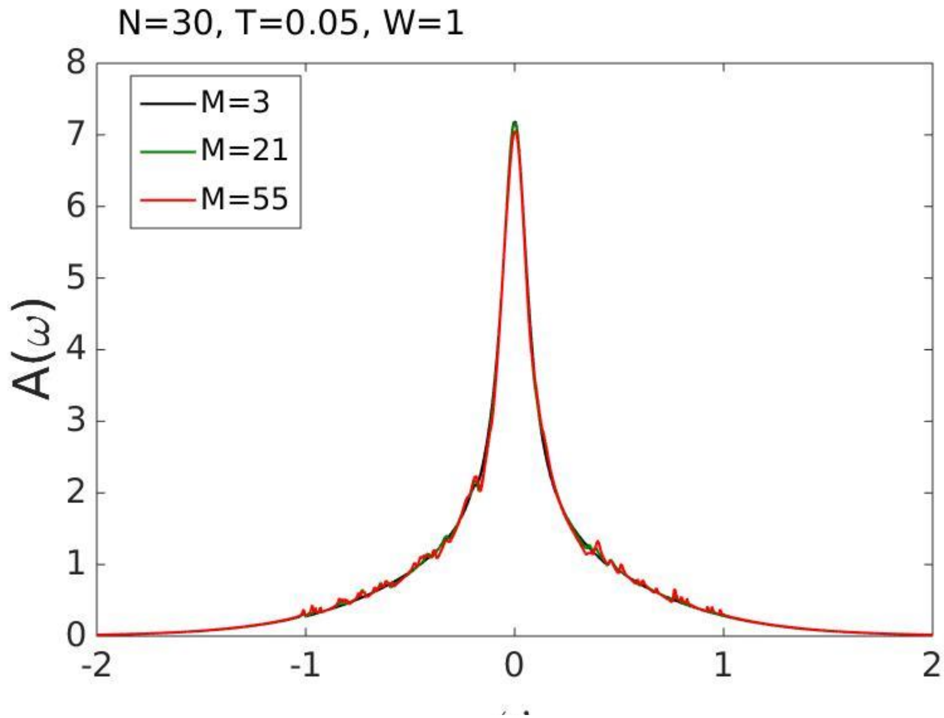
$\xi = 2$

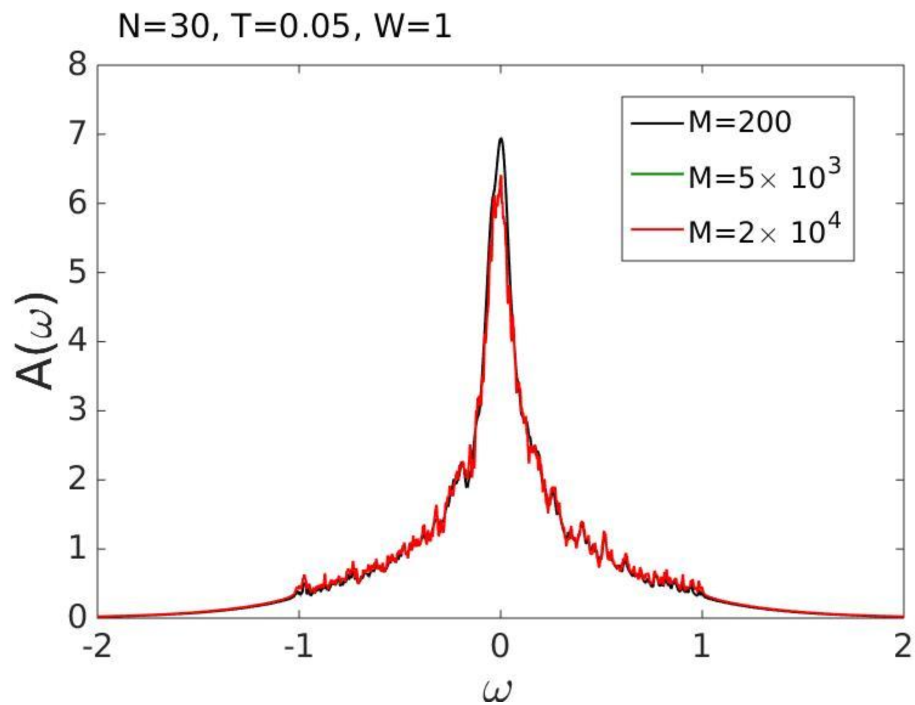
$N = 30$ bath sites

M localized sites



$$V(r) = Ve^{-\frac{r}{\xi}}$$

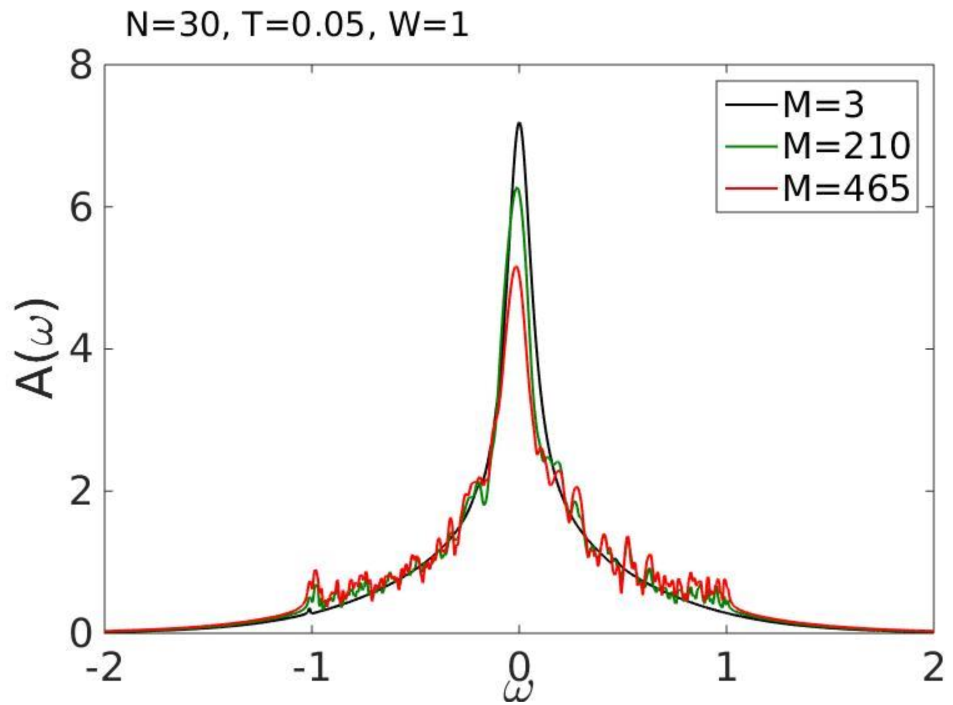




Coupling

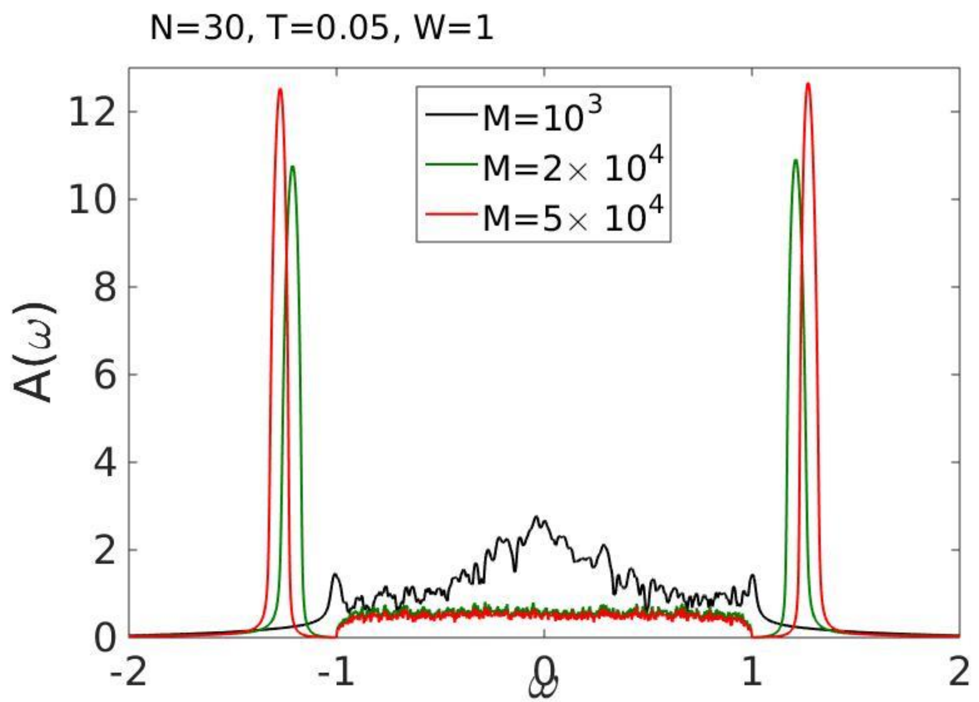
$V = 0.1$

$\xi = 10$



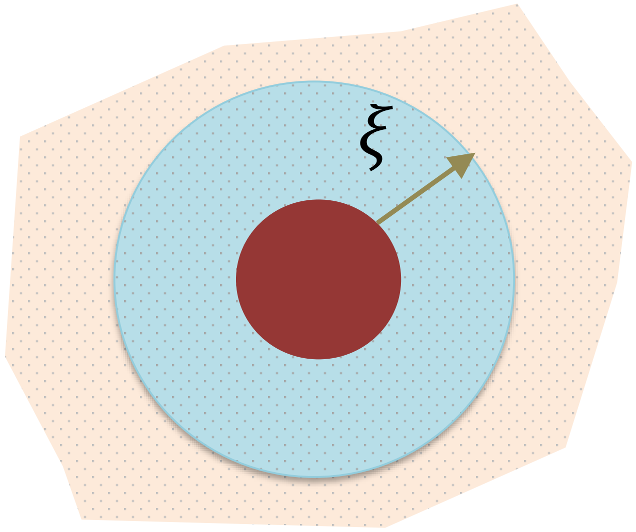
Thermal bubble is destroyed.

