The man who invented calculus:
the life and work of Madhava

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A short history of the history

Madhava's name relatively unknown to maths historians till recently – and his mathematical achievements misunderstood even today.
– First notice of "Kerala school" from outside Kerala: Charles Whish (Royal Asiatic Society, London, 1832). Uses words like 'quadrature', 'fluxion' and 'fluent' (Newton's words for derivative and integral) in describing the content. But NO Madhava.

– NO Madhava in European and American writings (critical editions, translations and commentaries) till as late as the 1960s (Whiteside).

– The first mention of his name ("Sangamagrama Madhavan") in print that I know of is from the 1940s (in the Foreword to the now famous **Malayalam** analytic commentary (*vyākhyā*) of Ramavarma Tampuran and Akhilesvara Ayyar of the **Malayalam** text Yuktibhasha (which will have a prominent place in these lectures).

– Soon after, K Kunjunni Raja and K V Sarma, both at Madras U., looked at the history of the "Kerala school of Hindu Astronomy" in some detail. Madhava figures in it as an astronomer (he did write a book on lunar astronomy), but not for his maths. Paramesvara, Madhava's disciple is given greater prominence.

– **Breakthrough:** The first public references in English, quite numerous, to Madhava by name are in Sarasvati Amma's pioneering history of Indian geometry (published in 1979, but written in 1959 as her thesis). More important: Madhava's main theorems described with proofs, all steps written out. Hints strongly that they are applications to the geometry of the circle of a new paradigm. Shies away from the word 'calculus' (I suspect for non-scholarly reasons), but not from' infinitesimal', differential','integral' etc.

– The story was taken up a little earlier by C T Rajagopal, mathematician at Madras U (and his collaborators including K Balagangadharan) who was fascinated by Madhava's algebraic and analytical advances seen from a modern angle; Nothing about Madhava + calculus.

But times are changing!

1. We are now pretty sure of the attribution of **most** of the new ideas and results of the "Kerala school" to Madhava himself; and
2. We have a much better appreciation of the general method (in the singular!) by which he got them: **infinitesimal calculus**, by any serious definition of the term.

"It seems fair to me to compare [Madhava] to Newton and Leibniz"
David Mumford (Review of Plofker's book, 2010).

But Plofker herself is part of a circle still avoiding the C-word.

That is one reason for my choice of title. I hope to convince you of its rightness in these lectures, given in his name in a space named as a tribute to him.

Sources – the Madhava lineage

Madhava 1350 - 1420-30 AD

→ Paramesvara 1360 – 1450
→ Damodara 1380 – ?
→ **Nilakantha (Somayaji)** 1444 – 1530
→ **Jyeshthadeva** 1470 – ?
→ Sankara (Variyar) 16th C.
→ Achyuta (Pisharati) ? – 1621

They wrote lots of books which are (almost) our only source of information about Madhava. He himself seems to have shared his wisdom by scattering around individual verses (*sūtras*) and of course in face-to-face teaching.

Especially valuable:

Jyeshthadeva's *Yuktibhāṣā* (=YB) in Malayalam prose has statements and complete proofs of all the main propositions attributed to Madhava. Also has some isolated verses quoted anonymously (or as by *ācārya* who is almost certainly Madhava). But his name does not occur in the book. Sankara has more such verses, some of them "as said by Madhava".

Nilakantha's *Āryabhaṭīya-bhāṣya* (=ABB) in expansive Sanskrit prose has quite a few explicit allusions to Madhava: "stated by Madhava", "verse(s) composed by Madhava", "proofs constructed by mathematician-teachers such as Madhava", etc.

Little doubt that results assigned to Madhava are really his. That is my default position. (But there are also some foundational passages in YB which I think are its own).

Some geography – the Nila school

We know not only the line of succession of these 6 or 7 generations of Madhava's brilliant descendants, but also where they lived and worked: in a cluster of temple-villages in the lower basin of the river now called Bharata, but known at that time by its classical name **Niḷā** (not Nīla).

