Parsimonius Model Estimation in the presence of correlated variables

ICTS Colloquium

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12th Mar, 2018

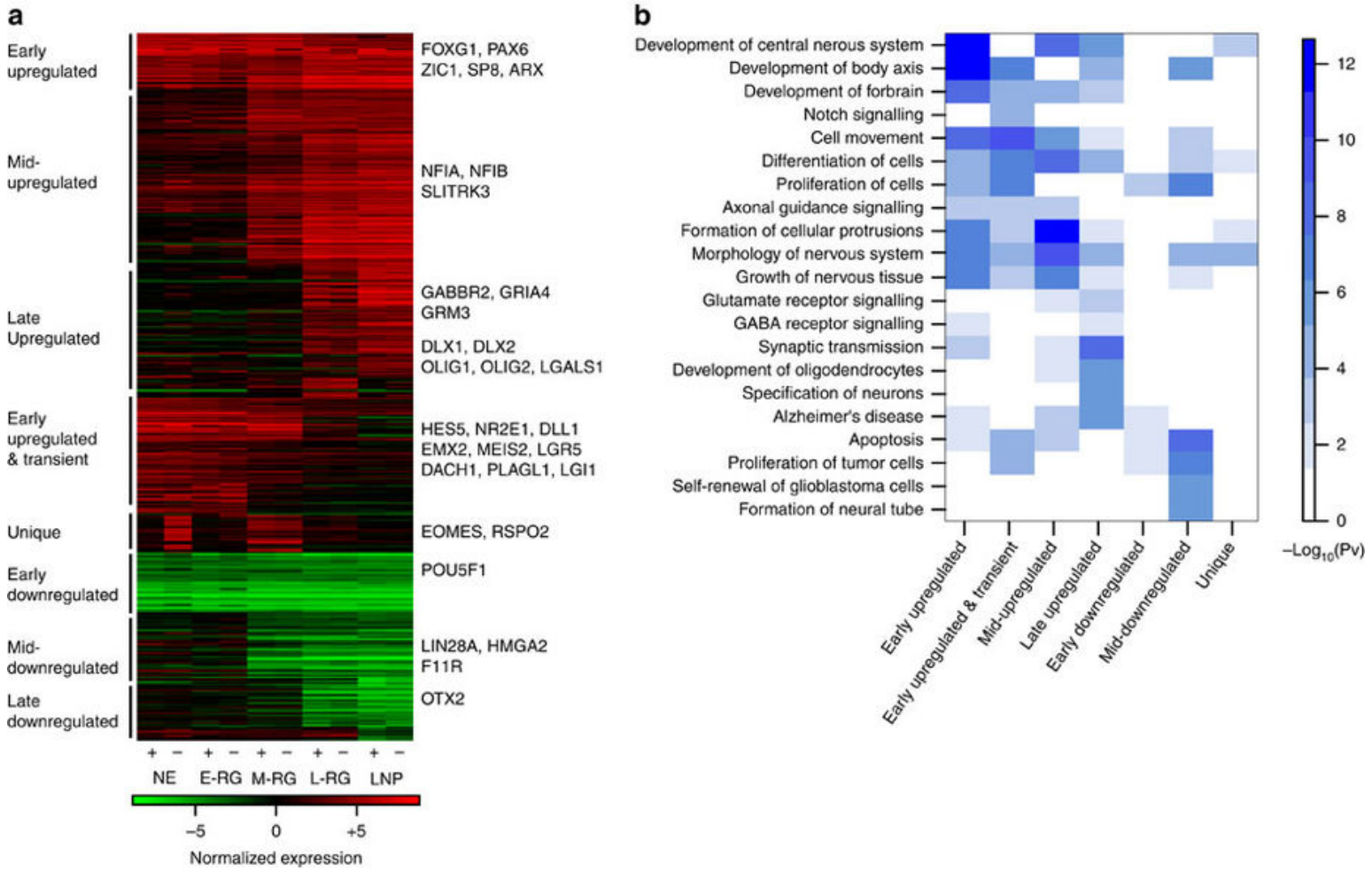
Identifying groups of strongly correlated variables through Smoothed Ordered Weighted L_1-norms

Joint Work

R. Sankaran and *Francis Bach*

AISTATS '17

Genomics


a
Early upregulated
FOXG1, PAX6
ZIC1, SP8, ARX
Mid-upregulated
NFIA, NFIB
SLITRK3
Late Upregulated
GABBR2, GRIA4
GRM3
DLX1, DLX2
OLIG1, OLIG2, LGALS1
Early upregulated & transient
HES5, NR2E1, DLL1
EMX2, MEIS2, LGR5
DACH1, PLAGL1, LGI1
Unique
EOMES, RSPO2
Early downregulated
POU5F1
Mid-downregulated
LIN28A, HMGA2
F11R
Late downregulated
OTX2
+ − + − + − + − + −
NE E-RG M-RG L-RG LNP
−5 0 +5
Normalized expression
b
Development of central nerous system
Development of body axis
Development of forbrain
Notch signalling
Cell movement
Differentiation of cells
Proliferation of cells
Axonal guidance signalling
Formation of cellular protrusions
Morphology of nervous system
Growth of nervous tissue
Glutamate receptor signalling
GABA receptor signalling
Synaptic transmission
Development of oligodendrocytes
Specification of neurons
Alzheimer's disease
Apoptosis
Proliferation of tumor cells
Self-renewal of glioblastoma cells
Formation of neural tube
Early upregulated
Early upregulated & transient
Mid-upregulated
Late upregulated
Early downregulated
Mid-downregulated
Unique
12
10
8
6
4
2
0
−Log10(Pv)

Text Analytics

Elite '13 Wing Y.
303 San Carlos, CA
129

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6/16/2013 3 photos 1 check-in here

Jen found this place on Yelp a few weeks ago, and we'd been meaning to go, but it closes at 630pm on Saturdays. So yesterday our plan was to make a quick stop before having early dinner. It was a great plan, but then we got there and they had spumoni and ice cream-filled cannolis. At that point there was no way we were leaving without spoiling our appetites, so we had a little pre-dinner dessert (in addition to taking 4 cannolis to go). The service was super-friendly.

