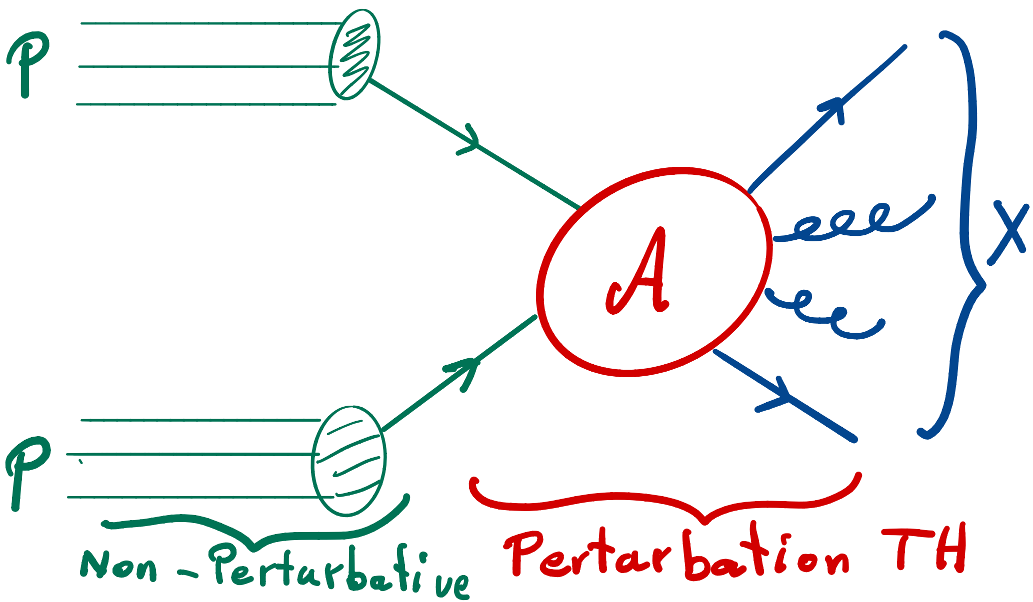
1. Scattering Amplitudes

- <u>GOAL</u>: Compute predictions for collider experiments like the LHC



Probo$(p,p \to X)$

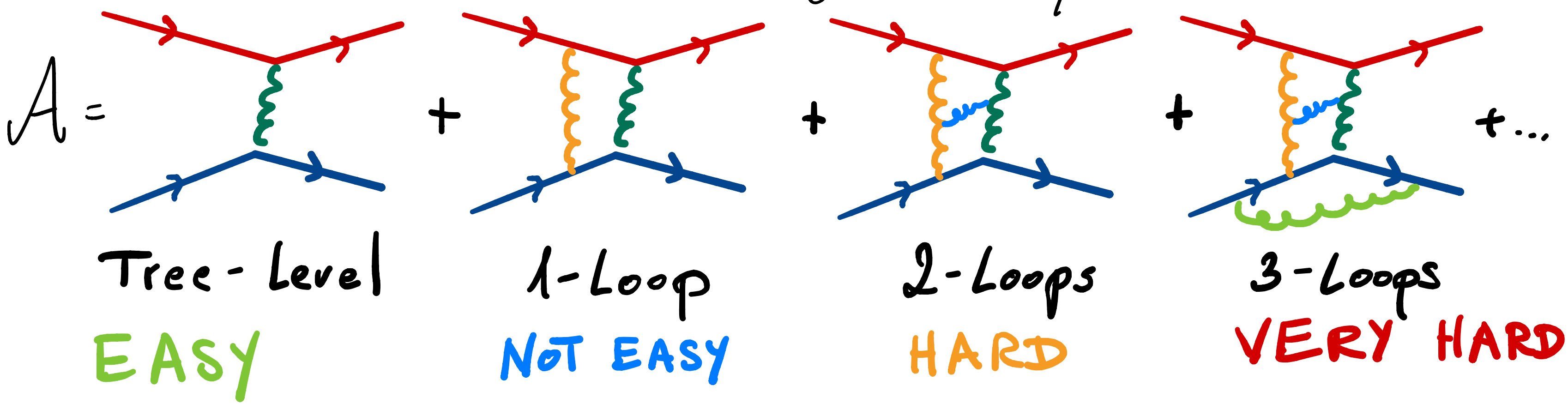
$$= f_1 \otimes f_2 \otimes \hat{\sigma}$$

$$\hat{\sigma} = \int dPS\, |A|^2$$

⇒ We want to compute scattering amplitudes with external gluon and quark states!

⇒ We only know how to do that in Perturbation Theory (PT):

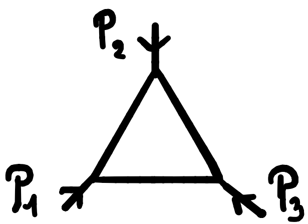
⇒ E.g. 2-to-2 scattering of quarks:



2. Feynman Integrals

- Beyond tree-level, we need to compute integrals.