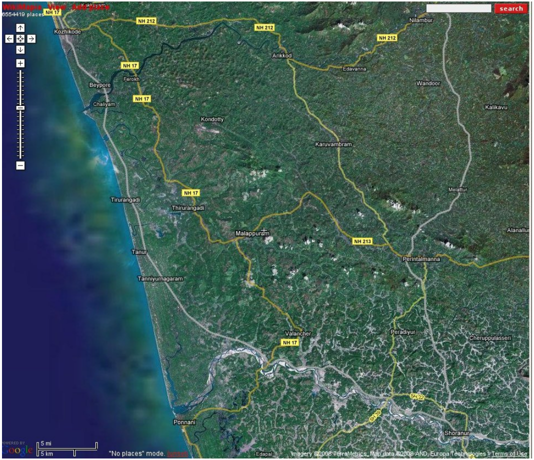
– Alattiyur, the focal point of mathematical/astronomical activities (famous these days for less scientific reasons). NOT Alattur (which is some 100 km away), as several historians have written.

– Trikkandiyur, the birth place of Nilakantha and Achyuta.

– Triprangode, the village of Jyeshthadeva.

That is one reason I prefer the name Nila school. The lazy label "Kerala school" is historically and geographically misleading – there was another ("Muziris") school of astronomy elsewhere in Kerala 500 years earlier.

The other reason: the Nila valley was the cradle of much of Kerala culture – language and literature (modern Malayalam found its final form there), the arts, ayurveda, etc – which didn't put astro and maths in a different compartment. The past is still present and I wanted to evoke that glory.



Nila as it flows by the mathematical villages

So, **where was Madhava himself from?** Patience!

Calculus themes, a first look

Almost everyone now knows that Madhava was the first to state and prove the infinite series expansions:

$$\pi/4 = 1 - 1/3 + 1/5 - \cdots ,$$
$$\sin x = x - x^3/3! + x^5/5! - \cdots$$

and the general arctangent series

$$x = 1 - \tan x + \tan^3 x/3 - \cdots .$$

for x **in the first octant**; Somewhat fewer people know that he **derived** formulae for the area and volume of the sphere:

$$A = Cd, \ V = Cd^2/6.$$

The breakthrough idea in all these is the same: calculus on the circle.

Inspiration: Aryabhata's discrete (finite) calculus

Rule of three (*trairāśikam*): If f is a linear function of x, the functional differences are proportional to differences in x, $(f(x_i) - f(x_{i-1}))/(x_i - x_{i-1}) =: \delta f_i/\delta x_i$ is constant. Divide the unit interval into n equal parts, $x_0 = 0, x_n = 1$; so $\delta x_i = 1/n, \delta f_i =$ constant.

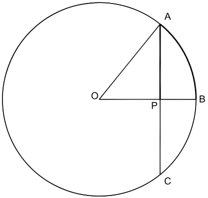
Generalise. If f is not linear, take the next natural step; think of linearity as the first approximation in a process of 'refining' and compute the second differences: $\delta^2 f_i := \delta f_i - \delta f_{i-1}$, *ad inf.* if necessary. If we know the kth differences, we can get the lower differences by iteratied summation:

$$\sum_1^m \delta^k f_i = \delta^{k-1} f_m - \delta^{k-1} f_0.$$

Discrete fundamental thorem: The whole is the sum (**discrete integral**) of the parts (**differences**). In particular, $f_n = f_0 + \delta f_n + \delta f_{n-1} + \cdots$.

Example: Aryabhata's table of sine differences

Take x to be an angle (arc of circle), $\delta x = 2\pi/96(\text{in radians}) =: \epsilon$ and f = sine.



Easy geometry: $\delta \sin(j\epsilon) = 2 \sin(\epsilon/2) \cos(j - 1/2)\epsilon$, and

$$\delta^2 \sin(j\epsilon) = -4 \sin^2(\epsilon/2) \sin(j-1)\epsilon. \qquad (*)$$

"Integrate" once to get the sine-difference table.

From differences to differentials

Aryabhata also says "Divide [the quadrant] into as many parts as you please" (*chindyāt . . . yatheṣṭāni*): an invitation to make ϵ as small – or n as large – as we please. For 900 years no one got the point. Madhava did, took the limit $n \to \infty$. In the limit, $2 \sin \epsilon/2 \to \epsilon (= \delta x)$, $(j-1)\epsilon \to j\epsilon = x$, and we have $\delta^2 \sin x = -(\sin x)(\delta x)^2$ or, in modern notation, the harmonic differential equation

$$\frac{d^2 \sin x}{dx^2} = -\sin x.$$

But Madhava solves the eqn (*) for general finite n and then takes the limit. Formidably challenging! For today, I reproduce his solution but **after** taking the limit, i.e., of the differential equation, in the form of the power series. Brings out the originality and elegance (and rigour) of his thinking.