Localized ??!!



Dynamical transition in bubble spectrum

Only sites within the localization length ξ strongly affects the bath

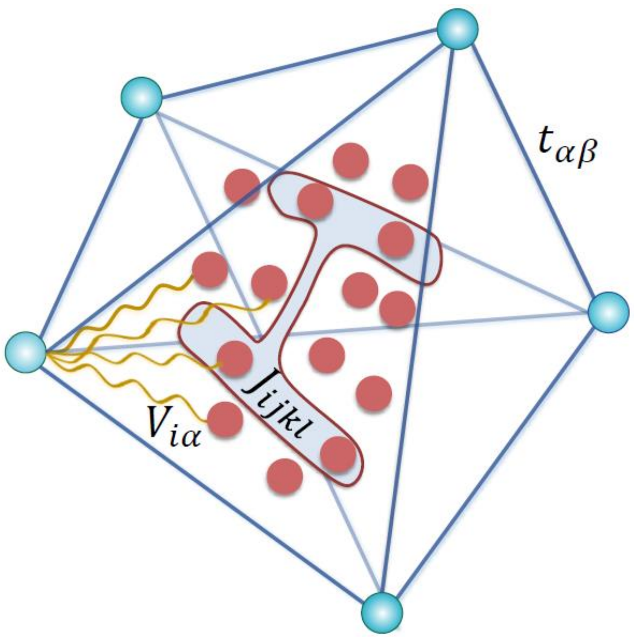


For $\xi \gg 1$,

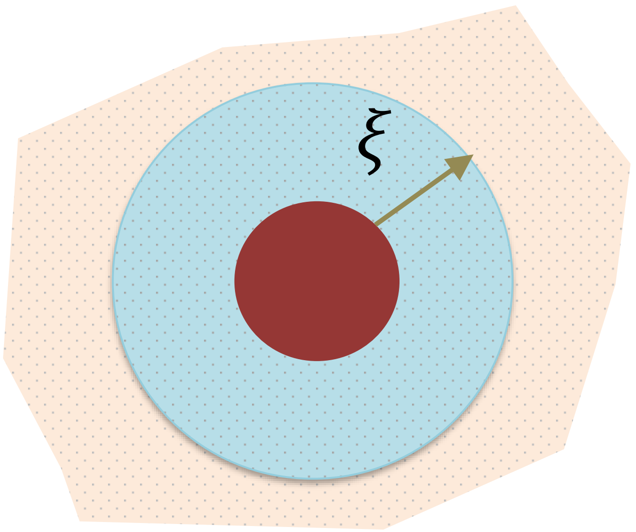
$\rightarrow M \approx \pi\xi^2$ sites "strongly" coupled with coupling strength V with N bath sites