As a huge fan of Mike's Pastry in Boston, I might have to say that I prefer this one. They fill the cannolis when you order them, so the shell is really fresh. I think the cheese filling is a bit lighter, but it's been over 5 years since I had a cannoli from Mike's. I need to arrange a side-by-side comparison somehow. Another bonus for Romolo's is that you don't have to wait in a huge line. We'll definitely be back.

In any case, we'd never been down 37th Street before, and probably never would have gone down that way, despite the fact that I pass by nearly every day during my commute.

chocolate chip, brickle, and pistachio

chocolatey cannoli shells

ice cream cannoli

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Linear Regression in High Dimensions

Setting

- Need to estimate a response variable
- Large number of features
- Most of them irrelevant

Principle of Parsimony

Use as few variables as possible to achieve good prediction

LASSO: The method of choice for feature selection

Ordered Weighted L1(OWL) norm

Submodular penalties

$\Omega_{\mathcal{S}}$: Smoothed OWL (SOWL)

LASSO: The method of choice for feature selection

Linear Regression in High dimension

Question?

Let $\mathbf{x} \in \mathbb{R}^d, y \in \mathbb{R}$ and

$$y = w^{*\top}\mathbf{x} + \epsilon \;\; \epsilon \sim N(0, \sigma^2)$$

From $D = \{(\mathbf{x}^i, y_i) | i \in [n], \mathbf{x}_i \in \mathbb{R}^d, y_i \in \mathbb{R}\}$ can we find w^*

$$X = \begin{bmatrix} \mathbf{x}^{1\top} \\ \mathbf{x}^{2\top} \\ \vdots \\ \mathbf{x}^{n\top} \end{bmatrix}, Y \in \mathbb{R}^n$$

Linear Regression in High dimension

Least Squares Linear Regression

$X \in \mathbb{R}^{n \times d}, y \in \mathbb{R}^n$:

$$w_{LS} = \text{argmin}_{w \in \mathbb{R}^d} \frac{1}{2} \|Xw - Y\|_2^2$$

Assumptions

- Labels centered : $\sum_{j=1}^n y_j = 0$.
- Features normalized : $x_i \in \mathbb{R}^d, \|x_i\|_2 = 1, x_i^\top 1_d = 0$.

Linear Regression in High dimension

- **Least squares estimate:**
  $w_{LS} = \left(X^\top X\right)^{-1} X^\top Y$ and $E(w_{LS}) = w^*$
- **Variance of Predictive error:**
  $\frac{1}{n} E(\|X(w_{LS} - w^*)\|^2) = \sigma^2 \frac{d}{n}$

Linear Regression in High dimension

$$rank(X) = d$$

- w_{LS} is unique
- Poor predictive performance, d is close to n

Linear Regression in High dimension

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$rank(X) < d$

- $d > n$
- w_{LS} is not unique.
- Difficult in model recovery. Use prior information available.
- Is it possible to reduce the model variance ? better scalings of n against d possible ?

Linear Regression in High Dimensions

$rank(X) < d$

- There are infinite solutions
- There exists at-least coordinate j such that

$$\hat{w}_j > 0, \text{ and } \tilde{w}_j < 0$$

where $\hat{w}, \tilde{w}$ are two least squares solution

Parsimony

Best subset Regression $X \in \mathbb{R}^{n \times d}, y \in \mathbb{R}^d, \Omega : \mathbb{R}^d \to \mathbb{R}$:

$$\min_{w \in \mathbb{R}^d} \frac{1}{2} \|Xw - Y\|_2^2 \quad \text{s.t. } \|w\|_0 \leq k$$

$$\|w\|_0 = \{i | w_i \neq 0, i \in [d]\}$$

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$$\|w\|_0 = \{i | w_i \neq 0, i \in [d]\}$$

NP hard!!!

Parsimony

- Suppose the true support, S, was known in advance

$$Y = Xw^* + \epsilon = X_S w_S^* + \epsilon$$

$$w_{S^c}^* = 0$$

- $w_S^{oracle} = \left(X_S^\top X_S\right)^{-1} X_S^\top Y$
- $\frac{1}{n} E(\|X(w^{oracle} - w^*)\|^2) = E(\|X_S \left(X_S^\top X_S\right)^{-1} X_S^\top \epsilon\|) = \sigma^2 \frac{k}{n}$
- **Best K-subset Regression**

$$\frac{1}{n} E(\|X(\hat{w} - w^*)\|^2) \leq (4 logd)\, \sigma^2 \frac{k}{n}$$

Regularized Linear Regression

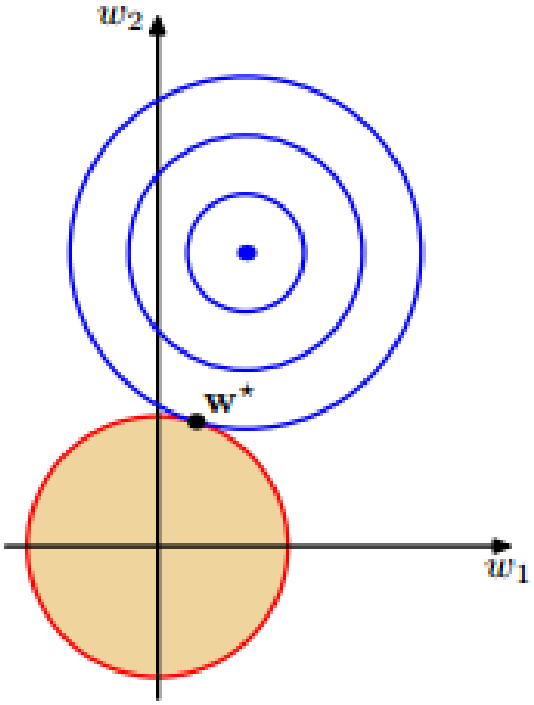
Regularized Linear Regression $X \in \mathbb{R}^{n \times d}, y \in \mathbb{R}^d, \Omega : \mathbb{R}^d \to \mathbb{R}$:

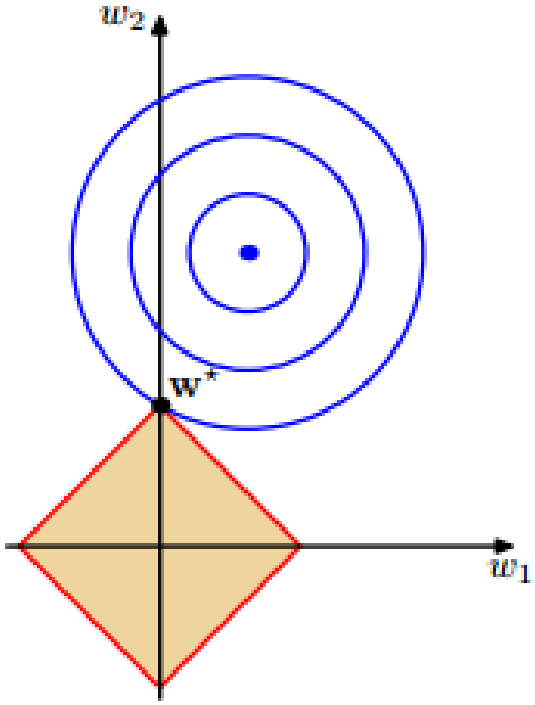
$$\min_{w \in \mathbb{R}^d} \frac{1}{2} \|Xw - Y\|_2^2 \quad \text{s.t. } \Omega(w) \leq t$$

Regularizer: $\Omega(w)$

- non-negative
- Convex function, typically a norm.
- Possibly non-differentiable.

Regularized regression





Ridge: $\Omega(w) = \|w\|_2^2$

- Does not promote sparsity
- Closed form solution

Lasso: $\Omega(w) = \|w\|_1$

- Encourages sparse solutions.
- Solve convex optimization problem

Figures from [Tibshirani, 1994]

Regularized Linear Regression

Regularized Linear Regression $X \in \mathbb{R}^{n \times d}, y \in \mathbb{R}^d, \Omega : \mathbb{R}^d \to \mathbb{R}$:

$$\min_{w \in \mathbb{R}^d} \frac{1}{2} \|Xw - Y\|_2^2 + \lambda \Omega(w)$$

- Equivalent to the constraint version
- Unconstrained

Lasso: Properties at a glance

Computational

- Proximal methods : IST, FISTA, Chambolle-Pock [Chambolle and Pock, 2011, Beck and Teboulle, 2009].
- Convergence rate : $O(1/T^2)$ in T iterations.
- Assumption: availability of *proximal operator* of Ω (Easy for ℓ_1).

Lasso Model Recovery[Wainwright, 2009, Theorem 1]

Setup

$$y = Xw^* + \epsilon, \epsilon_i \sim \mathcal{N}(0, \sigma^2), X \in \mathbb{R}^{n \times d}$$

Support of w^* be $S = \{j | w_j^* \neq 0 \; j \in [d]\}$

Lasso

$$\min_{w \in \mathbb{R}^d} \frac{1}{2} \|Xw - Y\|_2^2 + \lambda \|w\|_1$$

Lasso Model Recovery[Wainwright, 2009, Theorem 1]

Conditions[1].

$$\left\| X_{S^c}^\top X_S \left(X_S^\top X_S \right)^{-1} \right\|_\infty \leq 1 - \gamma, \text{Incoherence with } \gamma \in (0, 1],$$

$$\Lambda_{\min} \left(X_S^\top X_S \right) \geq C_{\min}$$

$$\lambda > \lambda_0 \triangleq \frac{2}{\gamma} \sqrt{\frac{2\sigma^2 \log d}{n}}$$

[1] Define $\|M\|_\infty = \max_i \sum_j |M_{ij}|$

Model Recovery

Under the Assumptions:

- **Support Set Recovery**

$$Support(\hat{w}) \subseteq S$$

- **Model Recovery**
  W.h.p, the following holds:

$$\boxed{\|\hat{w}_S - w_S^*\|_\infty \leq \lambda \left(\left\| \left(X_S^\top X_S \right)^{-1} \right\|_\infty + 4\sigma/\sqrt{C_{\min}} \right)}$$

- if $min_{i \in S} |w_i^*| \geq \lambda \left(\left\| \left(X_S^\top X_S \right)^{-1} \right\|_\infty + 4\sigma/\sqrt{C_{\min}} \right)$

$$sign(w^*) = sign(\hat{w})$$

Lasso Model Recovery: Special cases

Case: $X_{S^c}^\top X_S = 0$.

- The incoherence condition trivially holds, and $\gamma = 1$.
- The threshold λ_0 is lesser $\Rightarrow$ The recovery error is lesser.

Case: $X_S^\top X_S = I$.

- $C_{\min} = 1/n$, the largest possible for a given n.
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When does Lasso work well?

- Lasso prefers low correlation between support and non-support columns.
- Low correlation of columns within support lead to better recovery.

Lasso Model Recovery: Implications

Setting: Strongly correlated columns in X.