Solving the differential equation $f'' = -f$.

Integrate formally (fundamental theorem of calculus) twice:

$$f(x) = f(0) + xf'(0) - \int_0^x dy \int_0^y dz f(z)$$

Initial values: for $f=$ sin, $f(0) = 0$ and $f'(0) = 1$. Solve the integral equation by recursive iteration (*saṃskāram*): first guess is the first term; substitute it back into the equation and go on:

$$f(x) = x - x^3/3! + x^5/5! - \cdots .$$

(For $f(0) = 1,\ f'(0) = 0$, we get the cosine series.)

YB goes through the **identical** steps except that it is done at the (much more intricate) discrete level and ends up first with a trigonometric identity, whose limit as $n \to \infty$ is the power series.

Newton on the sine series

Begin as always with $d \sin x / dx = \cos x = \sqrt{1 - \sin^2 x}$. He already had the binomial series for fractional powers but would still need to integrate powers of $\sin x$ over x. So invert: $dx / d \sin x = (1 - \sin^2 x)^{-1/2}$, expand and integrate. But that is not what he wants.

If it is desired to find the sine from the arc given, of the equation found above (for arc $(= z)$ from sine $(= x)$), I extract the root, which will be

$$x = z - z^3/6 + z^5/120 - z^7/5040 + z^9/362880 \cdots .$$

Let it be noted here ... that when you know 5 or 6 of these roots, you will ... be able to prolong them at will by **observing analogies**.

The other calculus problems

– **Rectification of arcs and the value of** π: Find dx/dt as a function of $t := \tan x$ (geometry), for $(0 \leq x \leq \pi/4)$, expand in powers of t and integrate term by term. Done at the discrete level, neglect terms which make vanishing contributions in the limit. Every single step is given a *yukti*. Along the way, "first principles" integration of positive integral powers. Elementary remarks on convergence. (Question: What is the definition of the length of a curve?)

– **Surface area of sphere**: Almost exactly what is done in today's classrooms. Approximate the surface between two infinitesimally separated latitudes by a rectangle (i.e., ignore curvature), and add/integrate.

– Volume of sphere: Very similar logic.

The sphere formulae are the best exhibits in the case for calculus – no distractions from infinite series, etc.

The birthmarks of calculus

– Implement "local linearisation" by "division by ∞": divide the variable by a very large n (conventionally *parārdham* $= 10^{17}$), compute δf, add/integrate (*saṃkalitam*), and let $n \to \infty$. Keep track of neglected terms and make sure they go away. The fundamental theorem is a triviality (Aryabhata). Compare Leibniz (NOT Newton of the *Principia*).
– YB describes the limiting process very patiently; infinitesimal quantities are understood as limits of infinite sequences (D'Alembert's *métaphysique du calcul infinitésimal*) and given precise names: *śūnyaprāyam, aṇuprāyam*.
– Several technical tools of calculus appear in discrete form: e.g., integration by parts (Abel's resummation formula) though not the explicit Leibniz formula, not needed; multiple integrals of general order, etc..
– The idea of setting up difference/differential equations and solving them by turning them into integral equations. (The use of recursive methods in their solution has a very ancient pedigree).

etc., etc.

Calculus? Yes!

The framework ('**paradigm**') Madhava constructed was subtle, complete, general and absolutely new. No fundamental changes, only computational flexibility, needed to accommodate other functions and other curves, say general conics (which of course he hadn't heard of). And he started from scratch; nothing from the past, except Aryabhata's elliptic "divide as you please" verse and his functional differences, to show the way..

So why do some historians still use circumlocutions like "pre-calculus", "a form of calculus", etc.? The same people also claim that Archimedes "anticipated" calculus in his method of exhaustion. Not true, for all his greatness. (Heath, *The Works of Archimedes*, Ch VII, also Euclid XII.2 for area of circle). If the Greeks feared infinity, how could Euclid have lived with (supposedly) his own propositions on primes, Pythagorean triples, etc.?

Several possible explanations for the scepticism.
– YB, the "first textbook of calculus" was linguistically inaccessible to most till very recently and not easy to read even in English translation: very sophisticated maths in very rudimentary prose, no symbols, equations or even figures.