N SYK sites

$M \simeq \pi\xi^2$ peripheral sites



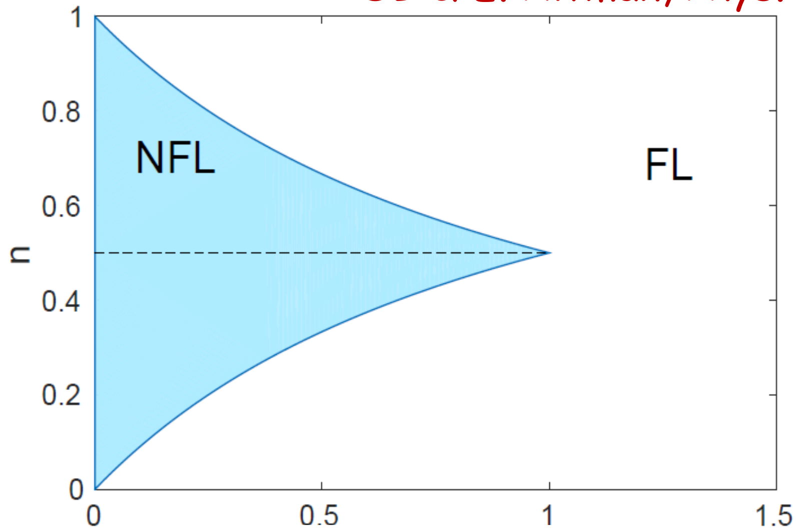
Ergodic bubble
in an Anderson insulator

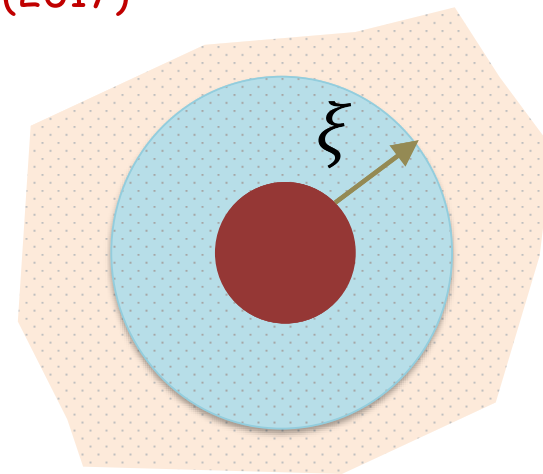


$$H = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{M}} \sum_{\alpha\beta} t_{\alpha\beta} a_\alpha^\dagger a_\beta$$

$$+ \frac{1}{(MN)^{1/4}} \sum_{i\alpha} (V_{i\alpha} c_i^\dagger a_\alpha + h.c.)$$

SB & E. Altman, Phys. Rev. B **95**, 134302 (2017)





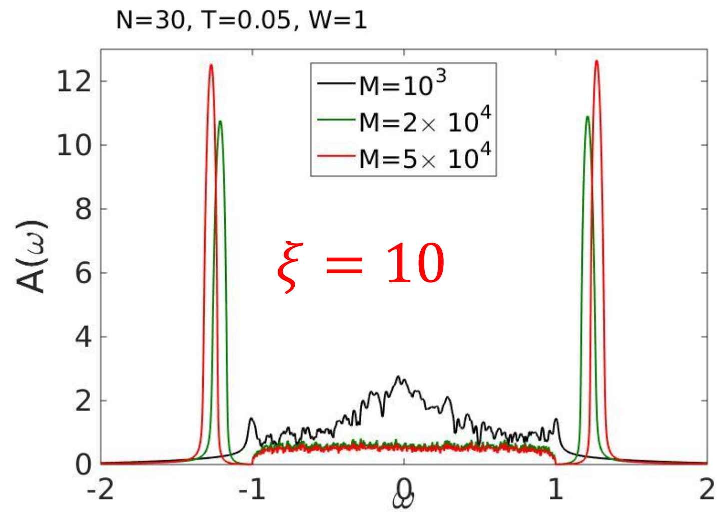
$$p = \frac{\pi \xi^2}{N}$$

$p > 1$ → Perturbative fixed point

- Fermi liquid state at infinite N.
- Finite N, quantum dot

→ Many body localization at low $T < T_c$

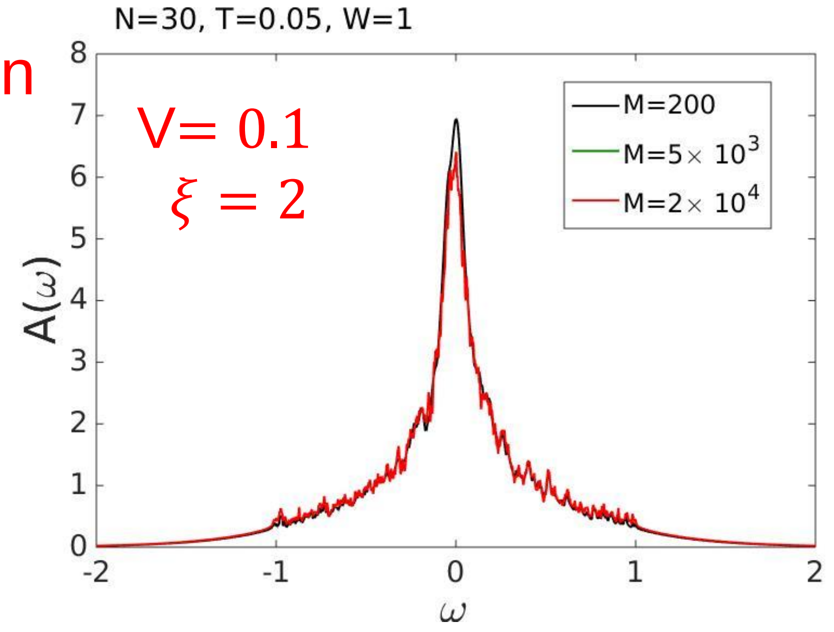
Altshuler, Gefen, Kamenev & Levitov, PRL (1997)



→ Ergodic bubble is destroyed for $\xi > N^{1/d}$

What happens at small localization length ξ ?

$$\frac{\pi\xi^2}{N} < 1$$



The assumption

$$\rho(\omega) = \tilde{\rho}(\omega)$$

seems to hold!

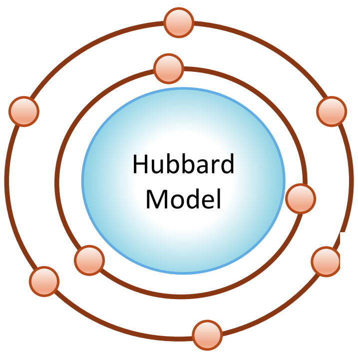
→ After adding $\sim N$ sites the bubble flows to the perturbative fixed point at low energies $\omega < \omega_N^*$.

