PennState

ICTS International Centre for Theoretical Sciences

Tata Institute of Fundamental Research

Adventures in Perturbation Theory

Jacob Bourjaily

Penn State University

& The Niels Bohr Institute

based on work in collaboration with

Herrmann, Langer, Trnka;

McLeod, von Hippel, Vergu, Volk, Wilhelm; …

The Niels Bohr International Academy

VILLUM FONDEN

erc

Organization & Outline

- ***Spiritus Movens*:** the *surprising* simplicity of QFT
- **Loop *Integrands***
  - ***generalized unitarity*** (generally speaking)
  - ***building bases big-enough*** (for *e.g.* the Standard Model)
  - ***non-planar power-counting*** (a modest proposal)
- **Loop *Integration*:** what makes an integral easy?
  - ***integration polemics*** (what constitutes *being integrated*?)
  - ***direct integration*** (made easy)
- **Loop *Integrals*:** their generic analytic structure
  - *a **bestiary** of Feynman integral Calabi-Yau geometries*

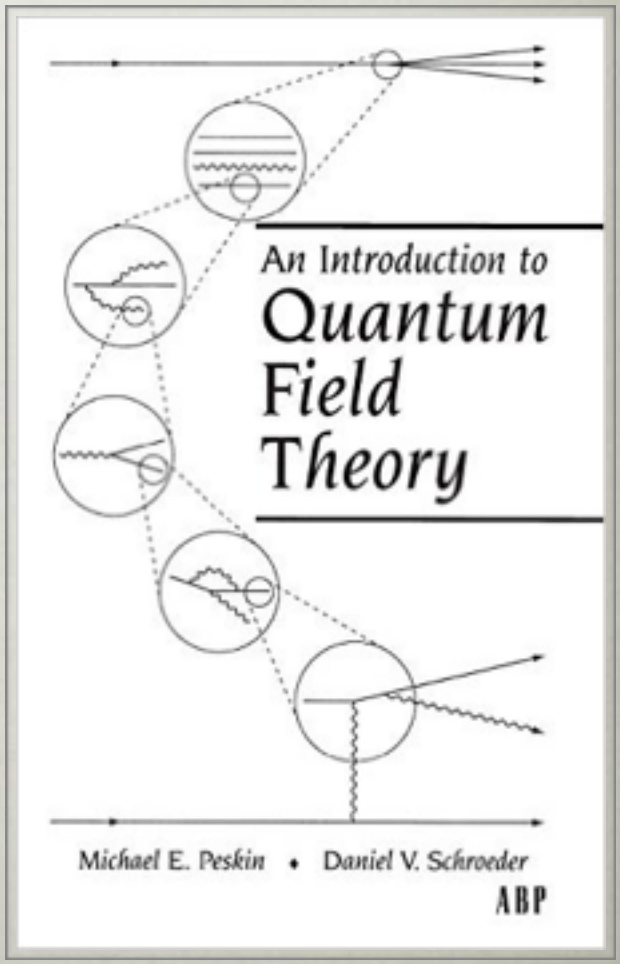
Spiritus Movens

the Surprising Simplicity of Scattering Amplitudes

Traditional Description of QFT

- **Quantum Field Theory**: the marriage of (special) *relativity* with *quantum mechanics*
- Theories (can be) specified by Lagrangians—or equivalently, by ***Feynman rules*** for virtual particles

$$\mathcal{L} \equiv -\frac{1}{4}\sum_i (F^a_{i\mu\nu})^2 + \sum_J \overline{\psi}_J (i\not{D})\psi_J$$



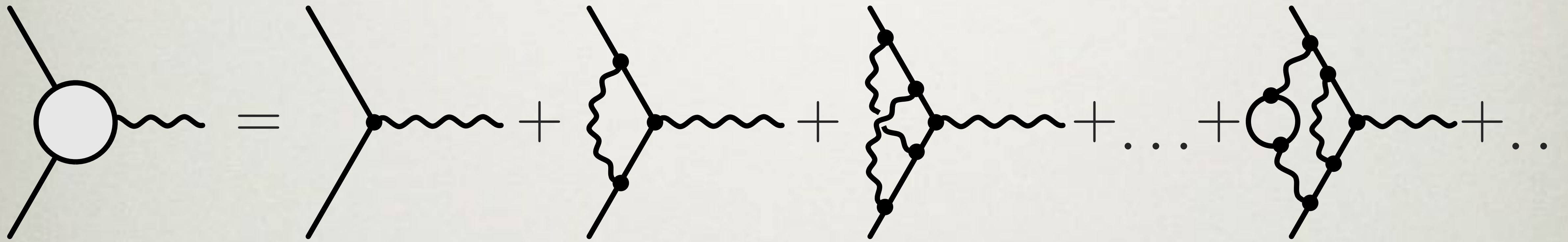
- Predicted probability (*amplitudes*) from path integrals (over virtual 'histories'):

$$\int \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\; e^{i\int d^4x \mathcal{L}}$$

Perturbation Theory and Loops

- Predictions (often) made perturbatively, according to the **loop** expansion:

$\alpha \approx 1/137.036$



$$g_e^{\text{thy}} = 2 + \frac{\alpha}{\pi}(1) + \frac{\alpha^2}{\pi^2}\left(\frac{3}{2}\zeta_3 - \pi^2\log(2) + \zeta_2 + \frac{197}{72}\right) + \ldots$$
$$= 2.00231930435801\ldots$$
$$g_e^{\text{exp}} = 2.00231930436146\ldots$$

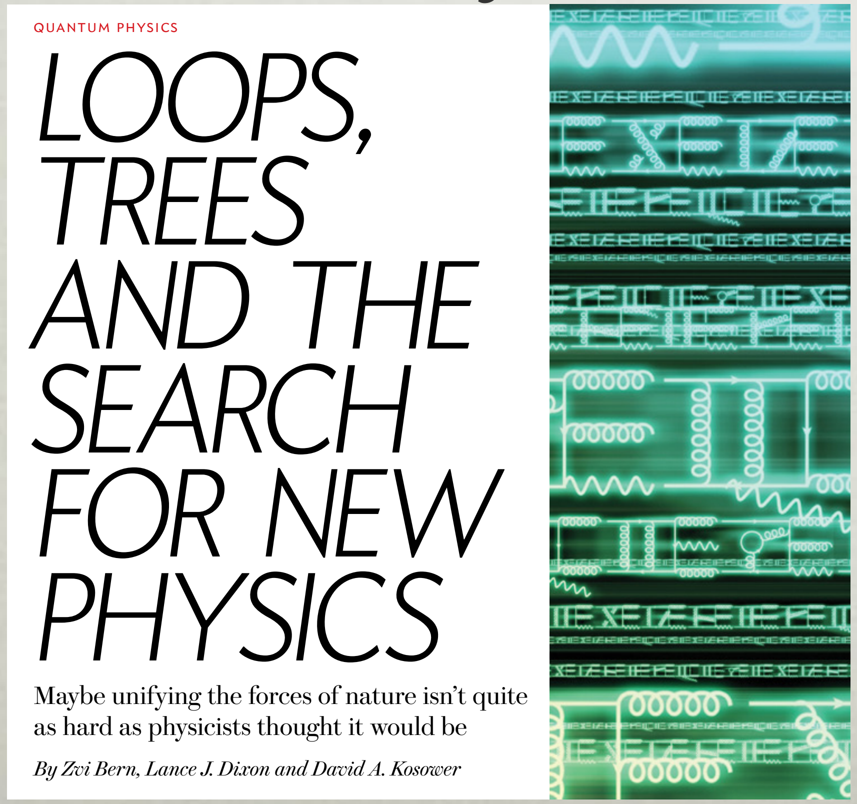
[Dirac (1933)]
[Feynman; Schwinger; Tomanaga (1947)]
[Petermann (1957)]
[Kinoshita (1990)]

- *the most precisely tested idea in all of science!*

Explosions of Complexity

- While ultimately correct, the Feynman expansion renders *all but the* ***most trivial*** predictions—
    - **involving the fewest particles**, at the
    - **lowest orders of perturbation**—