- <u>Example</u>



$$\sim \int \frac{d^4 k}{k^2 (k+p_1)^2 (k+p_1+p_2)^2} = ?$$

$p_1 + p_2 + p_3 = 0$

N.B.: In general, such integrals diverge and need to be regularised.

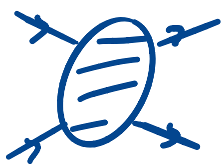
⇒ What does this integral evaluate to?

⇒ Lorentz invariance: Function of ratios:

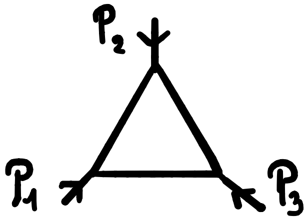
$$u = p_1^2/p_3^2 \qquad v = p_2^2/p_3^2$$

⇒ Is it a simple rational fantion of u and v?

⇒ No! → Unitarity:

Disc  $\sim 2\pi i$ 

Intermediate Particles on-shell



$$\sim \frac{2}{z-\bar{z}}\left[\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z})+\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}\right]$$

$$u = z\bar{z} \qquad v = (1-z)(1-\bar{z})$$

Log(arithm)

$$\log x = \int_1^x \frac{dt}{t} \qquad \log(1-x) = \int_0^x \frac{dt}{t-1}$$

Recursive structure!

Dilog(arithm)

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t}\log(1-t) = \int_0^x \frac{dt}{t}\int_0^t \frac{dt'}{1-t'}$$

PolyLog(arithm):

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t}\, \mathrm{Li}_{n-1}(t) \qquad \mathrm{Li}_1(x) = -\log(1-x)$$

Multiple PolyLogarithms (MPLs)

$$G(a_1,\dots,a_n;x) = \int_0^x \frac{dt}{t-a_1}\, G(a_2,\dots,a_n;t) \qquad a_n \neq 0$$

$$G(\underbrace{0,\dots,0}_{n \text{ times}};x) = \frac{1}{n!}\log^n x$$

$$G(a;x) = \log\left(1-\frac{x}{a}\right), \quad G(\underbrace{0,\dots,0}_{n-1 \text{ times}},1;x) = -\mathrm{Li}_n(x)$$

- Natural "invariant" attached to MPLs:

Weight = # integrations

Examples: $\log x \longrightarrow$ Weight 1

$\log x \cdot \log y \longrightarrow$ Weight 2

$\mathrm{Li}_n(x)\,,\ G(a_1,\dots,a_n;x) \longrightarrow$ Weight n

Zeta Value

$\pm i\pi = \underset{1}{\log}(-1)$ $\qquad \frac{\pi^2}{6} = \underset{2}{\mathrm{Li}_2}(1)$ $\qquad \zeta_n = \underset{n}{\mathrm{Li}_n}(1)$

Rational Number/funct $\longrightarrow$ Weight 0

$\Rightarrow$ GOAL: UNDERSTAND THE PROPERTIES OF THESE FUNCTIONS.

<u>N.B.</u>: BEYOND 1-LOOP, also other functions related to

- Elliptic Curves
- Modular Forms
- K3 / Calabi-Yau varieties
- ???

Why?

- → Evaluate the integrals
- → Numerical evaluation

Important for collider physics

- → Learn something about the structure of amplitudes
- → Bootstrap program

"Compute Amps w/o computing integrals"

- → Also show up in other places (e.g. Wave function of the universe).

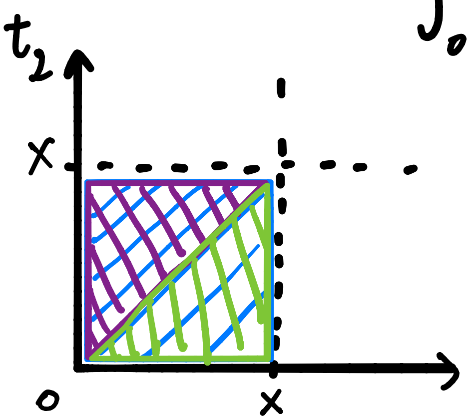
3. Properties of MPLs

3.1. Shuffle algebra

Consider the product of 2 MPLs of weight 1:

$$G(a;x)\,G(b,x) = \left(\int_0^x \frac{dt_1}{t_1-a}\right)\left(\int_0^x \frac{dt_2}{t_2-b}\right)$$