– The reliance on other (poor quality or secondary) sources.

– The historical burden of orientalism and the crude Indian response to it.

– 'Expected' but missing concepts and theorems. e.g., tangents to curves, Taylor series, etc. There are perfectly convincing reasons for their absence.

– "Only the circle and trigonometric functions". Well, Newton and Leibniz, living after the Cartesian synthesis, wouldn't have known a distribution if it blew up in their faces.

The key: no fear of infinity

The main agent of Madhava's paradigm shift was a precise idea of (numerical, not philosophical!) infinity. Numbers have no end:
"Endowed with multiplication (by 10) and the (consequent) positional variation, . . .we cannot know all the numbers and their order." (YB)
$\rightarrow$ Infinite sequences and their limits, e.g., $10^{-n} \rightarrow 0$ ('division by ∞').
$\rightarrow$ Infinite processes such as iterative refining, e.g., as in the sine series, resulting in
$\rightarrow$ (Convergent) infinite series and power series as well-defined mathematical objects. Both ABB (geometric series) and YB (arctangent series) write about the notion of convergence and how to ensure it.
$\rightarrow$ Formal inductive proofs (as in integrals of powers and elsewhere). (In YB there is, right at the beginning, a quasi-Peano 'axiom' about defining natural numbers by their property of succession).

No need to say that **these are all historical firsts.** (Will return to some of these issues later).

Geometry, analysis, algebra, . . .

Fundamentally, the viewpoint was geometric: the infinitesimal method was invented to deal with "curvature" rigorously. YB has a full paragraph on it: ".. the first [infinitesimal] arc has no curvature and is almost equal to the chord ... do not apply *trairāśikam* to [finite] arcs because the result will be gross."

But Madhava also did several stunningly modern things not directly involved in the making of calculus:
– Very generally, the understanding of a function as the assignment of a value to every point in its natural domain. Remember Aryabhata's table (and Bhaskara I's fallacy)!
– An abstract definition (YB) of polynomials as finite sequences of coefficients obeying appropriate rules of addition and multiplication – and also of rational functions – and a corresponding notation modelled on place-value numbers: e.g., $ax^2 - bx + c \leftrightarrow |a| - b|c| \text{ with } x \leftrightarrow |1|0|$.

– Rapidly convergent rearrangements of the basic π series, e.g.,

$$\frac{\pi}{16} = \frac{1}{1^5 + 4 \cdot 1} - \frac{1}{3^5 + 4 \cdot 3} + \frac{1}{5^5 + 4 \cdot 5} -$$

– Estimates of the remainder when the basic π series is terminated (Madhava's own verses): $\pi/4 = 1 - 1/3 \cdots \pm 1/j \mp 1/r_j$, in the form of (convergents of) the continued fraction

$$\frac{1}{r_j} = \frac{1}{2(j+1)+} \; \frac{4}{2(j+1)+} \; \frac{16}{2(j+1)+\cdots}$$

He himself used it to find π to 11 decimal places. In modern times the full expansion was completed by C T Rajagopal (with M S Rangachari) and also Whiteside (Newton's editor).

Nilakantha hints (in ABB, $\sim$ 1520) that his conjecture on the irrationality of π may be based on such an expansion.

. . . but no number theory

Madhava and the entire Nila school seem to have kept away from the very popular subject of the "Pell equation". ABB invokes the topic in passing in a long account of approximate square-root algorithms. YB has not one word on the subject (but plenty on Brahmagupta's other major (geometrical) interest, his beautiful treatment of cyclic quadrilaterals).

Who was Madhava?

Much misinformation in the past **despite** what we already knew:
– "Madhavan, Elanjippalli **empran** . . . Vatasseri Parameshvaran was his disciple" (Malayalam inscription on a ms.)
– ". . . *bakulādhiṣṭhita vihāra* . . . *gṛhanāma* . . . " (in his own words) (bakula is Sanskrit for a flowering tree = elanji in Malayalam, vihara = palli). So we know his house name.