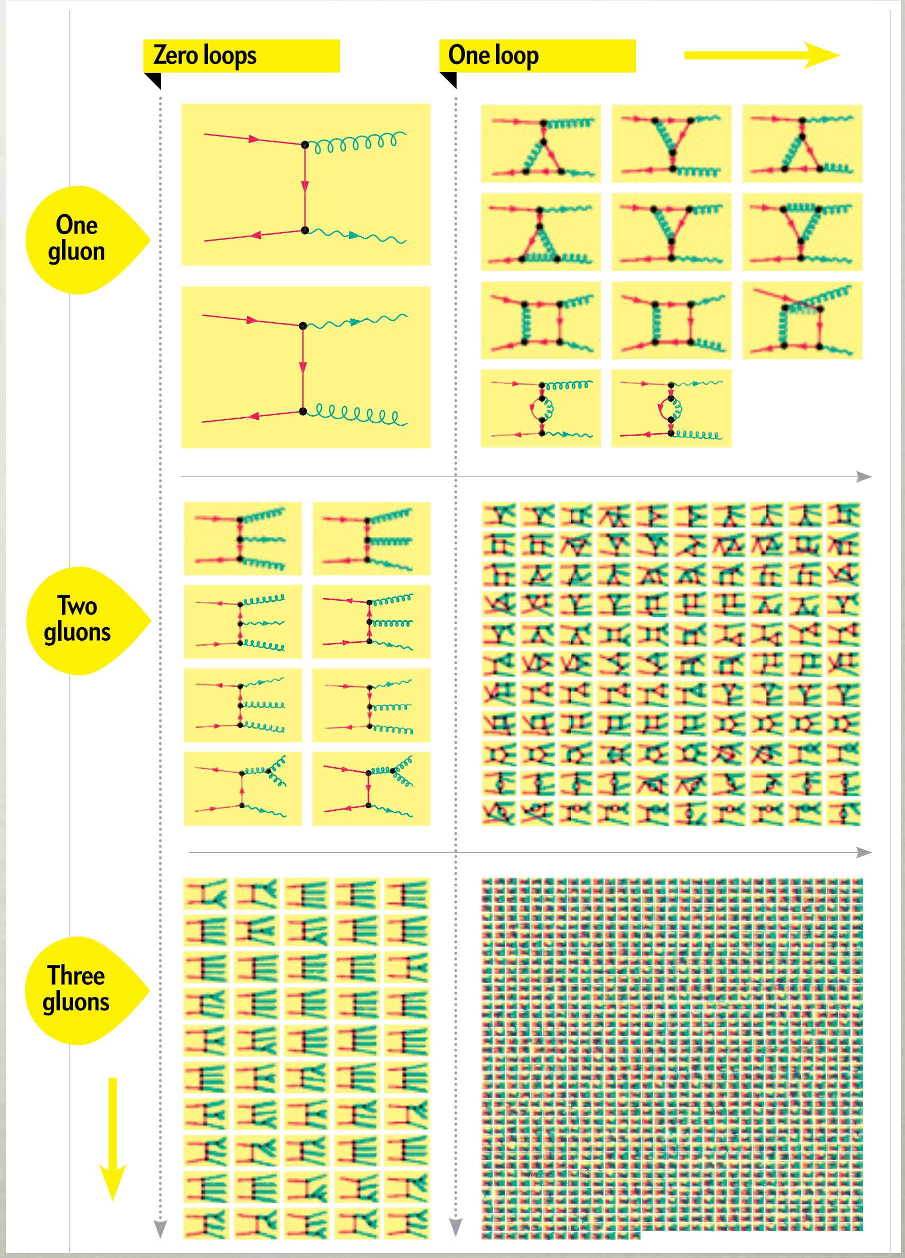
  computationally ***intractable*** or theoretically ***inscrutable***


QUANTUM PHYSICS

LOOPS, TREES AND THE SEARCH FOR NEW PHYSICS

Maybe unifying the forces of nature isn't quite as hard as physicists thought it would be

By Zvi Bern, Lance J. Dixon and David A. Kosower



[Bern, Dixon, Kosower, *Scientific American* (2012)]

Needs (Once) Beyond Our Reach

- Background amplitudes **crucial** for *e.g.* colliders

Supercollider physics [*Rev.Mod.Phys*. **56** (1984)]

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

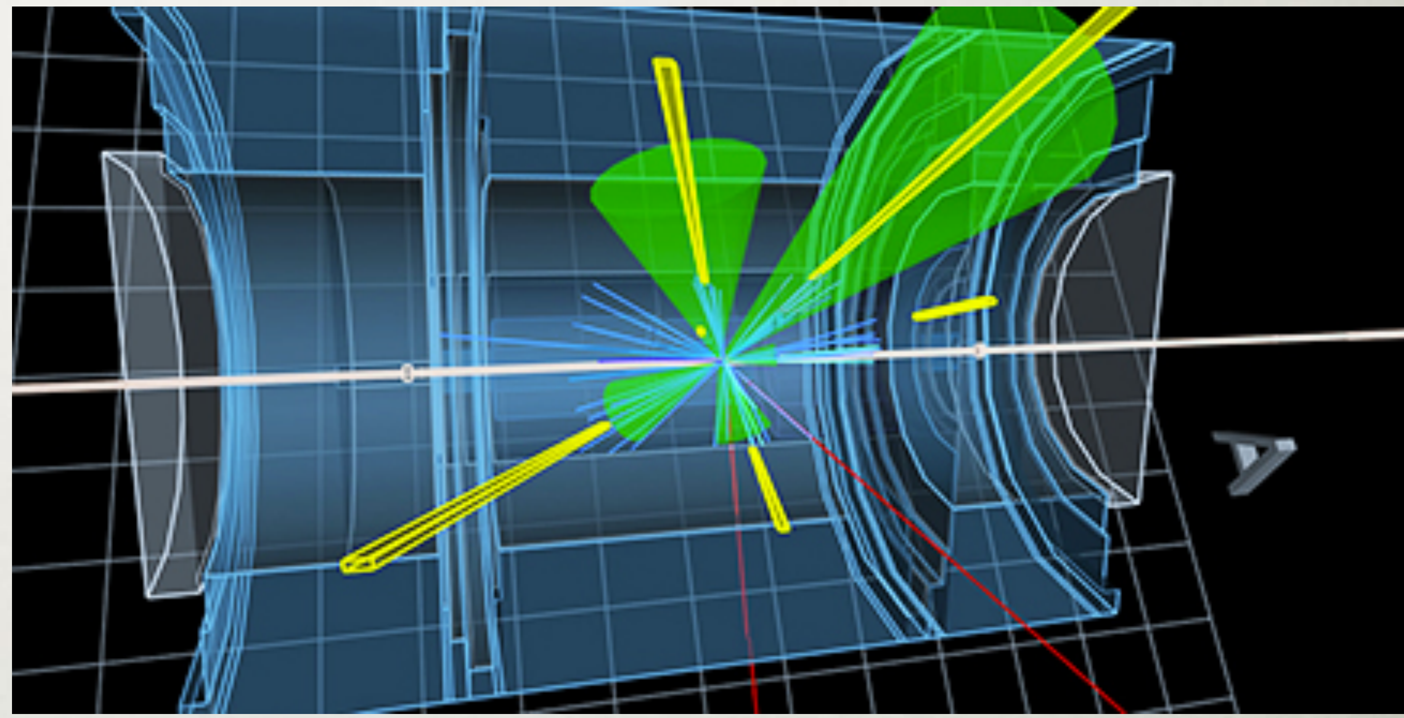
K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.



- Once considered ***computationally intractable***

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Needs (Once) Beyond Our Reach

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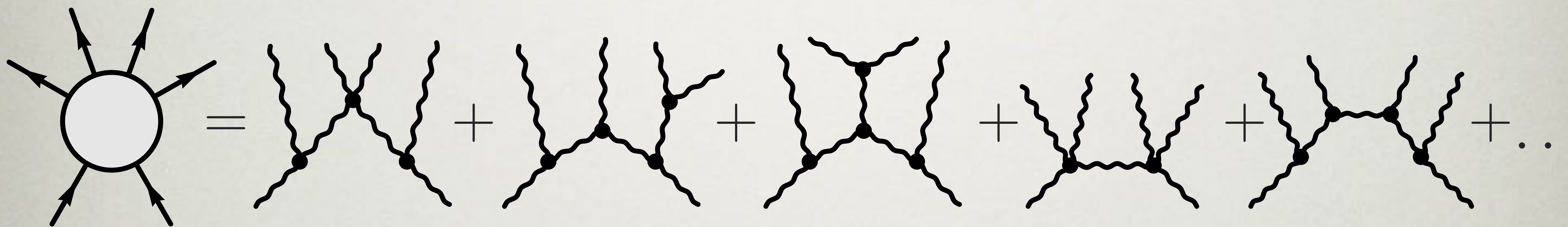

220 diagrams — thousands of terms

- Once considered ***computationally intractable***

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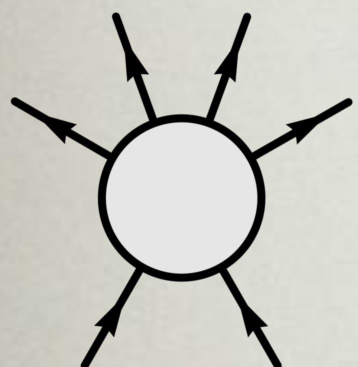