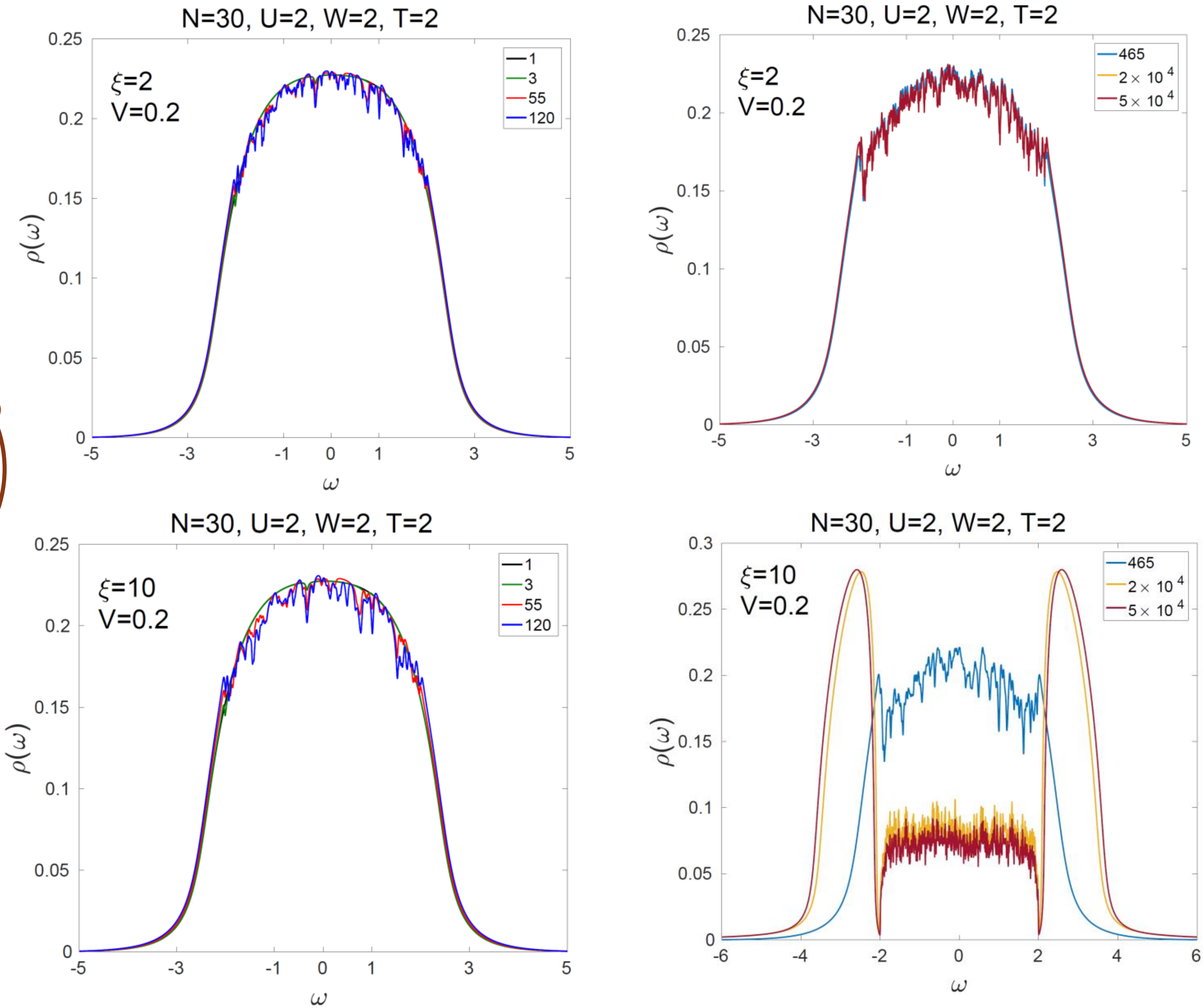
$$\omega_N^* \simeq (V^4/W^2 J)e^{-4\sqrt{N/\pi\xi^2}}$$

→ Non-perturbative effects of ergodic grain are not important?

→ Not an obstruction to MBL at low but finite temperature?

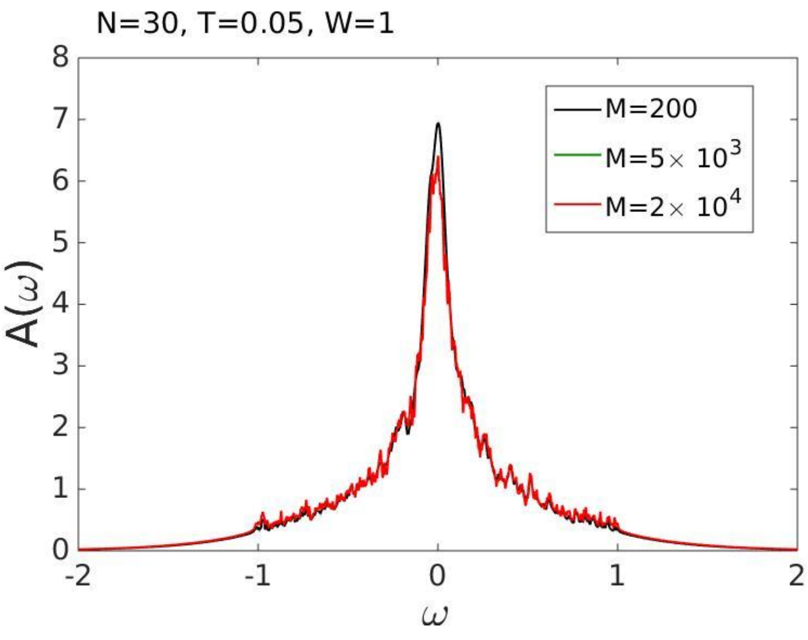
Ergodic grain:
Hubbard model with N sites





What happens at small localization length ξ ?

$$\frac{\pi\xi^2}{N} < 1$$



The assumption

$$\rho(\omega) = \tilde{\rho}(\omega)$$

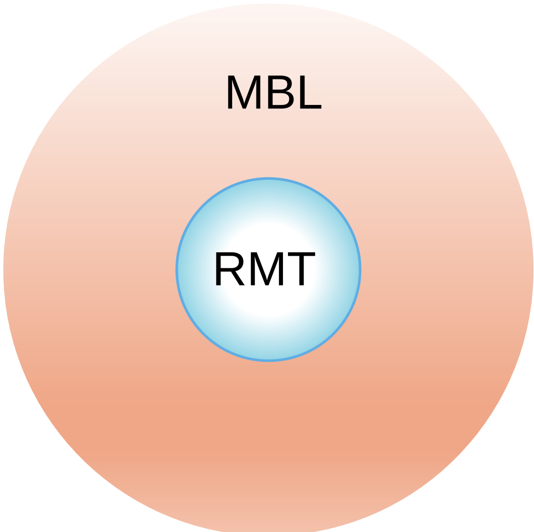
seems to hold!