- Correlation between feature i and feature j

$$\rho_{ij} \approx x_i^\top x_j$$

- Large correlation between X_S and $X_{S^c} \Rightarrow \gamma$ is small.
- Large correlation within $X_S \Rightarrow C_{\min}$ is small.

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- Hence w.h.p., lasso fails in **model recovery**.

In other words:

- Lasso solutions differ with the solver used.
- Solution is not unique typically.
- The prediction error may not be as worse though [Hebiri and Lederer, 2013].

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Requirements.

- Need consistent estimates independent of the solver.
- Preferably select all the correlated variables as a group.

Illustration: Lasso under correlation[Zeng and Figueiredo, 2015]

Setting: strongly correlated features.

- $\{1, \ldots, d\} = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_k$, $\mathcal{G}_m \cap \mathcal{G}_l = \emptyset, \forall l \neq m$
- $\rho_{ij} \equiv |x_i^\top x_j|$ very high (≈ 1) for pairs $i, j \in \mathcal{G}_m$.

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Toy Example

- $d = 40$, $k = 4$.
- $\mathcal{G}_1 = [1:10]$, $\mathcal{G}_2 = [11:20]$, $\mathcal{G}_3 = [21:30]$, $\mathcal{G}_4 = [31:40]$.

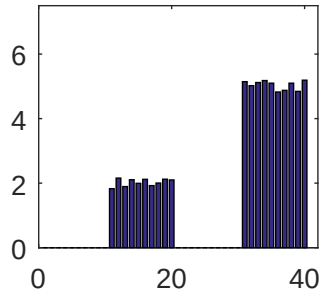

Figure: Original signal

Lasso: $\Omega(w) = \|w\|_1$.

- Sparse recovery.
- Arbitrarily selects the variables within a group.

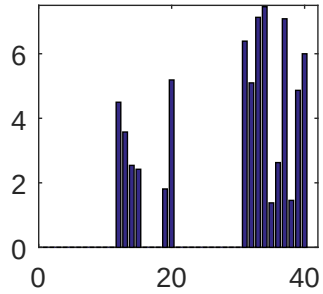

Figure: Recovered signal.

Possible solutions: 2-stage procedures

Cluster Group Lasso [Buhlmann et al., 2013]

- Identify strongly correlated groups $\mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_k\}$
  - Canonical Correlation.
- Group selection. $\Omega(w) = \sum_{j=1}^{k} \alpha_j \|w_{\mathcal{G}_j}\|_2$.
- Select all or no variables from each group.

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- Select all or no variables from each group.

Goal: Learn $\mathcal{G}$ and w simultaneously ?

Ordered Weighted L1(OWL) norm

Ordered Weighted ℓ_1 (OWL) norms

OSCAR[2] [Bondell and Reich, 2008]

$$\Omega_{\mathcal{O}}(w) = \sum_{i=1}^{d} c_i |w|_{(i)}$$

$$c_i = c_0 + (d - i)\mu,$$

$$c_0, \mu, c_d > 0.$$

[2] **Notation:** $|w|_{(i)}$: i^{th} largest in $|w|$.

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OWL [Figueiredo and Nowak, 2016]:

- $c_1 \geq \cdots \geq c_d \geq 0.$

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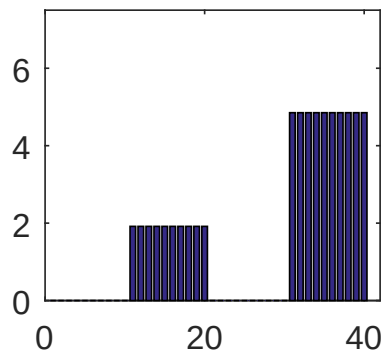

Figure: Recovered: OWL

OWL [Figueiredo and Nowak, 2016]:

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Oscar: Sparsity Illustrated

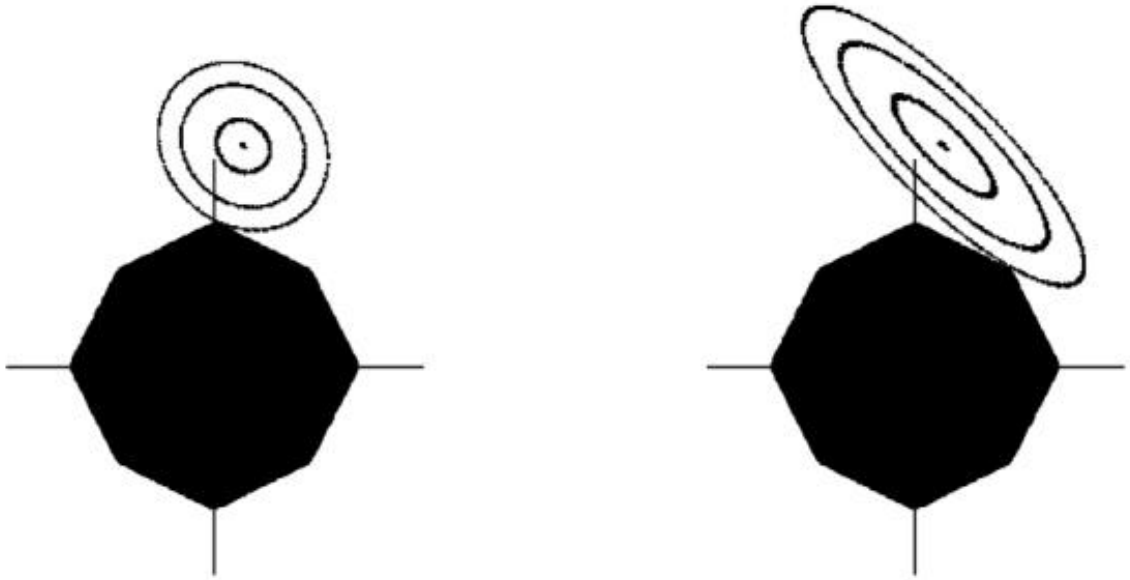

Figure: Examples of solutions:[Bondell and Reich, 2008]

- Solutions encouraged towards vertices.
- Encourages blockwise constant solutions.
- See [Bondell and Reich, 2008].