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON–GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[*Nucl.Phys*. **B269** (1985)]

- Once considered ***computationally intractable***

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

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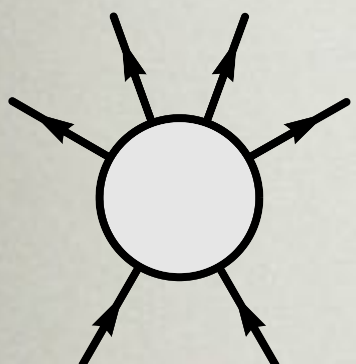

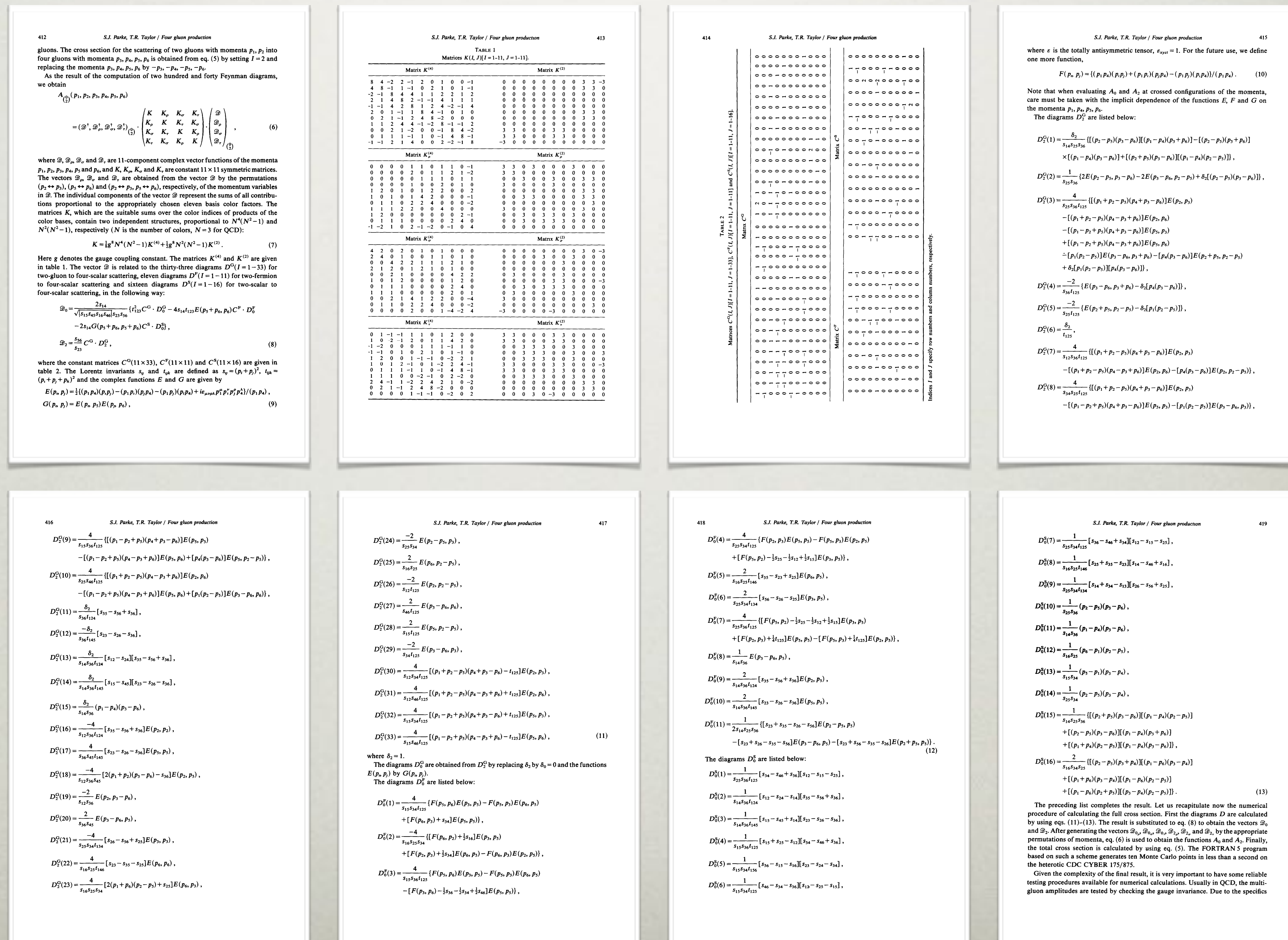

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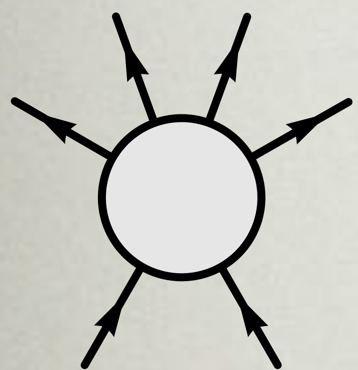
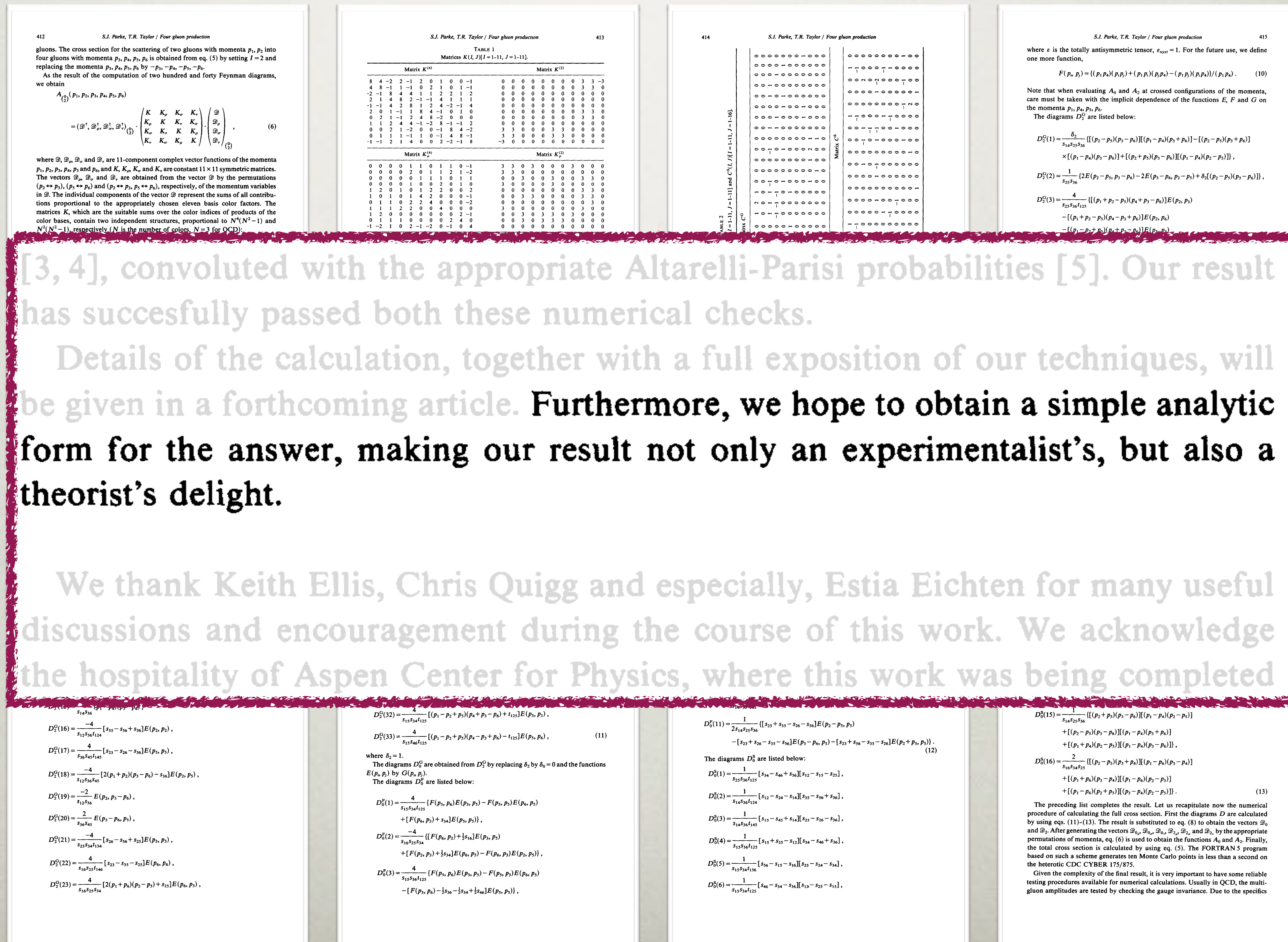