$$= \int_0^x \int_0^x \frac{dt_1\,dt_2}{(t_1-a)(t_2-b)}$$



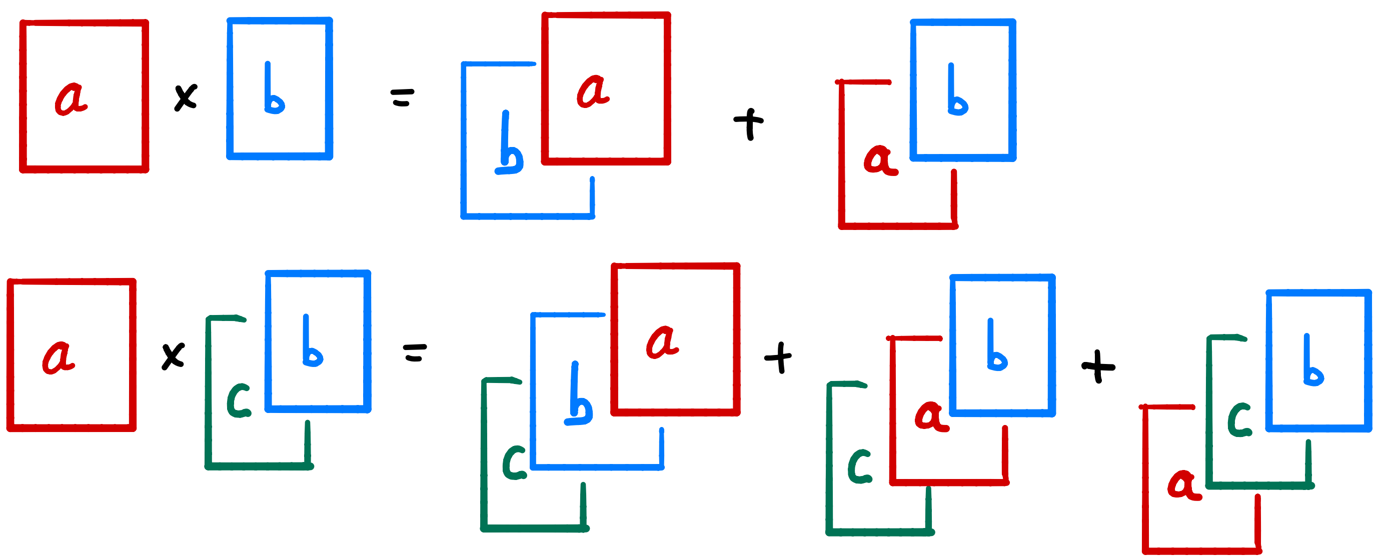
$$= \int_0^x \frac{dt_1}{t_1-a}\int_0^{t_1} \frac{dt_2}{t_2-b} + \int_0^x \frac{dt_2}{t_2-b}\int_0^{t_2} \frac{dt_1}{t_1-a}$$

$$= G(a,b;x) + G(b,a;x)$$

$$G(a;x)\,G(b;x) = G(a,b;x) + G(b,a;x)$$

(weight 1) x (Weight 1) = (Weight 2) + (Weight 2)

→ This generalises ! (Shuffle algebra)



$$G(a;x)\;G(b,c;x) = G(a,b,c;x) + G(b,a,c;x) + G(b,c,a;x)$$

<u>Property</u>: The multiplication of MPLs preserves the weight.

$\mathcal{A}_n$ = "$\mathbb{Q}$-vector space spanned by all MPLs of weight n"

$\mathcal{A}_0 = \mathbb{Q}$

$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ = "$\mathbb{Q}$-vector space of all MPLs"

$\mathcal{A}$ is graded algebra!

$\hookrightarrow \mathcal{A}_m \cdot \mathcal{A}_n \subseteq \mathcal{A}_{m+n}$

3.2. The coaction

- A is an algebra: vect. sp. with multipl. $\mu : A \otimes A \to A$

Associativity

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

$a \otimes b \otimes c$; $(ab) \otimes c$; $a \otimes (bc)$; $(ab)c = a(bc)$

Co-Associativity

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta \otimes id"] \\ A \otimes A \arrow[r, "id \otimes \Delta"] & A \otimes A \otimes A \end{tikzcd}$$

- A is a coalgebra: vect. sp. with coproduct $\Delta : A \to A \otimes A$

- Hopf algebra = Algebra and Coalgebra

  + some more axioms, e.g.

$$\Delta(a\cdot b) = \Delta(a)\cdot\Delta(b)$$

<u>N.B:</u> Multiplication on $A\otimes A$ is "componentwise":

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2) = (a_1\cdot a_2)\otimes(b_1\cdot b_2)$$

Example: A = $\mathbb{Q}$-vect. sp. spanned by all words in letter $a, b, c, d, \dots$

A_n = subspace of words of length n.