OWL-Properties

Grouping covariates [Bondell and Reich, 2008, Theorem 1], [Figueiredo and Nowak, 2016, Theorem 1]

$$|w_i| = |w_j|, \text{if } \lambda \geq \lambda_{ij}^0.$$

OWL-Properties

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$$|w_i| = |w_j|, \text{if } \lambda \geq \lambda_{ij}^0.$$

- $\lambda_{ij}^0 \propto \sqrt{1 - \rho_{ij}^2}$.
- $\lambda_{ij}^0 \propto \frac{1}{\sqrt{C}}, \quad C = min_i(c_{i+1} - c_i)$
- Strongly correlated pairs grouped early in the regularization path.
- Groups: $\mathcal{G}_j = \{i \mid |w_i| = \alpha_j\}$.

OWL-Issues

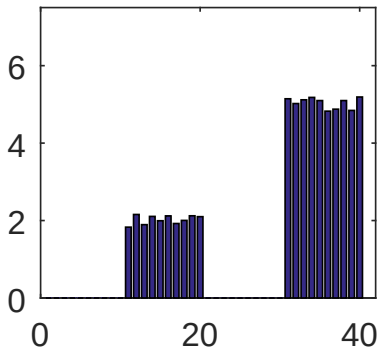

Figure: True model

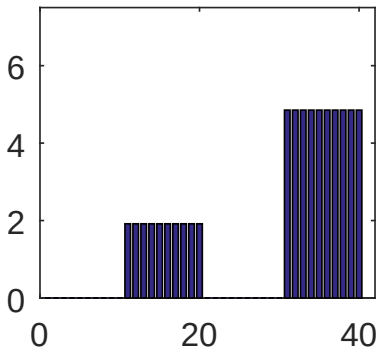

Figure: Recovered: OWL

- **Bias for piecewise constant $\hat{w}$**
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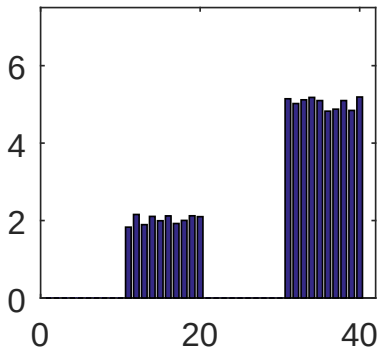

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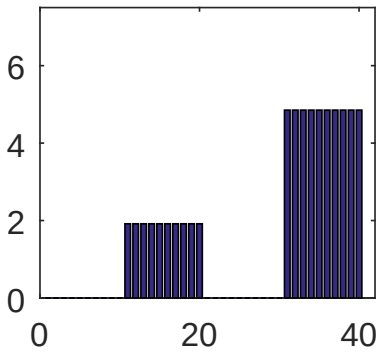

Figure: Recovered: OWL

- **Bias for piecewise constant $\hat{w}$**
  - Easily understood through the norm balls.
  - Requires more samples to consistent estimation.
- **Lack of interpretations for choosing c**

Preliminaries: Penalties on the Support

Goal

Encourage w to have desired support structure.

$$\text{supp}(w) \quad = \quad \{i | w_i \quad \neq \quad 0\}$$

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Penalty on support [Obozinski and Bach, 2012]:

$$\text{pen}(w) = F(\text{supp}(w)) + \|w\|_p^p, p \in [1, \infty].$$

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Relaxation(pen(w)): $\Omega_p^F(w)$

Tightest positively homogenous, convex lower bound.

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Example: $F(\text{supp}(w)) = |\text{supp}(w)|$

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Relaxation(pen(w)): $\Omega_p^F(w)$

Tightest positively homogenous, convex lower bound.

Example: $F(\text{supp}(w)) = |\text{supp}(w)| \Rightarrow \Omega_p^F(w) = \|w\|_1$. **Familiar!**

Message: The cardinality function always relaxes to the ℓ_1 norm.

Nondecreasing Submodular Penalties on Cardinality

Assumptions. Denote $F \in \mathcal{F}$, if $F : A \subseteq \{1, \ldots, d\} \to \mathbb{R}$ is:

1. **Submodular** [Bach, 2011].
   - $\forall A \subseteq B, F(A \cup \{k\}) - F(A) \geq F(B \cup \{k\}) - F(B)$.
   - Lovász extension: $f : \mathbb{R}^d \to \mathbb{R}$. (Convex extension of F to $\mathbb{R}^d$).
2. **Cardinality based**.
   - $F(A) = g(|A|)$ (Invariant to permutations).
3. **Non Decreasing**.
   - $g(0) = 0, g(x) \geq g(x-1)$.

Implication: $F \in \mathcal{F} \Rightarrow F$ completely specified through g.

Example: Let $V = \{1, \ldots, d\}$, define $F(A) = |A||V \setminus A|$.

Then $f(w) = \sum_{i<j} |w_i - w_j|$.

Ω_∞^F and Lovász extension

Result: Case $p = \infty$ [Bach, 2010]:

$$\boxed{\Omega_\infty^F(w) = f(|w|)}$$

- The ℓ_∞ relaxation coincides with the Lovász extension in the positive orthant.
- To work with Ω_∞^F, may use existing results of submodular function minimization.
- Ω_p^F not known in closed form for $p < \infty$.