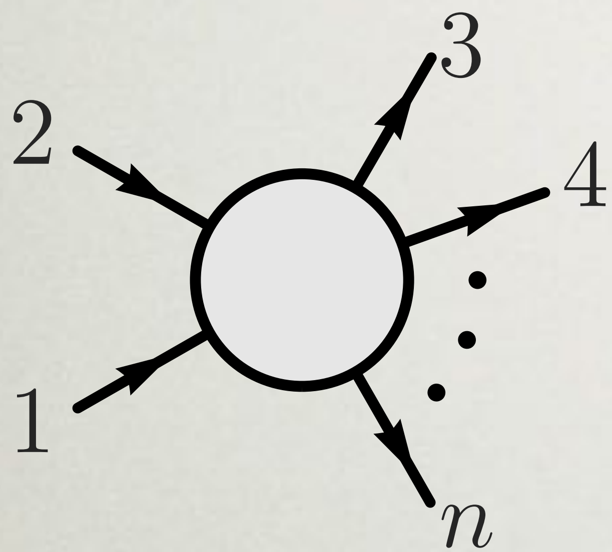
[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. **Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.**

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed

Discovery of Shocking Simplicity

- Within six months, Parke-Taylor stumbled on a simple guess—unquestionably a *theorist's delight*:



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,\cdots\,\langle n\,1\rangle}$$

Amplitude for *n*-Gluon Scattering [*PRL* **56** (1986)]

Stephen J. Parke and T. R. Taylor

Fermi National Accelerator Laboratory, Batavia, Illinois 60510

(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

$$p_a^\mu \equiv \sigma^\mu_{\alpha\dot{\alpha}}\lambda_a^\alpha\widetilde{\lambda}_a^{\dot{\alpha}}$$

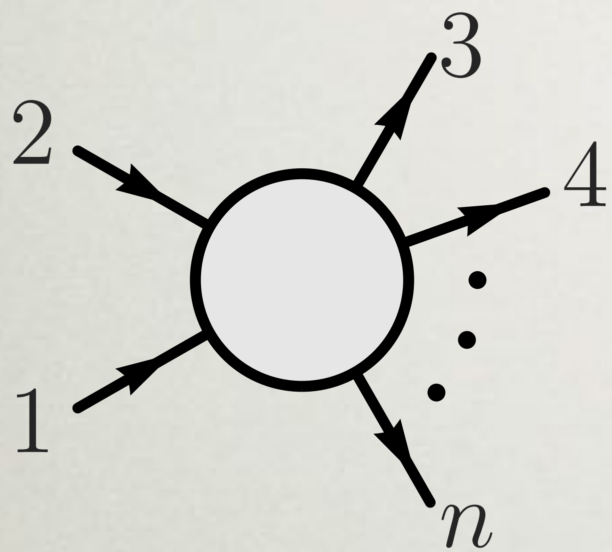
$$\langle a\,b\rangle \equiv \det(\lambda_a, \lambda_b)$$

$$[a\,b] \equiv \det(\widetilde{\lambda}_a, \widetilde{\lambda}_b)$$

[van der Waerden (1929)]

Perturbations of Parke-Taylor

- What about beyond the leading order?



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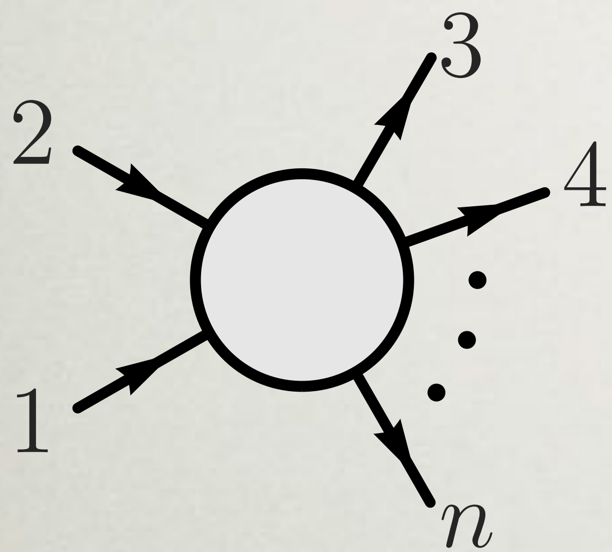


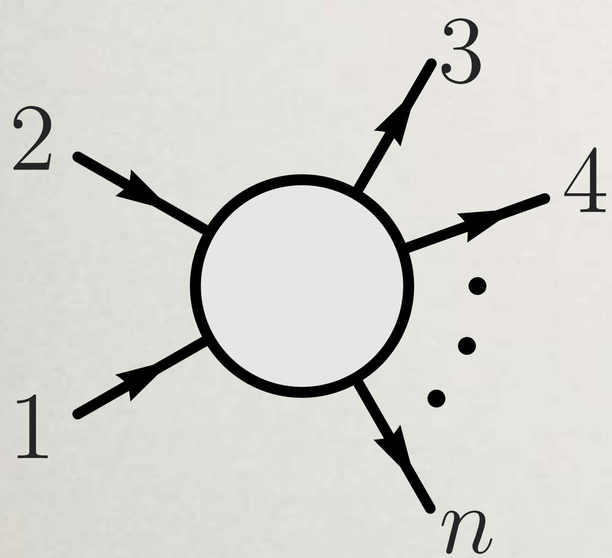
$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,\cdots\,\langle n\,1\rangle} \times$$

$$\Big\{ 1 + \ldots \qquad\qquad\qquad\qquad \Big\}$$

Perturbations of Parke-Taylor

- What about beyond the leading order?

[Bern, Dixon, Dunbar, Kosower (1994)]



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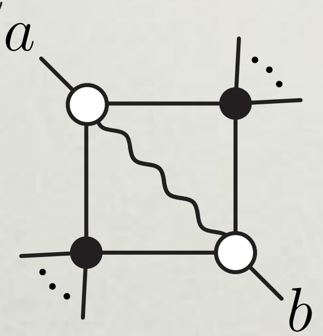
$$\left\{ 1 + \sum_{a<b} \right.$$