Multipl. = shuffle product:

E.g.: $a \cdot bc = abc + bac + bca$

Coproduct: "Deconcatenation"

E.g.: $\Delta(abc) = 1 \otimes abc + a \otimes bc$
$+ ab \otimes c + abc \otimes 1$

Let as check that $\Delta(a\cdot b) = \Delta(a)\cdot\Delta(b)$

$$\Delta(a\cdot b) = \Delta(ab) + \Delta(ba)$$

$$= 1\otimes ab + a\otimes b + ab\otimes 1$$

$$+ 1\otimes ba + b\otimes a + ba\otimes 1$$

$$\Delta(a)\cdot\Delta(b) = (1\otimes a + a\otimes 1)(1\otimes b + b\otimes 1)$$

$$= 1\otimes(a\cdot b) + b\otimes a + a\otimes b + (a\cdot b)\otimes 1$$

$$= 1\otimes ab + 1\otimes ba + b\otimes a + a\otimes b$$

$$+ ab\otimes 1 + ba\otimes 1$$

- MPLs form a shuffle algebra.
- MPLs form a Hopf algebra [GONCHAROV]

<u>Example:</u> $\Delta(\log x) = 1 \otimes \log x + \log x \otimes 1$

$$\Delta(\log x \log y) = 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1$$

$$\Delta(\mathrm{Li}_2(x)) = 1 \otimes \mathrm{Li}_2(x) - \log(1-x) \otimes \log x + \mathrm{Li}_2(x) \otimes 1$$

$$\Delta(\mathrm{Li}_n(x)) = 1 \otimes \mathrm{Li}_n(x) + \mathrm{Li}_n(x) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(x) \otimes \frac{\log^k x}{k!}$$

$$\Delta\left(Li_n(x)\right) = \underbrace{1 \otimes Li_n(x)}_{1 \otimes \zeta_n} + \underbrace{Li_n(x) \otimes 1}_{\zeta_n \otimes 1} + \underbrace{\sum_{k=1}^{n-1} Li_{n-k}(x) \otimes \frac{\log^k x}{k!}}_{\cdots \otimes 0 = 0}$$

$$Li_n(1) = \zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

$$\Delta(\zeta_n) = 1 \otimes \zeta_n + \zeta_n \otimes 1$$

$$\zeta_2 = \frac{\pi^2}{6} \qquad \zeta_4 = \frac{\pi^4}{90} \rightsquigarrow \zeta_4 = \frac{2}{5}\zeta_2^2$$

$$\rightsquigarrow \Delta(\zeta_4) = \frac{2}{5}\Delta(\zeta_2)^2$$

$$= 1 \otimes \zeta_4 + \zeta_4 \otimes 1 + \frac{4}{5}\, \zeta_2 \otimes \zeta_2$$

???

We need $\zeta_2 \otimes \zeta_2 = 0$

1) Work "mod π": $\zeta_2 = \frac{\pi^2}{6} = 0 \mod \pi$

2) Work "mod π" only in second entry [Brown]

This gives:

$$\Delta(\zeta_2) = \zeta_2 \otimes 1, \quad \Delta(\zeta_4) = \zeta_4 \otimes 1, \dots$$

$$\Delta(\underbrace{i\pi}_{= \log(-1+i0)}) = i\pi \otimes 1$$

$\rightsquigarrow$ We will motivate this more later

Technical interlude

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \underbrace{\mathcal{H}}_{= \text{"}\mathcal{A} \bmod \pi\text{"}}$$ [Brown]

Δ : Coaction $\qquad$ $\mathcal{A}$: comodule

$$\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$$

Δ : Coproduct $\qquad$ $\mathcal{H}$: Hopf algebra

3.3. Properties of the coaction

Q: What is the "meaning" of the different entries in the coaction?

* $\Delta\left(\frac{\partial}{\partial x} \mathrm{Li}_2(x)\right) = \Delta\left(-\frac{1}{x}\log(1-x)\right)$

$$= -\frac{1}{x}\,\Delta(\log(1-x))$$

[$\frac{1}{x}$ is a rational number]

$$= -\frac{1}{x}\left(\log(1-x)\otimes 1 + 1\otimes\log(1-x)\right)$$

$$\left(\mathrm{id}\otimes\frac{\partial}{\partial x}\right)\Delta(\mathrm{Li}_2(x)) = 1\otimes\frac{\partial}{\partial x}\mathrm{Li}_2(x) - \log(1-x)\otimes\frac{\partial}{\partial x}\log x + \mathrm{Li}_2(x)\otimes\frac{\partial}{\partial x}1$$

$$= 1\otimes\left(-\frac{1}{x}\log x\right) - \log(1-x)\otimes\frac{1}{x} + 0$$

$$\Delta\left(\frac{\partial}{\partial x}F\right) = \left(\mathrm{id}\otimes\frac{\partial}{\partial x}\right)\Delta(F)$$

Derivatives only act in 2nd entry!