Equivalence of OWL and Lovász extensions: Statement

Proposition [Sankaran et al., 2017]: $F \in \mathcal{F}, \Omega_\infty^F(w) \Leftrightarrow \Omega_\mathcal{O}(w)$

1. Given $F(A) = f(|A|)$, $\Omega_\infty^F(w) = \Omega_\mathcal{O}(w)$, with $c_i = f(i) - f(i-1)$.
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  - Explains piecewise constant solutions of OWL.
- Motivates $\Omega_p^F(w)$ for $p < \infty$.

$\Omega_{\mathcal{S}}$: Smoothed OWL (SOWL)

SOWL: Definition

Smoothed OWL

$$\Omega_{\mathcal{S}}(w) := \Omega_2^F(w).$$

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Variational form for Ω_2^F [Obozinski and Bach, 2012].

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Use OWL equivalance: $f(|\eta|) = \Omega_\infty^F(\eta) = \sum_{i=1}^d c_i |\eta|_{(i)}$,

$$\Omega_{\mathcal{S}}(w) = \min_{\eta \in \mathbb{R}_+^d} \underbrace{\frac{1}{2} \sum_{i=1}^{d} \left(\frac{w_i^2}{\eta_i} + c_i \eta_{(i)} \right)}_{\Psi(w,\eta)}. \qquad \text{(SOWL)}$$

OWL vs SOWL

Case: $c = 1_d$.

- $\Omega_{\mathcal{S}}(w) = \|w\|_1$.
- $\Omega_{\mathcal{O}}(w) = \|w\|_1$.

Case: $c = [1, \underbrace{0, \ldots, 0}_{d-1}]^\top$.

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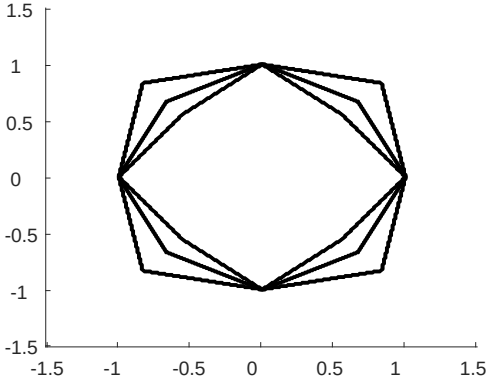
Norm Balls



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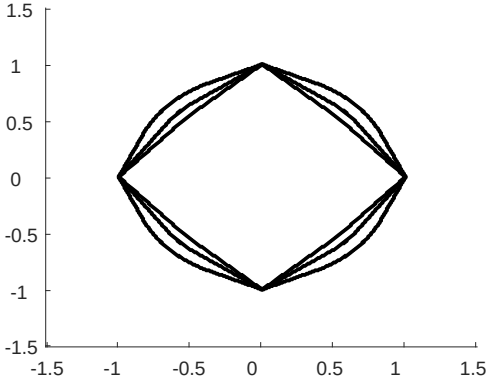

: SOWL

Figure: Norm balls for OWL, SOWL, for different values of c

Group Lasso and $\Omega_{\mathcal{S}}$

SOWL objective (eliminating η):

$$\Omega_{\mathcal{S}}(w) = \sum_{j=1}^{k} \left(\|w_{\mathcal{G}_j}\| \sqrt{\sum_{i \in \mathcal{G}_j} c_i} \right).$$

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Key differences:

- Groups defined through $\eta_w = [\underbrace{\delta_1, \ldots, \delta_1}_{\mathcal{G}_1}, \ldots, \underbrace{\delta_k, \ldots, \delta_k}_{\mathcal{G}_k}]$.
- Influenced by the choice of c.

Open Questions:

1. Does $\Omega_{\mathcal{S}}$ promotes grouping of correlated variables as $\Omega_{\mathcal{O}}$?
   - Are there any benefits over $\Omega_{\mathcal{O}}$?
2. Is using $\Omega_{\mathcal{S}}$ computationally feasible ?
3. Theoretical properties of $\Omega_{\mathcal{S}}$ vs Group Lasso?

Grouping property $\Omega_{\mathcal{S}}$: Statement

Learning Problem: LS-SOWL

$$\min_{w \in \mathbb{R}^d, \eta \in \mathbb{R}^d_+} \underbrace{\frac{1}{2n}\|Xw - y\|_2^2 + \frac{\lambda}{2}\sum_{i=1}^{d}\left(\frac{w_i^2}{\eta_i} + c_i \eta_{(i)}\right)}_{\Gamma^{(\lambda)}(w,\eta)}.$$

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Theorem: [Sankaran et al., 2017]

Define the following:

- $\left(\hat{w}^{(\lambda)}, \hat{\eta}^{(\lambda)}\right) = \text{argmin}_{w,\eta}\Gamma^{(\lambda)}(w, \eta)$.
- $\rho_{ij} = x_i^\top x_j$.
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There exists $0 \leq \lambda^0 \leq \frac{\|y\|_2}{\sqrt{\tilde{c}}}(4 - 4\rho_{ij}^2)^{\frac{1}{4}}$, such that $\forall \lambda > \lambda^0$, $\hat{\eta}_i^{(\lambda)} = \hat{\eta}_j^{(\lambda)}$

Grouping property $\Omega_{\mathcal{S}}$: Interpretation

1. Variables η_i, η_j grouped if $\rho_{ij} \approx 1$ (Even for small λ).
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   - Similar to Figueiredo and Nowak [2016, Theorem 1], which is for absolute values of w.
2. SOWL differentiates grouping variable η and model variable w.
3. Allows model variance within group.
   - $\hat{w}_i^{(\lambda)} \neq \hat{w}_j^{(\lambda)}$ as long as c has distinct values.

Illustration: Group Discovery using SOWL

Aim: Illustrate group discovery of SOWL.

- Consider $z \in \mathbb{R}^d$, Compute $\text{prox}_{\lambda\Omega_{\mathcal{S}}}(z)$.
- Study the regularization path.