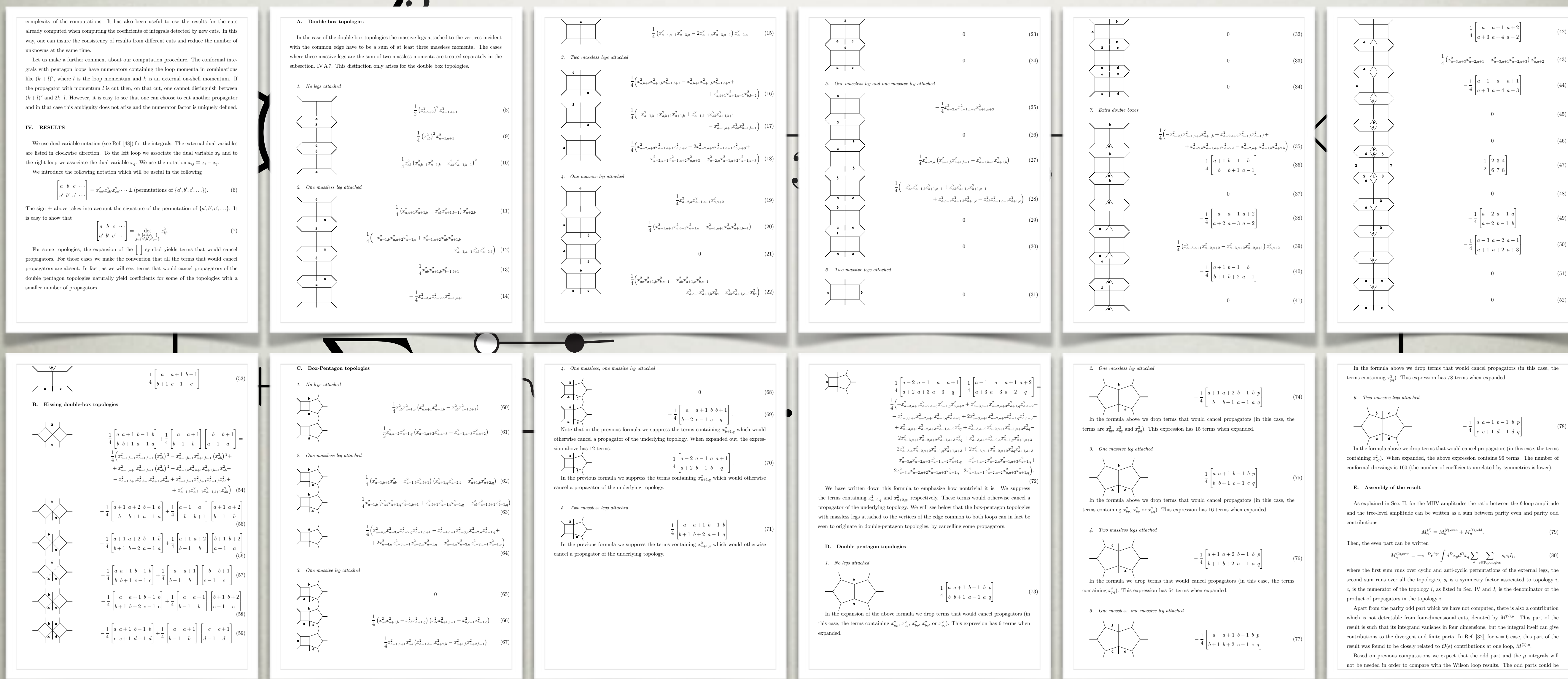


$$\left. + \ldots \right\}$$

Perturbations of Parke-Taylor

- What about beyond the leading order?

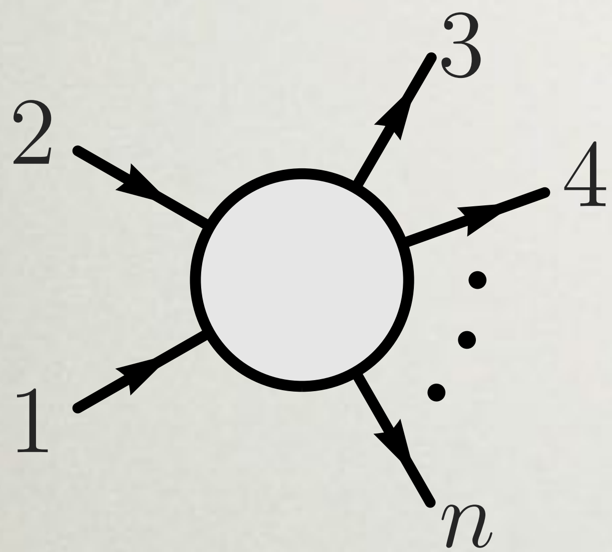
[Vergu (2009)]



Perturbations of Parke-Taylor

- What about beyond the leading order?

[Arkani-Hamed, **JB**, Cachazo, Trnka (2010)]



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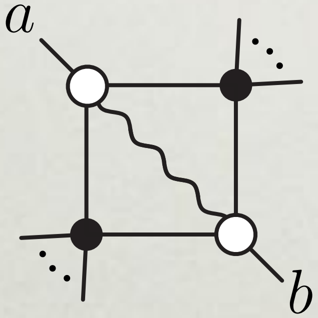
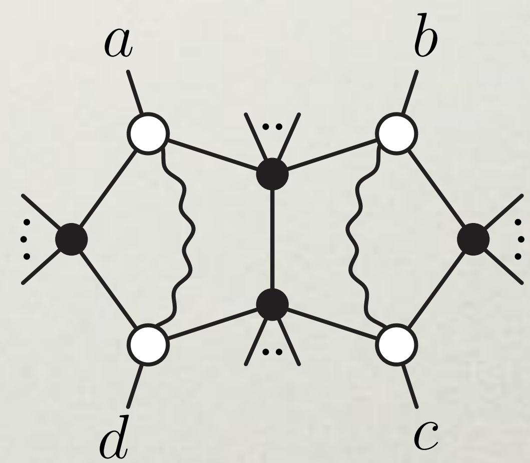


$$+ \sum_{a<b<c<d}$$



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Perturbations of Parke-Taylor

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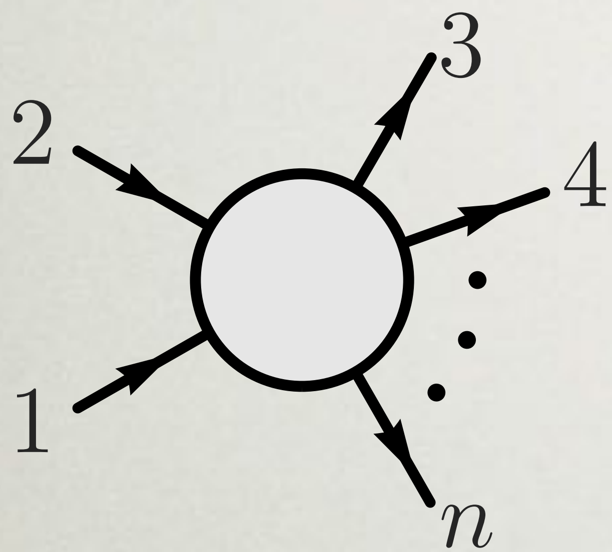

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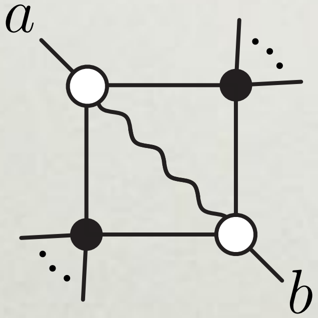
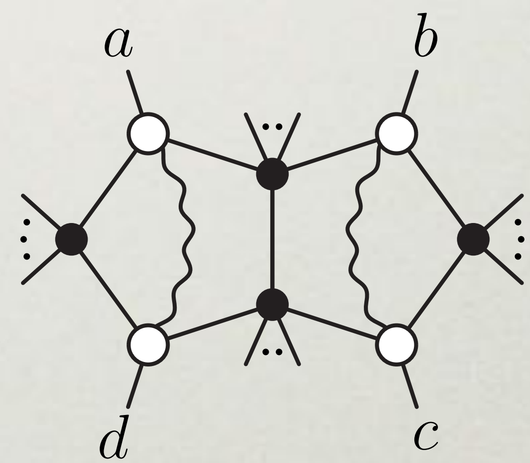
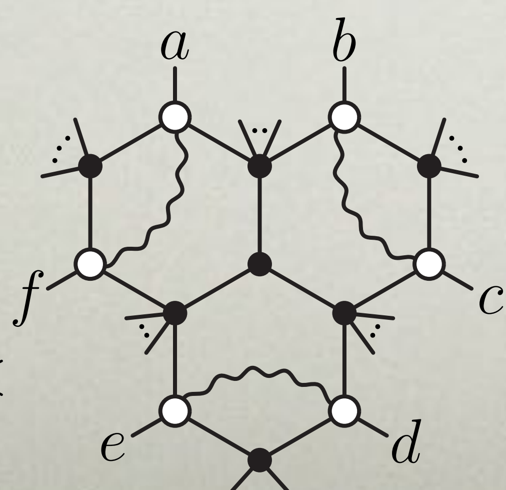