Similarly:

$$\Delta\left(\mathrm{Disc}_{x_0}F\right) = \left(\mathrm{Disc}_{x_0}\otimes\mathrm{id}\right)\Delta(F)$$

Discontinuities only act in 1st entry!

cf. $\mathrm{Disc}_0 \log x = 2\pi i$ and $\Delta(2\pi i) = 2\pi i \otimes 1$

Interlude: Discontinuities

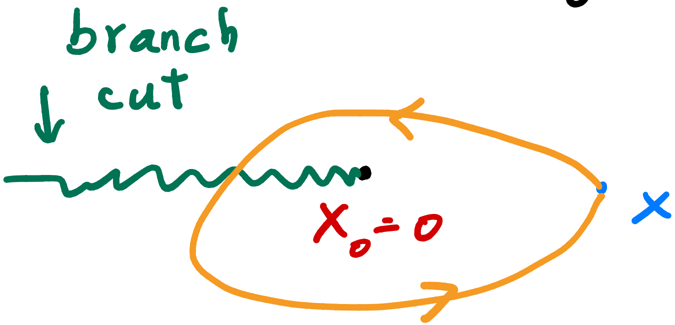
Let $F(x)$ be an analytic function. We define

$$\mathrm{Disc}_{x_0} F(x) = F^{\circlearrowleft}(x) - F(x)$$

where $F^{\circlearrowleft}(x)$ is obtained from $F(x)$ by analytic continuation along a small circle around x_0.

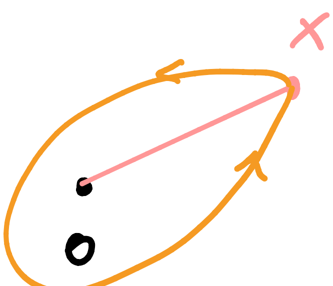
$\Rightarrow$ If F has a branch cut starting at $x = x_0$, then $\mathrm{Disc}_{x_0} F(x) \neq 0$.

Example: • $\log x$



$$\mathrm{Disc}_0 \log x = \log(x e^{2\pi i}) - \log x = 2\pi i$$

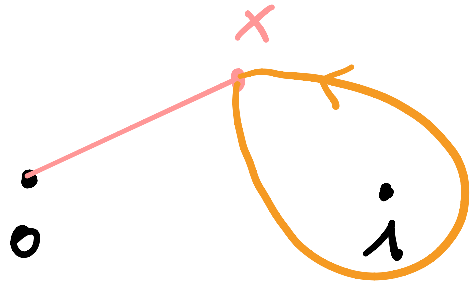
• $$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \log(1-t)$$



$$\mathrm{Disc}_0 \mathrm{Li}_2(x) = -\left(\int_\gamma - \int_0^x\right) \frac{dt}{t} \log(1-t)$$

$$= -\oint_{|x|=\varepsilon} \frac{dt}{t} \log(1-t)$$

$$= -\mathrm{Res}_{t=0} \frac{1}{t} \log(1-t) = 0$$



$$\mathrm{Li}_2(x) = -G(0,1;x)$$
$$= -G(0;x)\,G(1;x) + G(1,0;x)$$
$$= -\log x \log(1-x) + \int_0^x \frac{dt}{t-1} \log t$$

* $\mathrm{Disc}_1 \left(\log x \log(1-x)\right)$

$$= \log x \ \mathrm{Disc}_1 \log(1-x)$$
$$= 2\pi i \log x$$

* $\mathrm{Disc}_1 \int_0^x \frac{dt}{t-1} \log t = \mathrm{Res}_{t=1} \frac{\log t}{t-1} = 0$

$$\mathrm{Disc}_1 \mathrm{Li}_2(x) = -2\pi i \log x$$

$$\mathrm{Disc}_1 Li_2(x) = -2\pi i \log x$$

Coaction?

$$\Delta(\mathrm{Disc}_1 Li_2(x)) = -\Delta(2\pi i \log x)$$

$$= -\Delta(2\pi i)\,\Delta(\log x)$$

$$= -\,(2\pi i \otimes 1)(1 \otimes \log x + \log x \otimes 1)$$

$$= -(2\pi i \log x) \otimes 1 - 2\pi i \otimes \log x$$

$$(\mathrm{Disc}_1 \otimes \mathrm{id})\Delta(Li_2(x)) = \underbrace{\mathrm{Disc}_1 Li_2(x)}_{-2\pi i \log x} \otimes 1 - \cancel{\mathrm{Disc}_1 1 \otimes Li_2(x)}$$