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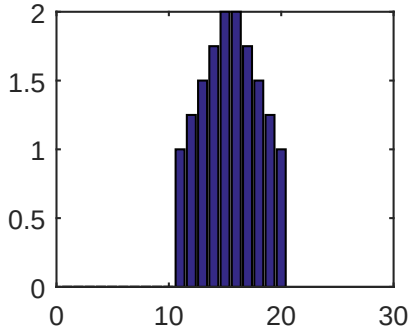

Figure: Original signal

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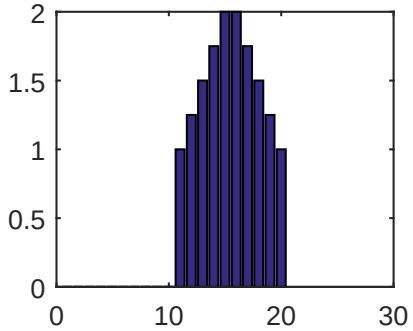

Figure: Original signal

Question: How do we identify grouping in w ?

SOWL Group Discovery

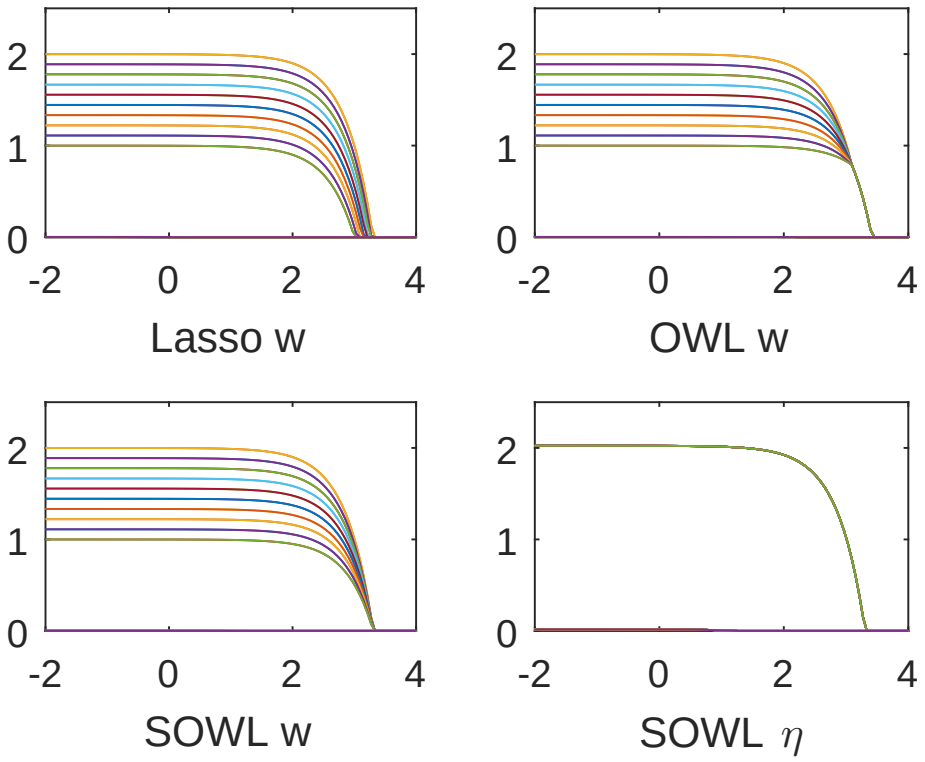

Figure: x-axis : λ (log scale), y-axis: $\hat{w}$.

- Lasso does not have grouping.
- OWL does grouping, but only later in the path.
- SOWL identifies groups through η very early (bottom left).
- SOWL w is not piecewise constant.

Proximal Methods: A brief overview

Proximal operator

$$\text{prox}_{\Omega}(z) = \text{argmin}_w \frac{1}{2}\|w - z\|_2^2 + \Omega(w)$$

- Easy to evaluate for many simple norms.
- $\text{prox}_{\lambda\ell_1}(z) = sign(z)\left(|z| - \lambda\right)_+$.
- Generalization of Projected Gradient Descent

FISTA [Beck and Teboulle, 2009]

Initialization

- $t^{(1)} = 1,\ \tilde{w}^{(1)} = x^{(1)} = 0.$

Steps : $k > 1$

- $w^{(k)} = \mathrm{prox}_{\Omega}\left(\tilde{w}^{(k-1)} - \frac{1}{L}\nabla f(\tilde{w}^{(k-1)})\right).$
- $t^{(k)} = \left(1 + \sqrt{1 + 4\left(t^{(k-1)}\right)^2}\right)/2.$
- $\tilde{w}^{(k)} = w^{(k)} + \left(\frac{t^{(k-1)}-1}{t^{(k)}}\right)\left(w^{(k)} - w^{(k-1)}\right).$

Guarantee

- Convergence rate $O(1/T^2)$.
- No additional assumptions than IST.
- Known to be optimal for this class of minimization problems.

Computing $\text{prox}_{\Omega_{\mathcal{S}}}$

Problem:

$$\text{prox}_{\lambda\Omega}(z) = \text{argmin}_w \frac{1}{2}\|w - z\|_2^2 + \lambda\Omega(w).$$