$$+ \sum_{a<b<c<d}$$



$$+ \sum_{\substack{a<b\leq c<\\<d\leq e<f}}$$



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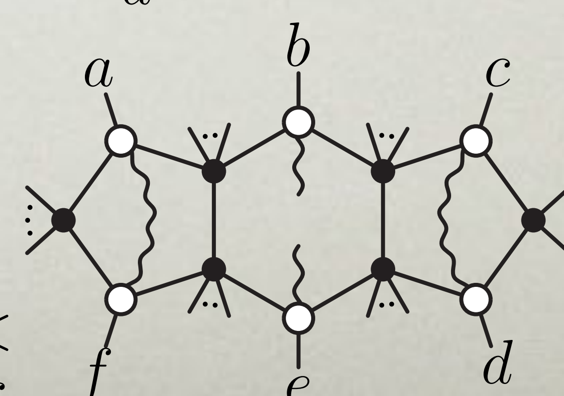
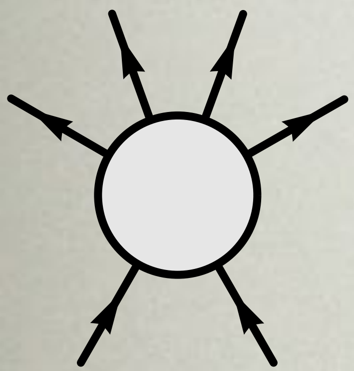
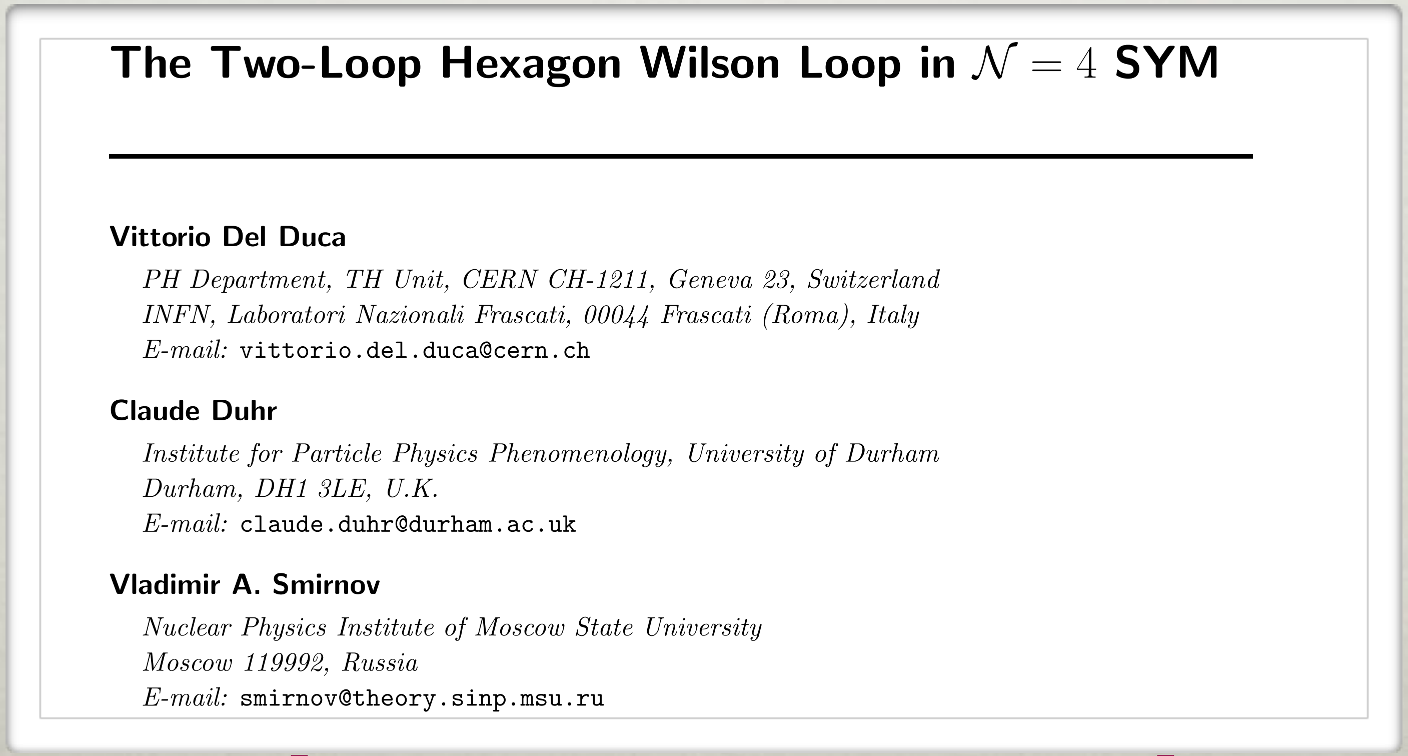


$$+ \ldots\Bigg\}$$

What About After Integration?

- Integrate the Parke-Taylor 2-to-4 amplitude in sYM
  - divergences exponentiate, leaving a finite *remainder*
- *Heroically* computed by Del Duca, Duhr, Smirnov in 2010, in terms of 'Goncharov' polylogarithms





The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

[Del Duca, Duhr, Smirnov (2010)]

What About After Integration?

- Integrate the Parke-Taylor 2-to-4 amplitude in sYM
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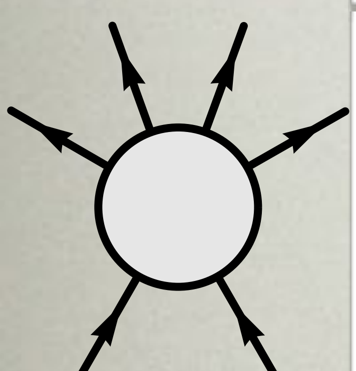
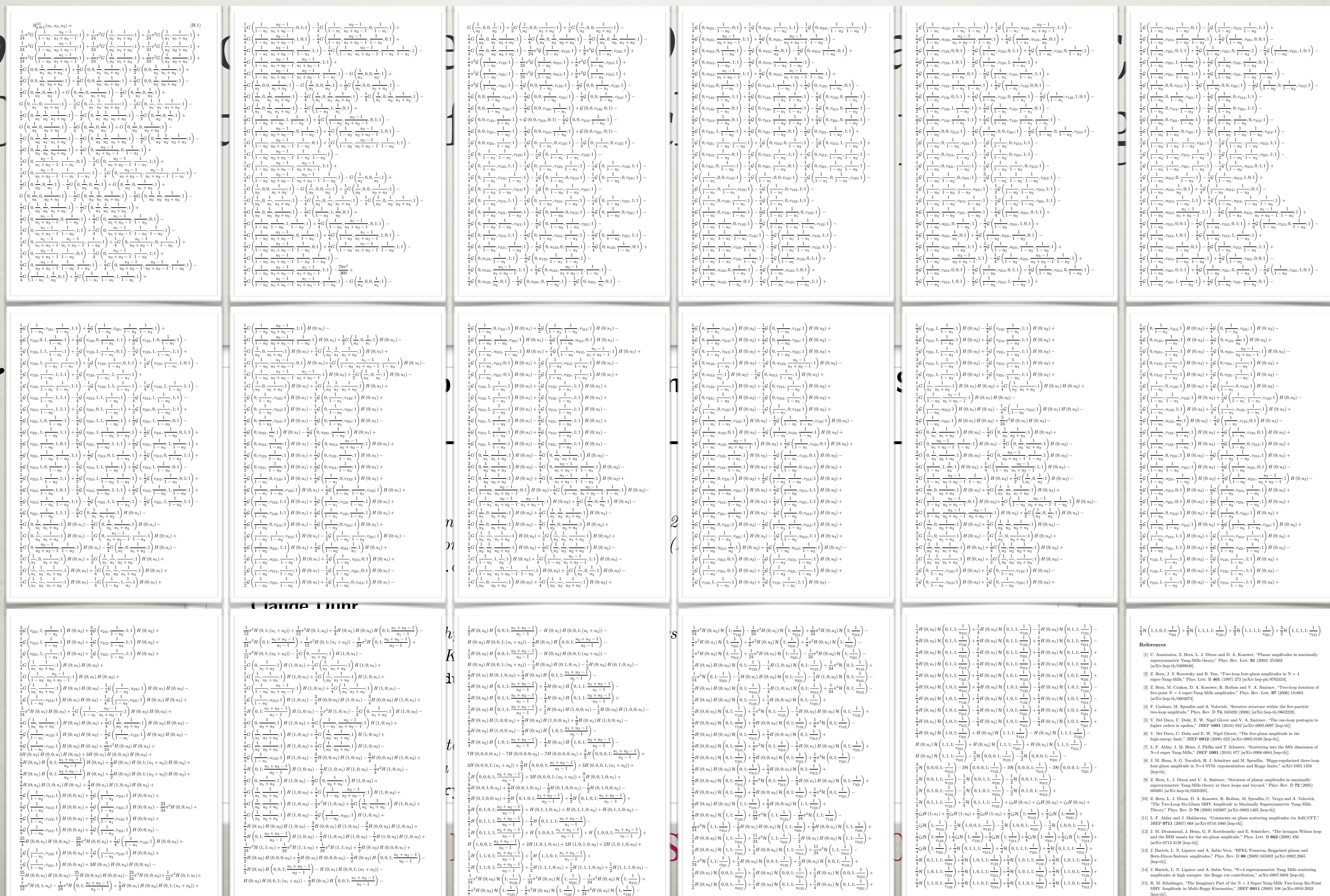
- *Hero* … ov
  in 20 … ns





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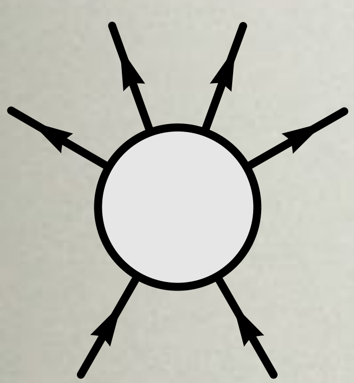