Thermal spectral function is insufficient

Need to look into spectral properties of eigenstates.

Exact diagonalization

RMT coupled to Anderson insulator



$$H = H_b + H_l + H_{bl}$$

- RMT H_b(N) , N interacting spins $\boldsymbol{S}_i$
  GOE random $2^N \times 2^N$ matrix
  Level spacing $\delta_\epsilon \sim W/2^N$

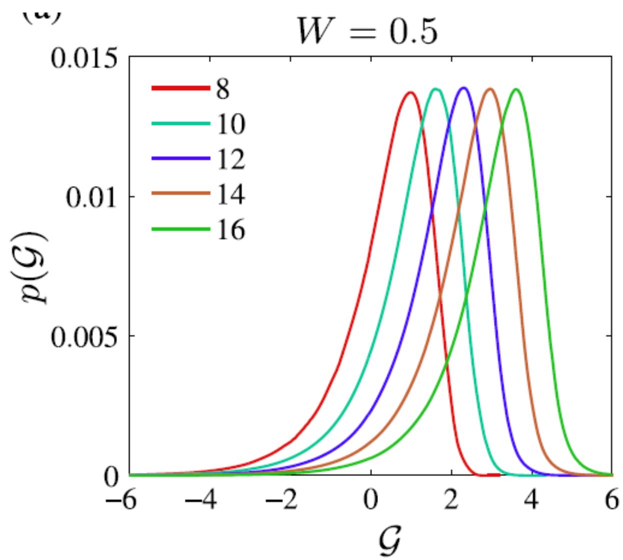
- $H_l = \sum_\alpha h_\alpha S_\alpha^z + \cdots$ $\qquad h_\alpha \in [-W, W]$

- Coupling, $H_{bl} = \sum_{i\alpha}(V_\alpha S_i^+ S_\alpha^- + h.c.)$ $\qquad V_\alpha \sim V \exp(-r_\alpha/\xi)$

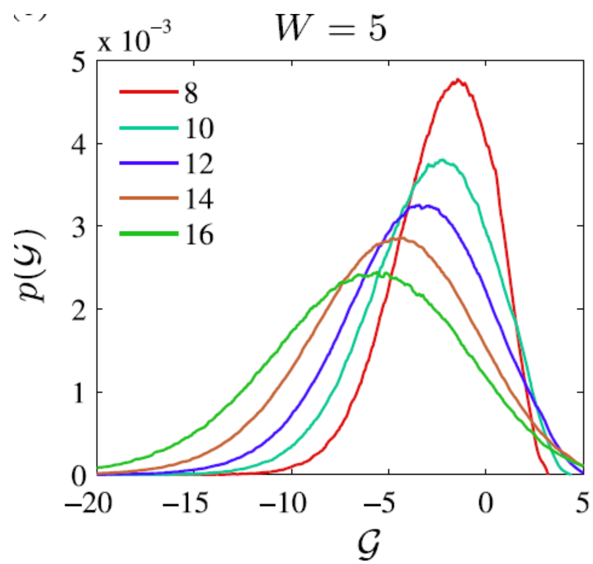
Many-body 'Thouless conductance', effect of local perturbation on eigenstates

$$\mathcal{G}(\epsilon, N) = \ln \frac{|V_{n,n+1}|}{E'_{n+1} - E'_n}$$

Serbyn et al. PRX (2015)



Delocalized

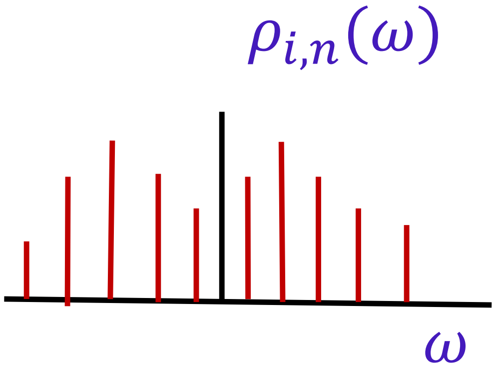


Localized

XXZ spin chain

Spectral signatures of MBL

- Local spectral function of an eigenstate



$\rho_{i,n}(\omega)$

← Single-particle Green's function

$G^R_{i,n}(t) = -i\langle\Psi_n\left|\{S_i^+(t), S_i^-(0)\}\right|\Psi_n\rangle\theta(t)$

- Thermal spectral function

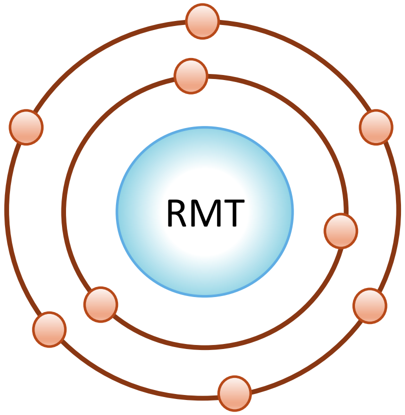
Infinite temperature ($\beta \to 0$), $\rho_i^{th}(\omega) = (1/D)\sum_n \rho_{i,n}(\omega)$