$$- \underbrace{\mathrm{Disc}_1 \log(1-x)}_{2\pi i} \otimes \log x$$

3.4. Symbols

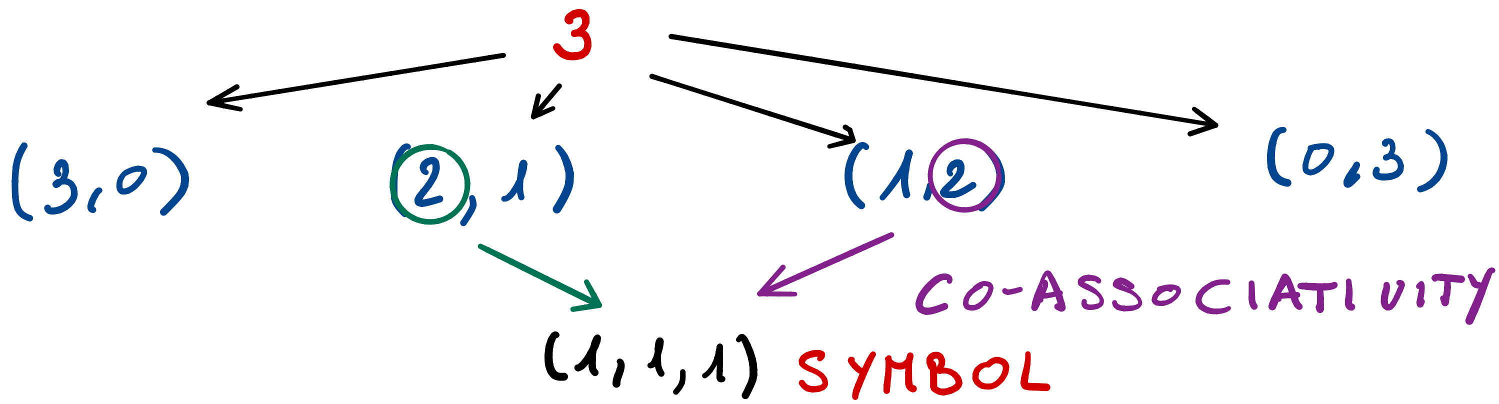
<u>Recap</u>: Coaction : Decompose an MPL into MPLs of lower weights.

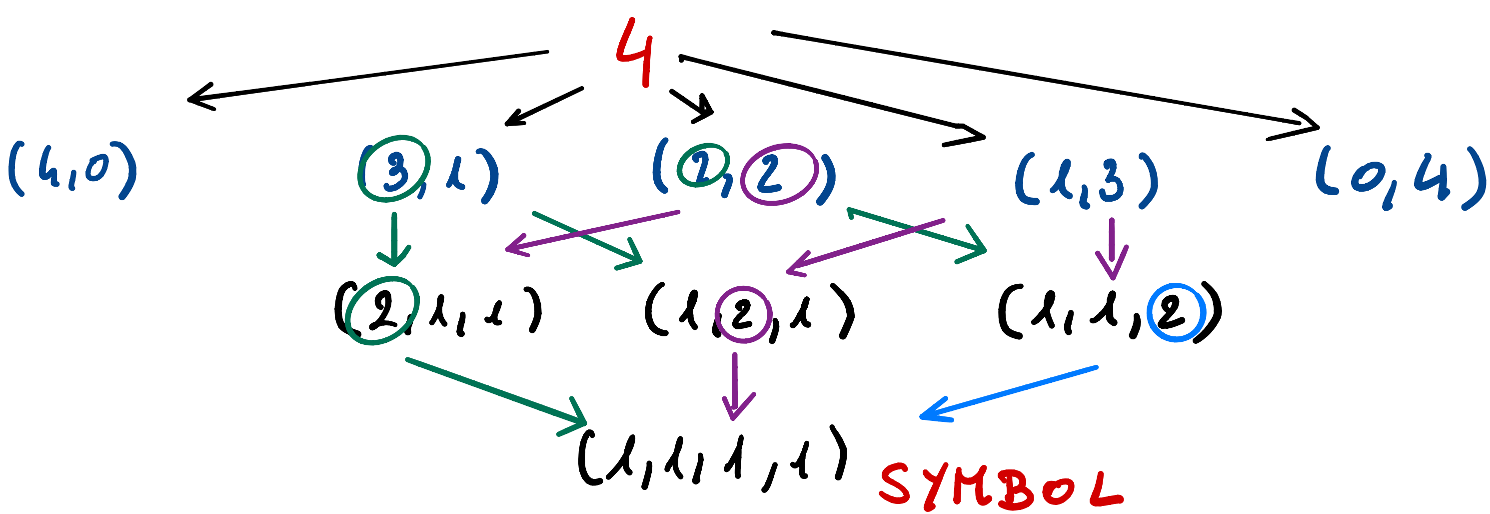
<u>Example</u>: $\underset{2}{\Delta(\mathrm{Li}_2(x))} = \underset{(0,2)}{1 \otimes \mathrm{Li}_2(x)} + \underset{(2,0)}{\mathrm{Li}_2(x) \otimes 1}$

$$- \underset{(1,1)}{\log(1-x) \otimes \log x}$$

$$\underset{3}{\Delta(\mathrm{Li}_3(x))} = \underset{(3,0)}{\mathrm{Li}_3(x) \otimes 1} + \underset{(0,3)}{1 \otimes \mathrm{Li}_3(x)}$$

$$+ \underset{(2,1)}{\mathrm{Li}_2(x) \otimes \log x} - \frac{1}{2} \underset{(1,2)}{\log(1-x) \otimes \log^2 x}$$