Classical Polylogarithms for Amplitudes and Wilson Loops

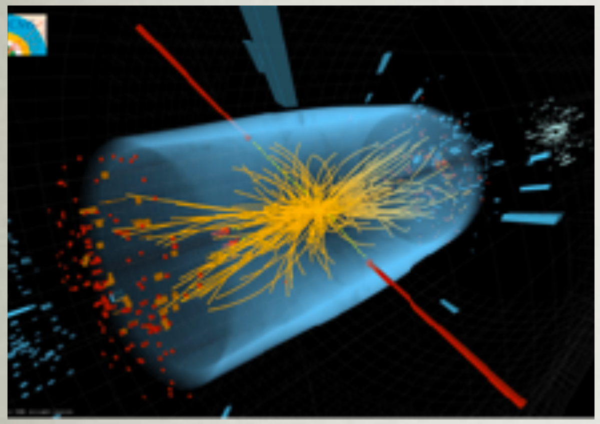
A. B. Goncharov,[1] M. Spradlin,[2] C. Vergu,[2] and A. Volovich[2]

[1] *Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2] *Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives. [Goncharov, Spradlin, Vergu, Volovich (2010)]

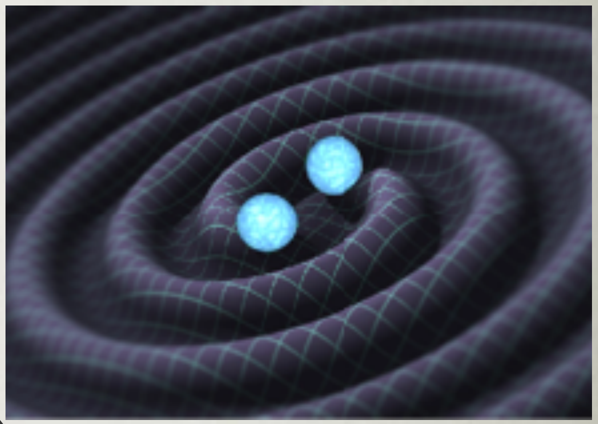
$$R(u_1,u_2,u_3)=\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{J^4}{24}+\frac{1}{2}\zeta_2(J^2+\zeta_2)$$

Amplitudes: a Virtuous Cycle



Compute Something

beyond the reach of recent imagination



Exploit Simplicity

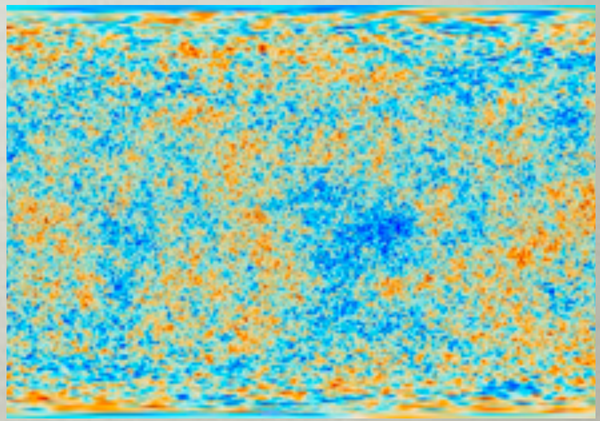
to build **more powerful** computational technology

Discover Simplicity

beyond expectations

Understand Why

study, understand, **explain** it, & explore its consequences

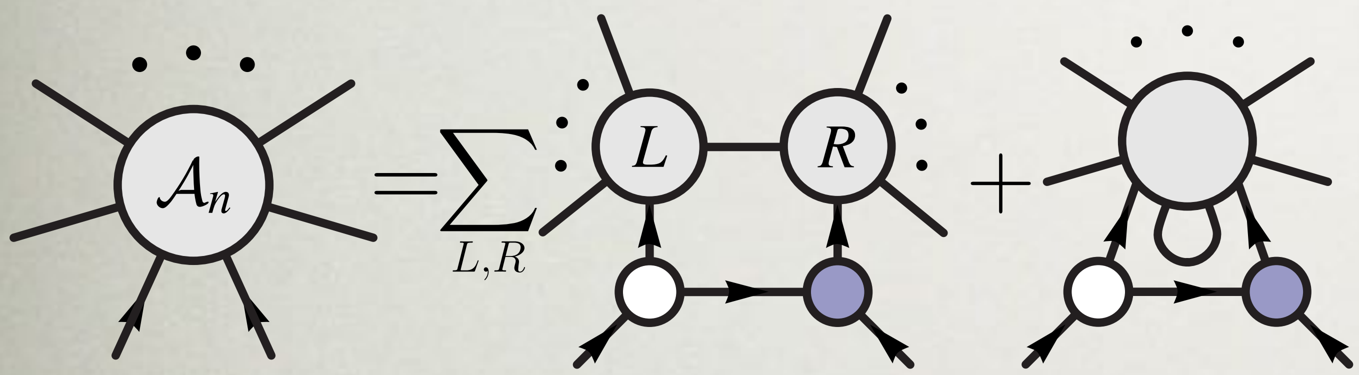


Constructing Integrands for Loop Amplitudes (constructively)

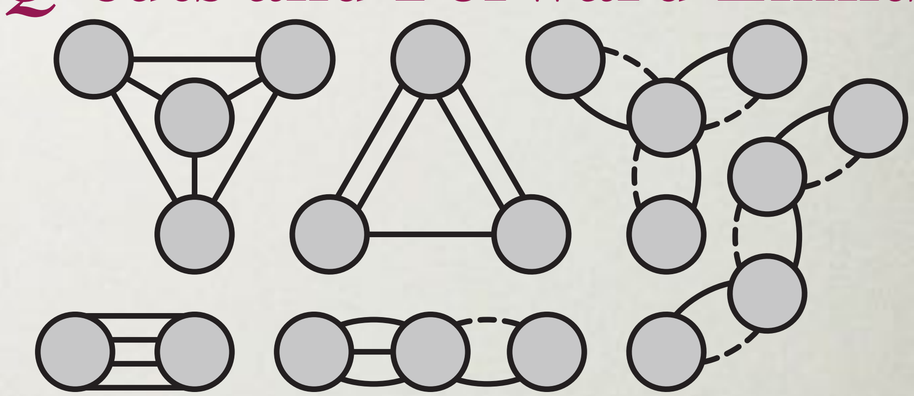
Novel Representations of Integrands

- **Powerful new tools** now exist for ***understanding*** and ***computing*** integrands in perturbation theory