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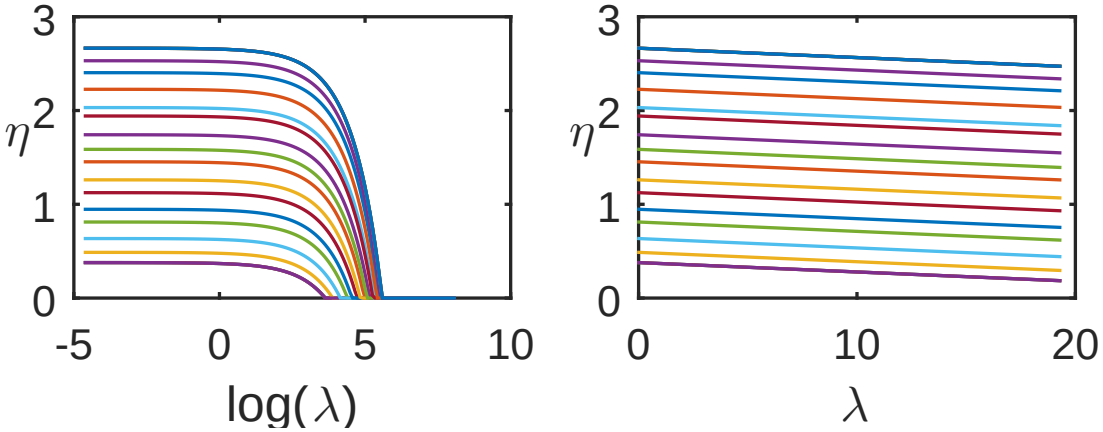
Key idea: Ordering of $\eta_w^{(\lambda)}$ remains same for all λ.



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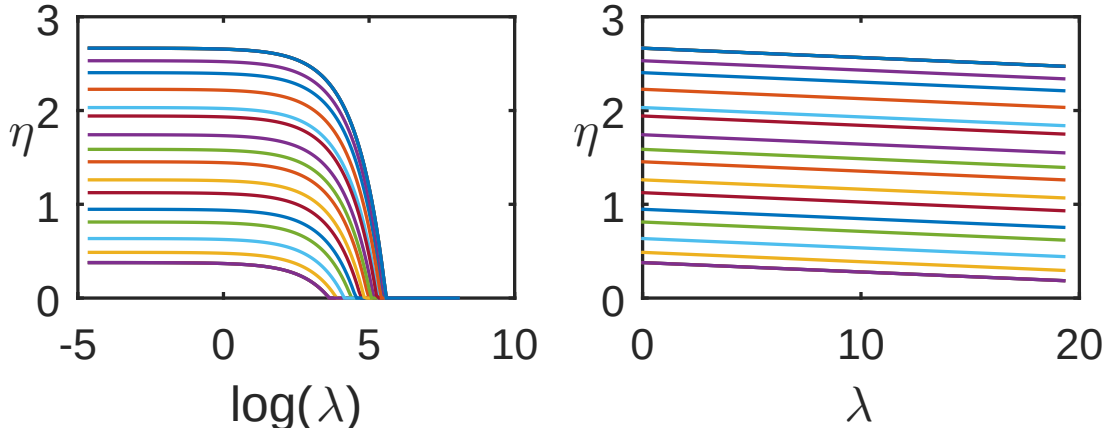

- $\eta_w^{(\lambda)} = (\eta_z - \lambda)_+$.
- Same complexity as computing the norm $\Omega_{\mathcal{S}}(O(d \log d))$.
- True for all cardinality based F.

Random Design

Problem setting: LS-SOWL.

- True model: $y = Xw^* + \varepsilon$, $\left(X^\top\right)_i \sim \mathcal{N}(\mu, \Sigma)$, $\varepsilon_i \in \mathcal{N}(0, \sigma^2)$.
- Notation: $\mathcal{J} = \{i | w_i^* \neq 0\}, \eta_{w^*} = [\underbrace{\delta_1^*, \ldots, \delta_1^*}_{G_1}, \ldots, \underbrace{\delta_k^*, \ldots, \delta_k^*}_{G_k}]$.

[3] D: Diagonal matrix, defined using groups $\mathcal{G}_1, \ldots, \mathcal{G}_k$, and w^*

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Result[Sankaran et al., 2017]

1. $\hat{w} \to_p w^*$.
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- Similar to Group Lasso [Bach, 2008, Theorem 2].
- Learns the weights, without explicit groups information.

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Quantitative Simulation: Predictive Accuracy

Aim: Learn $\hat{w}$ using LS-SOWL, evaluate prediction error.

[4] The experiments followed the setup of Bondell and Reich [2008]

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Data:[4] $w^* = [0_{10}^\top, 2_{10}^\top, 0_{10}^\top, 2_{10}^\top]^\top$.

- $n = 100$, $\sigma = 15$ and $\Sigma_{i,j} = 0.5$ if $i \neq j$ and 1 if $i = j$.

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Models with group variance:

- Measure $E[\|x^\top(\tilde{w}^* - \hat{w})\|_2]$.
  - $\tilde{w}^* = w^* + \tilde{\epsilon}$,
  - $\tilde{\epsilon} \sim \mathcal{U}[-\tau, \tau]$,
  - $\tau = 0, 0.2, 0.4$.

[4] The experiments followed the setup of Bondell and Reich [2008]

Predictive accuracy results

Algorithm	Med. MSE	MSE (10th Perc).	MSE (90th Perc)
LASSO	46.1 / 45.2 / 45.5	32.8 / 32.7 / 33.2	60.0 / 61.5 / 61.4
OWL	27.6 / 27.0 / 26.4	19.8 / 19.2 / 19.2	42.7 / 40.4 / 39.2
El. Net	30.8 / 30.7 / 30.6	21.9 / 22.6 / 23.0	42.4 / 43.0 / 41.4
$\Omega_{\mathcal{S}}$	**23.9** / **23.3** / **23.4**	**16.9** / **16.8** / **16.8**	**35.2** / **35.4** / **33.2**

Table: Each column has numbers for $\tau = 0, 0.2, 0.4$.

Summary

1. Proposed a new family of norms $\Omega_{\mathcal{S}}$.
2. Properties:
   - Equivalent to OWL in group identification.
   - Efficient computational tools
   - Equivalences to Group Lasso.
3. Illustrations on performance through simulations.

Questions ?

Thank you !!!

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