3
(3,0)
(2,1)
(1,2)
(0,3)
CO-ASSOCIATIVITY
(1,1,1)
SYMBOL


4
(4,0)
(3,1)
(2,2)
(1,3)
(0,4)
(2,1,1)
(1,2,1)
(1,1,2)
(1,1,1,1)
SYMBOL

Symbol = $(1,1,\dots,1)$ part of the coaction

$\rightsquigarrow$ "Invariant" attached to a polylog

Examples:

$$\mathcal{S}(\log x) = \log x \to x$$

$$\mathcal{S}(\mathrm{Li}_2(x)) = -\log(1-x) \otimes \log x \to -(1-x) \otimes x$$

$$\mathcal{S}(\mathrm{Li}_n(x)) = -\log(1-x) \otimes \underbrace{\log x \otimes \dots \otimes \log x}_{n-1 \text{ times}}$$

$$\to -(1-x) \otimes x \otimes \dots \otimes x$$

- Properties of $\mathcal{S}$ are inherited from coaction:

$$\ldots \otimes (a \cdot b) \otimes \ldots = \ldots \otimes a \otimes \ldots + \ldots \otimes b \otimes \ldots$$

$[\log(a \cdot b) = \log a + \log b]$

$$\ldots \otimes (\pm 1) \otimes \ldots = 0$$

$[\log(\pm 1) = 0 \mod \pi]$

$$\mathcal{S}(\zeta_n) = 0$$

$[\Delta(\zeta_n) = 1 \otimes \zeta_n + \zeta_n \otimes 1]$

↳ no non-trivial decomposition

$$\mathcal{S}(F \cdot G) = \text{Shuffle of } \mathcal{S}(F) \text{ and } \mathcal{S}(G)$$

E.g.: $\mathcal{S}(\log x \log y) = x \otimes y + y \otimes x$

Assume $\mathcal{S}(F) = a_1 \otimes \ldots \otimes a_n$, then

- "Discontinuities only act in 1st entry"

  $$\mathcal{S}(\mathrm{Disc}_x F) = (\mathrm{Disc}_x \log a_1)\,(a_2 \otimes \ldots \otimes a_n)$$

- "Derivatives only act in 2nd entry"

  $$\mathcal{S}\left(\frac{\partial}{\partial x} F\right) = \left(\frac{\partial}{\partial x} \log a_n\right)\,(a_1 \otimes \ldots \otimes a_{n-1})$$

- Leads to a recursive way to compute symbol
If F_n has weight n, and

$$dF_n = \sum_i F_{i,n-1} \, d\log R_i \qquad \text{[Pure Function]}$$

then

$$\mathcal{S}(F_n) = \sum_i \mathcal{S}(F_{i,n-1}) \otimes R_i$$

<u>Example:</u> $Li_2(x) = -\int_0^x \frac{dt}{t} \log(1-t)$

$$d\,Li_2(x) = -\frac{dx}{x}\log(1-x) = -\log(1-x)\, d\log x$$

$$\longrightarrow \mathcal{S}(Li_2(x)) = -(1-x) \otimes x$$

4. Applications

4.1. Relations among MPLs

There are many relations among MPLs of different arguments.

$\rightsquigarrow$ How to find/prove them?

<u>Conjecture:</u> There are no relations among MPLs of different weights

Example:

$$T = \mathrm{Li}_2(1-x) + \log(1-x)\log x = \ ?$$

Useful Notation: $\Delta_{p_1, p_2, \ldots, p_k} = (p_1, \ldots, p_k)$-part of coaction.

$$\Delta_{1,1}(T) = -\log(1-x) \otimes \log x + [\log(1-x) \otimes \log x + \log x \otimes \log(1-x)]$$

$$= \log x \otimes \log(1-x)$$

$$= \Delta_{1,1}(-\mathrm{Li}_2(x))$$

Does $\Delta_{1,1}\left(Li_2(1-x)+\log(1-x)\log x\right)=\Delta_{1,1}\left(-Li_2(x)\right)$

imply $Li_2(1-x)+\log(1-x)\log x=-Li_2(x)$??

NO! Kernel of $\Delta_{1,1}$ is not empty, e.g.

$$\Delta_{1,1}(\zeta_2)=\Delta_{1,1}\left(\frac{\pi^2}{6}\right)=0.$$

$\longrightarrow$ Can be recovered by looking at some special value:

@ $x=1$: $\quad \zeta_2+0 \;=\; -0+C$

unknown constant

$$\Longrightarrow \quad Li_2(1-x)+\log(1-x)\log x=-Li_2(x)+\zeta_2$$

Why do we care about these relations?

- Simplify complicated expressions.

  cf. 6-pt. Two-loop remainder function in planar $\mathcal{N}=4$ Super Yang-Mills

  17 pages $\longrightarrow$ 1 line

  [Del Duca, CD, Smirnov] [Goncharov, Spradlin, Vergu, Volovich]

- "Optimize" results for numerical evaluation.

  Important for phenomenology.

4.2. Analytic structure of Amplitudes

Assumption: We have an L-loop amplitude that can be expressed in terms of MPLs and rational/algebraic functions.