Hilbert space dimension D

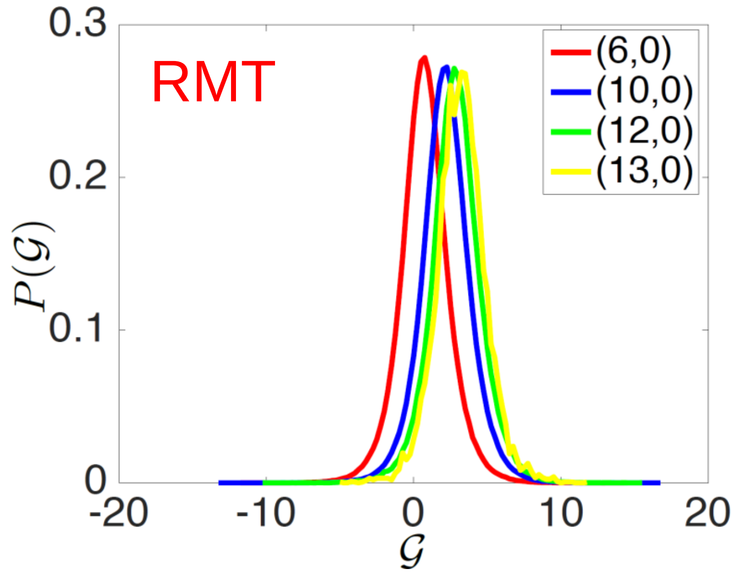
- Typical spectral function

$$\rho_i^{typ}(\omega) = \exp\left(\frac{1}{D}\sum_n \ln \rho_{i,n}(\omega)\right)$$

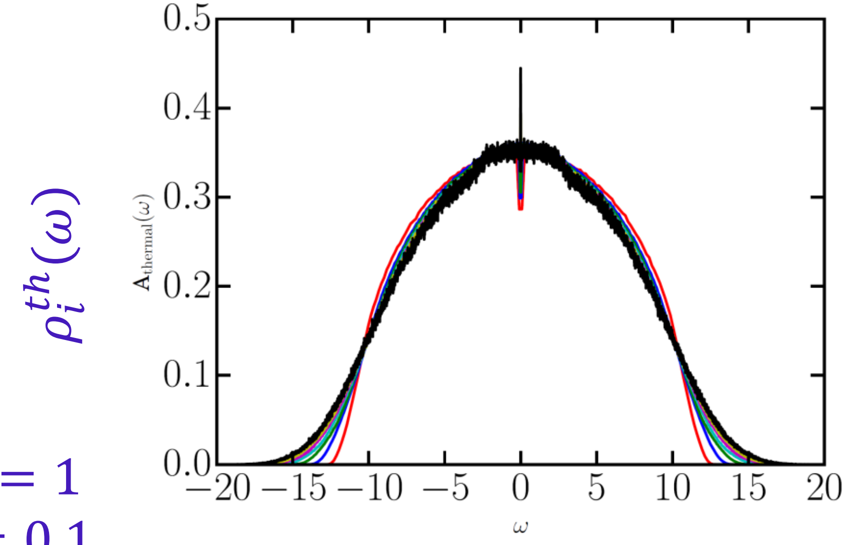
$\to 0$ In the MBL state



Thouless conductance

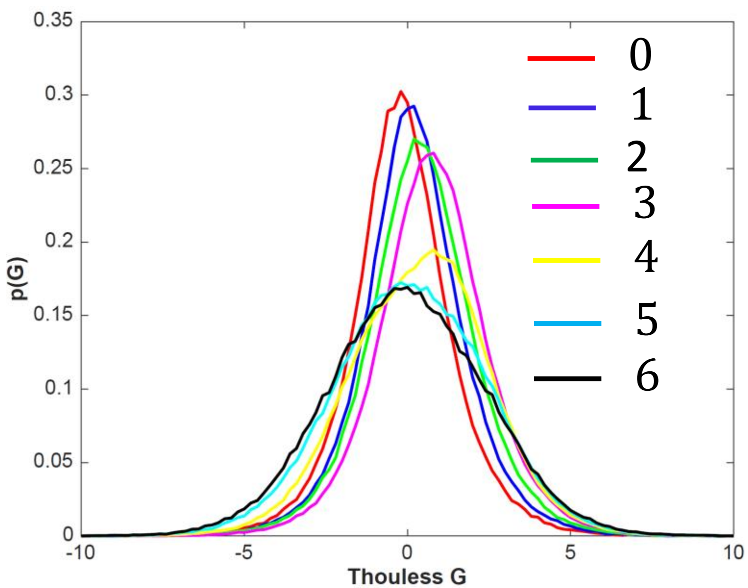


Thermal spectral function

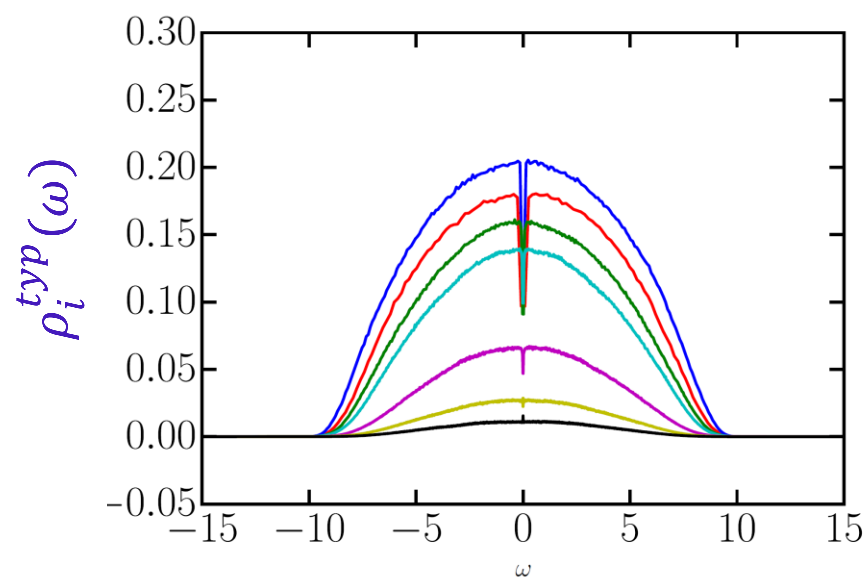


$W = 1$
$V = 0.1$
$\xi = 0.4$

RMT (N=8) + M(=1,…, 6) spins

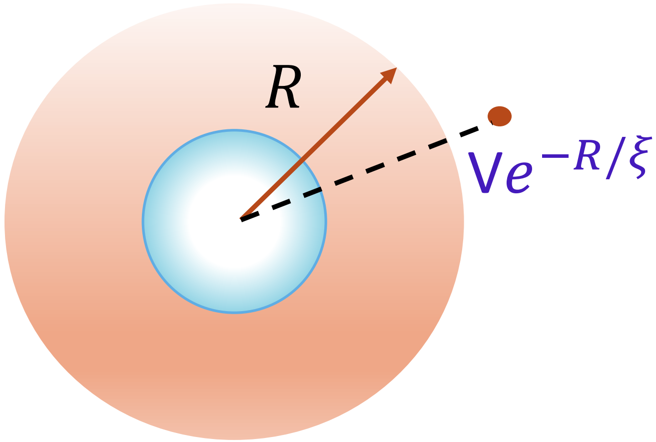


Typical spectral function



Bubble is localized due to back reaction of the insulator

How does the instability argument fail?