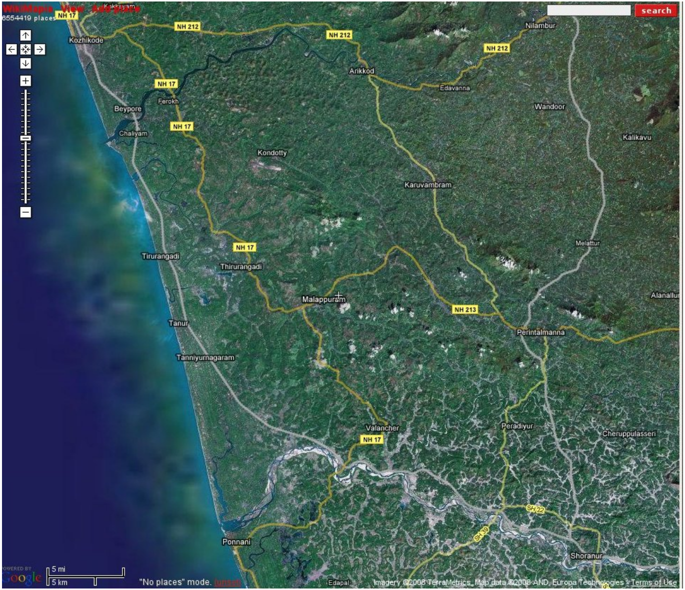
At that time Brahmans often literally translated Malayalam place and house names into Sanskrit (as above), Most confusing example: (Paramesvara's village) Alattiyur ↔ Asvatthagrama ↔ Vatasseri (his supposed house name), all mean the same, 'the village of ficus/banyan trees'.

Sangamagrama has never been convincingly identified.
But translate it back into Malayalam: sangamagrama $\leftrightarrow$ kudalur = **Kudallur** which was a flourishing village, now a town, on the Nila, right inside our calculus country, about 10 km as the crow flies from Alattiyur.

First conclusion: **Madhava was of Elanjipalli house in Kudallur village.**

Empran = Emprantiri is the name given to early Brahman migrants to the Tulu coast south of Gokarna who later moved to Kerala. Remember the Parasurama legend of the 64 Brahman settlements on the west coat; there is some history in support of this. Brahman migrants (Buddhists and Jainas came even earlier) in large numbers from N. India (mainly Valabhi and Ahichhatra) were welcomed by kings and chieftains of the west coast from as early as the 5th C. AD.

Second conclusion: **Madhava belonged to a family which moved from Tulunad to Kerala**.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Kerala discovers Bhaskaracharya (or Bhaskara I meets Bhaskara II)

S. Konkan was part of the extended Vijayanagara empire in the 14th C. Kerala was not. Vijayanagara was also a refuge for men of learning at a time N India had lost the tradition after Muhammad Ghori's depradations.

A community of astro-mathematicians in the Konkan and Deccan (the Daivajna) had become the keepers of the teachings of Bhaskara II (12th C.). Bhaskara himself was probably a Daivajna as was Narayana Pandita, a fine but underrated mathematician of the 14th C, a true apostle of Bhaskara, very likely from Tulunad (many reasons). My guess is that he was the carrier of Bhaskara's teachings to Tulunad and that Madhava took that culture to his new home. The times match.

Forget the details; before Madhava's arrival, no Bhaskara II in Kerala (the earlier Muziris school, followers of Bhaskara I, had stagnated and lapsed into astrology), after that, plenty of Bhaskara II. (**The first ever commentary on Lilavati is by Madhava's own pupil Paramesvara**)

Madhava's legacy

Madhava's arrival was a transformative event. He took the best minds he found among the astrological community of the Nila basin and turned them into the first of a long line of brilliant astronomer/mathematicians. They in turn acknowledged his genius, and it seems that was enough for them. He seems to have had no contemporary impact outside his own circle, his very name lost to history until the 20th C.

Achyuta Pisharati, the last of his direct lineage, died in 1621. Though traditional astro-math scholarship continued in fits and starts and still continues, we must think of that as marking a major break in a very ancient tradition. We can speculate about what brought about the break but it would seem that Portuguese colonial incursions, especially bloody on the Nila sands, was the immediate trigger. Eventually, the English brought another tradition to India and it wasn't long before it produced a Ramanujan who, once again, transformed the mathematical landscape. After all, what are 3 centuries in a story that has gone on for 4 millennia?