→ What can we say about this amplitude?

Conjecture: An L-loop amplitude in 4 space-time dimensions has weight at most $2L$.

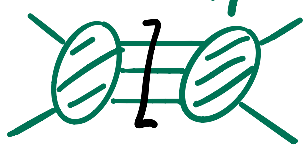
- What else can we say about such an amplitude $\mathcal{A}$?

  $\longrightarrow$ act with coaction:

$$\Delta(A) = \sum_i L_i \otimes R_i$$

$$\longrightarrow \Delta(\mathrm{Disc}\, A) = \sum_i (\mathrm{Disc}\, L_i) \otimes R_i$$

Unitarity



1st entries must have only branch cuts dictated by Unitarity ("1st entry condition"; "weakest version" of "Cosmic Galois principle")

• In special QFTs we can say more, e.g:

<u>Conjecture</u>: In $\mathcal{N}=4$ Super Yang-Mills in 4D an L-loop amplitude only involves functions of weight $2L$ (if it can be expressed in terms of MPLs)

Concretely, if an L-loop Amplitude in $\mathcal{N}=4$ SYM can be expressed in terms of MPLs, then

$$A_L^{\mathcal{N}=4} = \sum_i \underbrace{R_i}_{\text{Algebraic}} \, T_i$$

Pure function of weight $2L$.

4.3. Bootstrap Program

- **Planar Limit:** In an $SO(N)$ gauge theory with $N \to \infty$ with $g^2 N$ fixed, only planar diagrams contribute.

- **Technical aside:** We consider ratios

$$R_N = \frac{A_N^{\mathcal{N}=4,\,\mathrm{planar}}}{A_N^{\mathrm{BDS}}} = 1 + a \cdot 0 + a^2 R_2 + a^3 R_3 + \dots$$

IR divergent

Ratio is Finite (and dual conformal invariant)

$N = \#$ external particles

- What do we know about R?
  - $R_N^{1\text{-loop}} = 0$, for all N.
  - $R_4 = R_5 = 0$, at all loops.
  - Conjecture: $R_N^{\text{MHV}, L\text{-loop}}$ is a pure function of weight $2L$.

Recall: MHV = helicities $--++\ldots+$

- What can we say about MHV amplitudes in planar $\mathcal{N}=4$ SYM?

- <u>Bootstrap program:</u> Use knowledge of analytic structure to write down unique pure function that can be the amplitude
- We only sketch the main idea.
  - * Assume that symbol of R_N only involves certain entries.

<u>Example:</u> $\mathcal{S}(R_6^{2\text{-loop}})$ has all entries drawn from a set of 9 elements ("letters")

→ There is strong evidence that this holds to all loops

* Construct a space of functions of weight n

(C1) Whose symbols have their entries drawn from this 9-letter set.

(C2) Has the correct branch cuts to be an amplitude (Unitarity)

$\Rightarrow$ Functions of different weights of these spaces are not independent

Define:

* $\tilde{V}_n$ = vec. sp. of pure functions of weight n that satisfy (C1).

* V_n = vec. sp. of pure functions of weight n that satisfy (C1) & (C2).

Clearly: $V_n \subseteq \tilde{V}_n$

Then, if $F \in V_n$:

[1st entry condition]

$$\underset{\substack{\rotatebox{90}{$\in$} \\ V_n}}{\Delta(F)} = \sum_i \underset{\substack{\rotatebox{90}{$\in$} \\ V_k}}{F_i} \otimes \underset{\substack{\rotatebox{90}{$\in$} \\ \tilde{V}_{n-k}}}{G_i}$$

[N.B.: By now we now more than just 1st entry → extended Steinmann relations]

- let $\{B_i^{(n)}\}$ and $\{\tilde{B}_i^{(n)}\}$ be bases for V_n and $\tilde{V}_n$
- Assume we know bases up to weight n.
- Find a basis of V_{n+1}, by requiring

$$\Delta\left(B_i^{(n+1)}\right) \in \sum_{k=0}^{n+1} V_k \otimes \tilde{V}_{n-k}$$

- If we have a basis of V_{2L}, we must have

$$R_6^{2L\text{-loops},\,MHV} = \sum_i c_i \, B_i^{(2L)} \qquad c_i \in \mathbb{Q}$$

- Determine c_i by requiring some physics input. (e.g. correct collinear limits, etc.)

- Particularly successful for $N=6$ and $N=7$
  ["Hexagon-Bootstrap", "Heptagon-Bootstrap", Dixon et al.]

  → $N=6$ through 7-loops (MHV)
  $N=7$ through 4-loops

- For $N>7$, more complicated

  ⇝ What is the exact form of letters?

  → Recent work by

[Drummond, Foster, Gurdogan, Kalousios; He, Li, Zhang; Mago, Schreiber, Spradlin, Volovich; Henke, Papathanasiou]