Level spacing $\delta_R \sim \delta_\epsilon \exp(-R^d)$

Matrix element

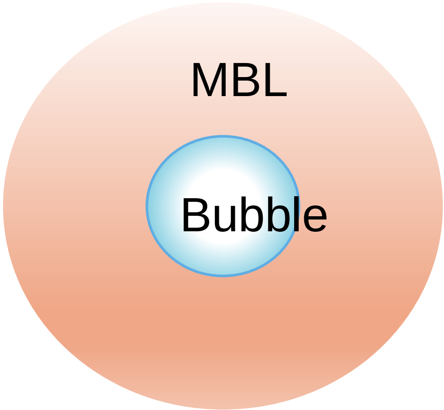
$$J_R \sim V \left| \langle \widetilde{\Psi}_b \left| c_i^\dagger \right| \widetilde{\Psi}_{b'} \rangle \right| \exp(-R/\xi)$$
$$\sim V\sqrt{\delta_\epsilon \rho(\omega)} \exp(-\mathrm{R}/\xi)$$

$$\frac{J_R}{\delta_R} \sim \frac{V}{\sqrt{\delta_\epsilon}} \exp\left(\frac{R^d}{2} - \frac{R}{\xi}\right) \exp(\frac{1}{2}\ln\rho(\omega))$$

$$\left(\frac{J_R}{\delta_R}\right)_{typ} \sim \frac{V}{\sqrt{\delta_\epsilon}} \exp\left(\frac{R^d}{2} - \frac{R}{\xi}\right) \sqrt{\rho_{typ}(\omega)} \to 0$$

When enough number of Anderson sites are coupled to the bubble.

Conclusions



- Arguments for instability of localized state in the presence of thermal bubble in $d > 1$

- Solvable toy models → controlled calculation of spectral function. SYK or Hubbard model coupled to Anderson localized sites.

→ Bubble gets destroyed via a dynamical transition for large localization length, $\xi > N^{1/d}$.

- For small localization length, $\xi < N^{1/d}$, instability argument fails due to vanishing of typical spectral function due to back reaction of the insulator.

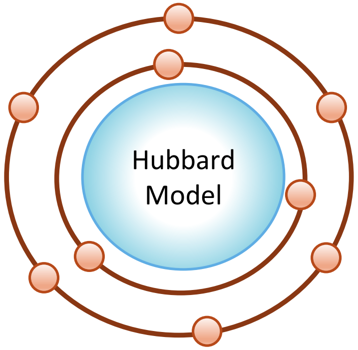
Thermal bubble is not an obstruction of MBL in $d > 1$.

Thank you!

Hubbard model bubble

'Large dimensional' Hubbard model with N sites as bath.

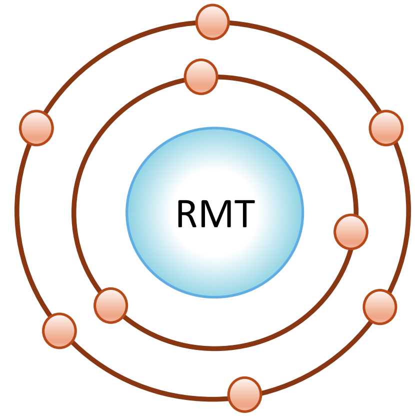
$$H_b = \frac{1}{\sqrt{N}} \sum_{ij\sigma} t_{ij,\sigma} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

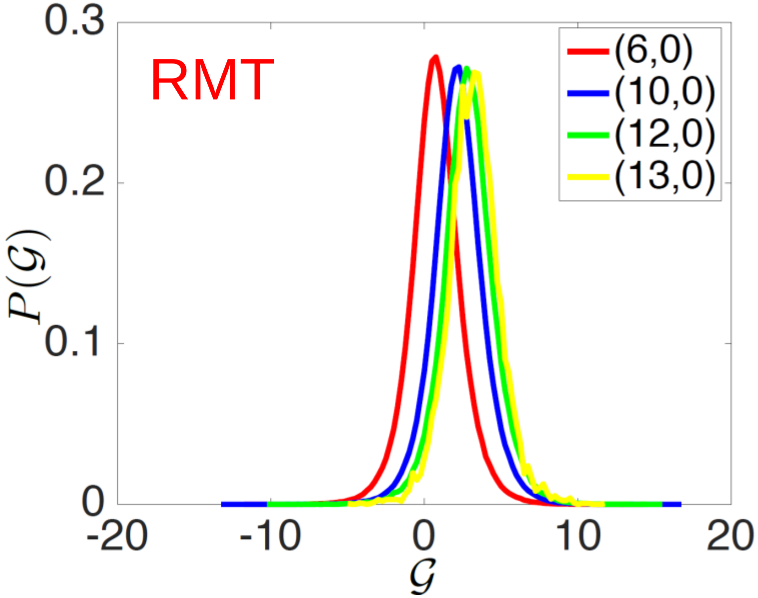


Non-interacting part – single-particle GOE random matrix

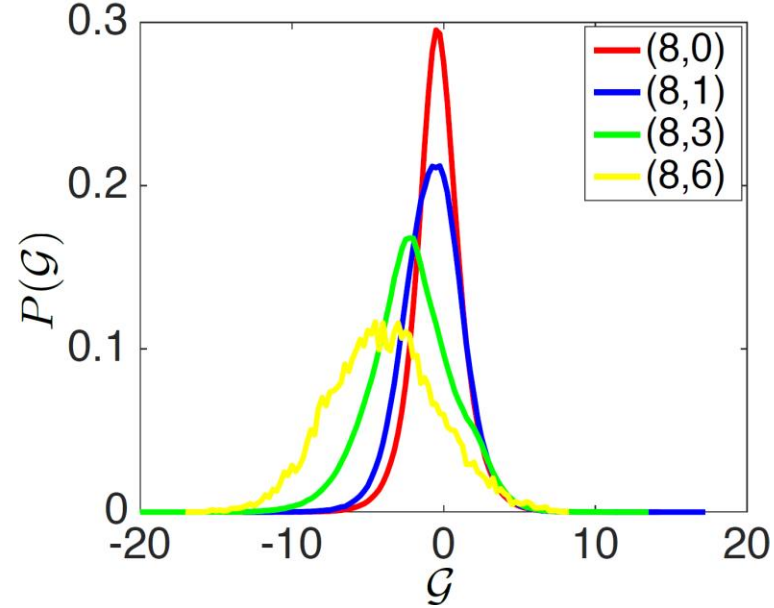
Solved via single-site dynamical mean field theory (DMFT)