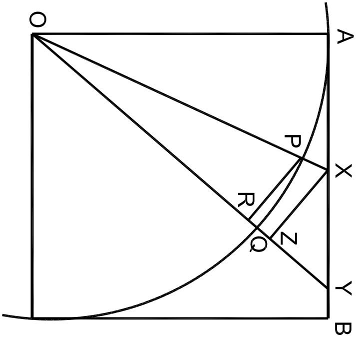
"The ocean of true and false knowledge"

In these days of make-believe history, I have to add a footnote. What I have described is what is in the technical writings, not in the imaginative literature of *itihāsa* and *purāṇa* (to quote Bhaskara I on the Aryabhatiya). I have tried to be as faithful as I can, adding nothing of my own and distorting nothing.

What comes out of it all is that there is a typically Indian style of doing maths and an unbroken thematic continuity, whether done in Bihar or in Kerala. And what nourished this pan-Indian activity at its best is the ready acceptance of fresh stimuli from time to time. The story of Kerala maths, as that of the Aryabhatan revolution, is a wonderful case for how a culture needs to be open, even if that won't produce a Madhava every century. Kerala was never a part of a kingdom or empire based outside of itself. That didn't stop people from the banks of the Ganga (infiltrators if you will) from ending up there in large numbers, bringing their talents with them, never mind who ruled which patch of the lands they had to traverse.

Postscript

Not far from Kudallur, there is today a splendid ancient house known as Kudallur Mana. The Kudallur family is now scattered around the world but family history has it that they are descended from a famous astronomer and that the family base was originally Kudallur village. Several of the later generations were equally famous as linguists and philosophers of the (materialistic) Lokayata school. The only old image in the house is of Patanjali, the great commentator on Panini. There is a large but now empty library, the manuscripts having been dispersed, quite a few (but not all) finding their way to the collection of the Sanskrit College in Trivandrum. Some, inscribed with the Malayalam equivalent of "ex libris Kudallur mana", are on astro and maths; one, of Arayabhatiya, is dated 1552. Jyeshthadeva was probably still alive!



Drawing a tangent to a circle involves no infinitesimal ideas. It is just the normal to the radius.

"Now I show the method of carrying out these operations. When units are added **successively** to a number, they (the results) will be higher and higher numbers starting with it. [similarly for successive removal of units]. In this way all numbers get their individual identity (*svarūpam*, self-form) . . . So, by recalling the identities of increasing and decreasing numbers, we get the results of addition to and subtraction from each number. That will be evident **if we reflect on it**. If we know how to count forward and backward, we get the results of addition and subtraction".

—YB after describing the decimal numbers and elementary operations with them.

Calculus spoken here

YB, surface area of sphere (extract):
" . . . Through the middle of the sphere, imagine drawn two circles, one along east-west ('equator') and the other along north-south (a 'meridian'). Then imagine circles, one slightly to the south and the other slightly to the north of the equator ('latitudes'). Their separation from the equator should be the same for all parts. These two will be slightly smaller than the first one. Then imagine slightly smaller and smaller circles as described above, all of them at equal distance. **Their separations along the meridian should be made equal**. Now imagine that the circle-shaped (annular) gap between two suuccessive circles is cut at one place, removed and straightened. Then, of the wto circles on eithet side of the gap, the bigger one will be the base (*bhūmi*) and the smaller one the face (*mukham*) of a trapezium whose flanks will be the arc segment along the meridian of two successive latitudes. Now cut out the part outside the altitude, turn it upside down and join it to the opposite flank. The result is

a rectangle whose length is half the sum of the base and the face and whose width is the altitude. In this way think of all the gaps as rectangles. Their widths are all equal. The lengths have varying measures. . . . The widths being equal, add the lengths of all and multiply by the width. Thus will arise the area of the sphere.

Now the method to know how many gaps there are and what their lengths and width are. The radii of the latitude circles are half chords of a circle of radius equal to the the radius of the sphere ($r \sin\theta$, $\theta =$ colatitude) . . . . (and so on)."

YB on what a function is

"It can never happen that each of the denominators ($r(j-2)$ and $r(j)$) is equal to twice the odd number (j). Here is why. Suppose the first denominator is double the odd number above it (i.e., $r(j-2) = 2j$). Then the second denominator has to be double the next odd number (i.e., $r(j) = 2(j+2)$) since it must be expressed in the same manner (or, following the same rule). . . . There is no way in which both the denominators can be equal to twice the odd number."