Use iterative perturbation theory (IPT) as impurity solver





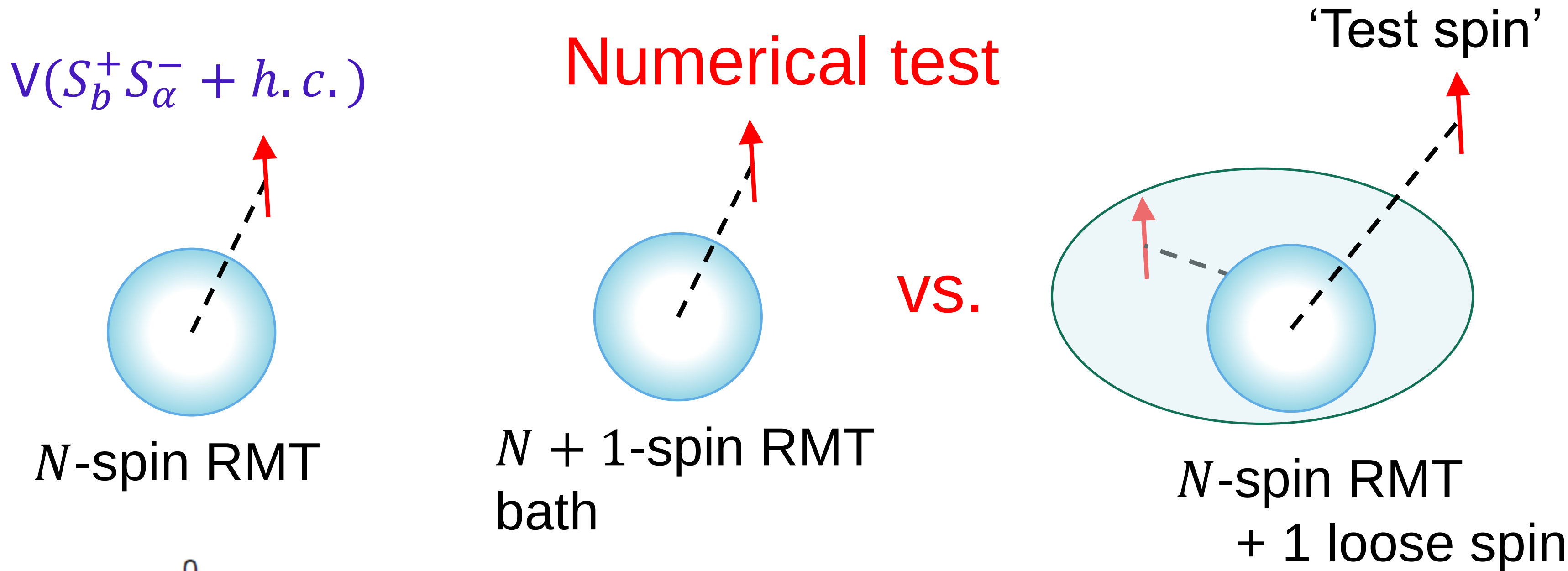
RMT + M spin

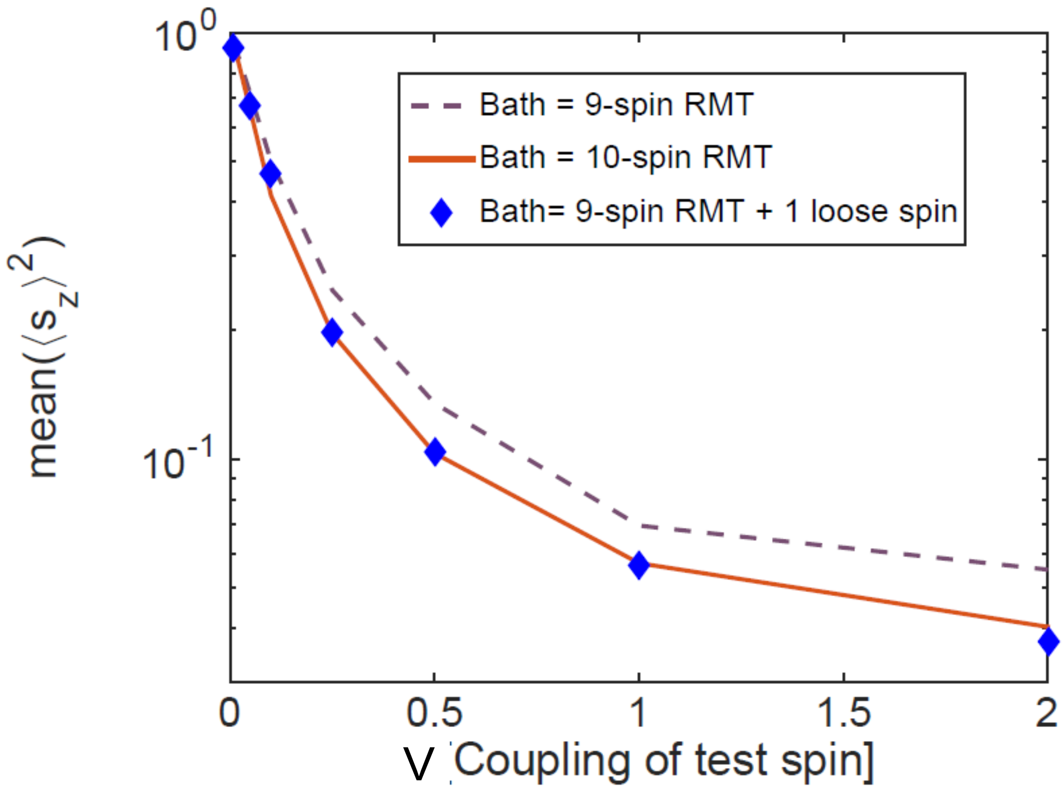


$$\frac{J_R}{\delta_R} \sim \exp\left(\frac{\pi R^2}{2} - \frac{R}{\xi}\right)$$

$$\rightarrow \pi R^2 = M > M_{min} \sim 1/\xi^2$$

Numerical test





Is the test spin thermalized ?

$\langle S^z \rangle \to 0$