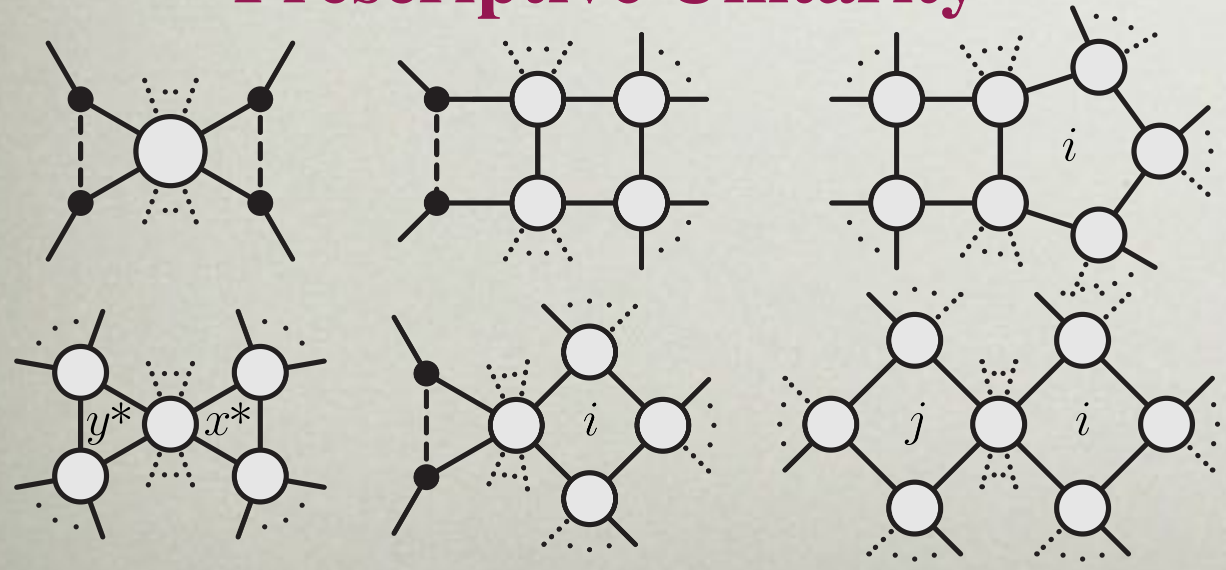
Recursion Relations



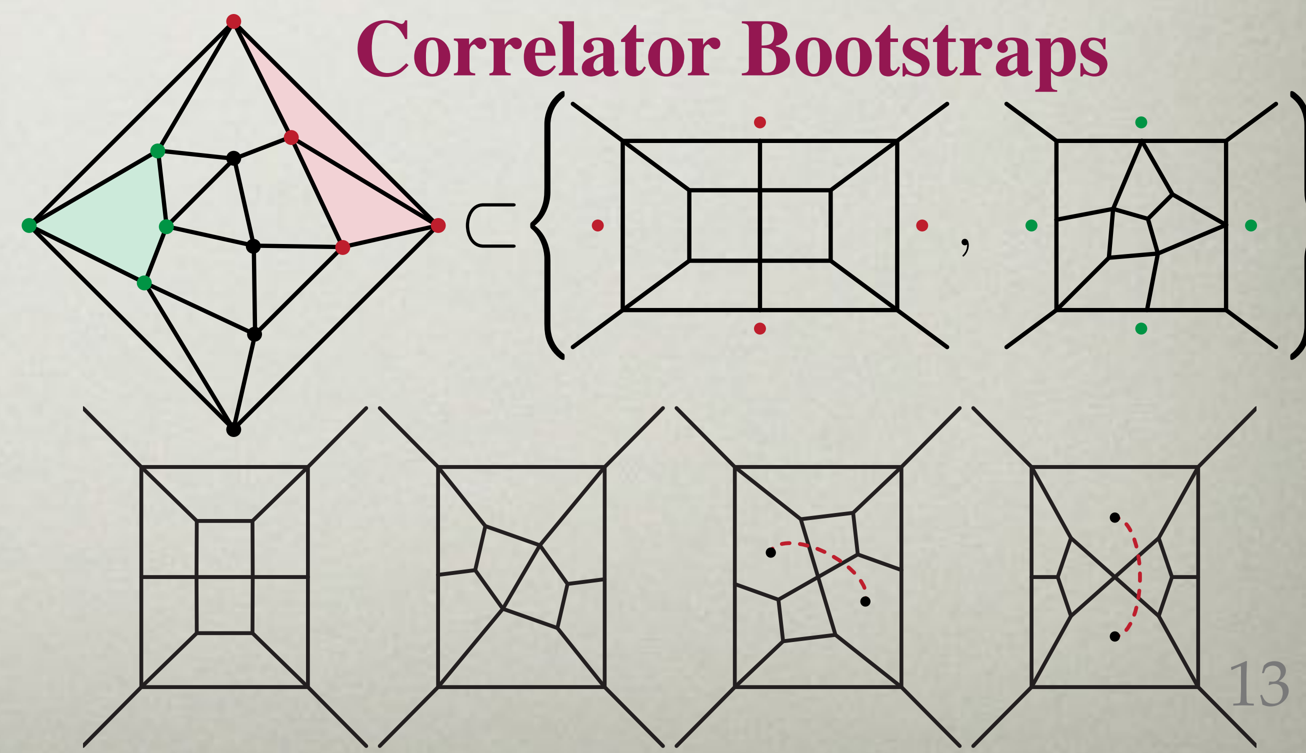
Q-cuts and Forward Limits



Prescriptive Unitarity



Correlator Bootstraps



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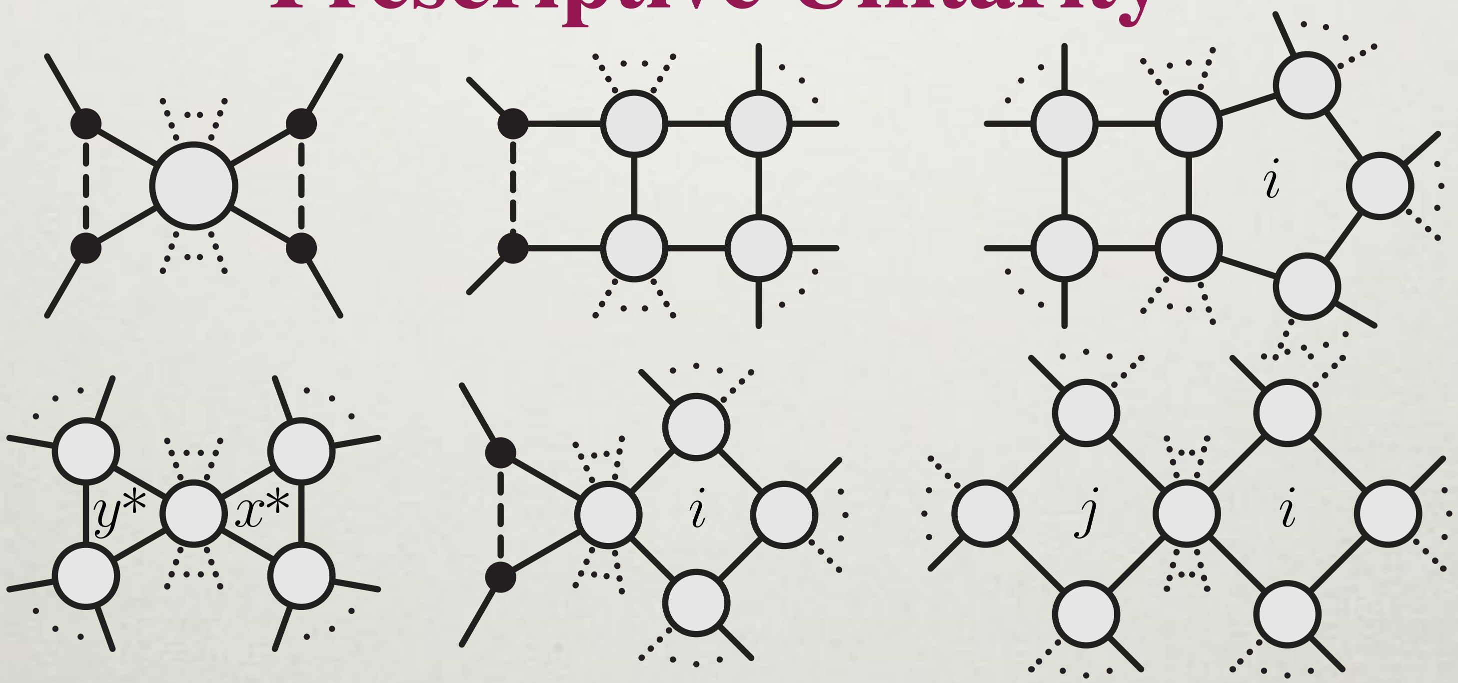
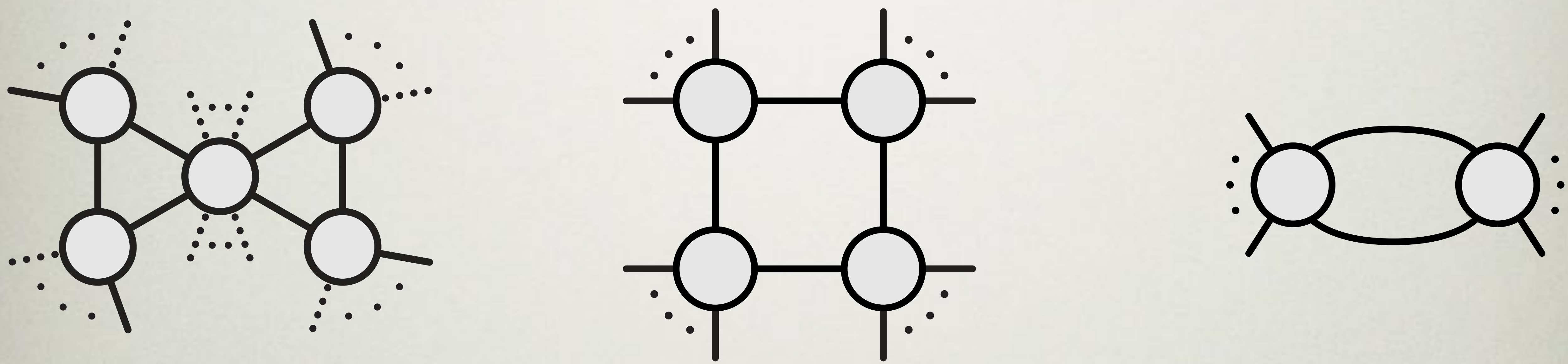


The Cuts of Loop Amplitudes

- **On-Shell Functions:** scattering amplitudes, and functions built thereof—as *networks* of amplitudes



- **Locality:** amplitudes independent, so multiplied
- **Unitarity:** internal particles unseen, so summed

$$f_\Gamma \equiv \prod_{i\in I}\Big(\sum_{h_i,c_i}\int d^{d-1}\text{LIPS}_i\Big)\prod_{v\in V}\mathcal{A}_v$$

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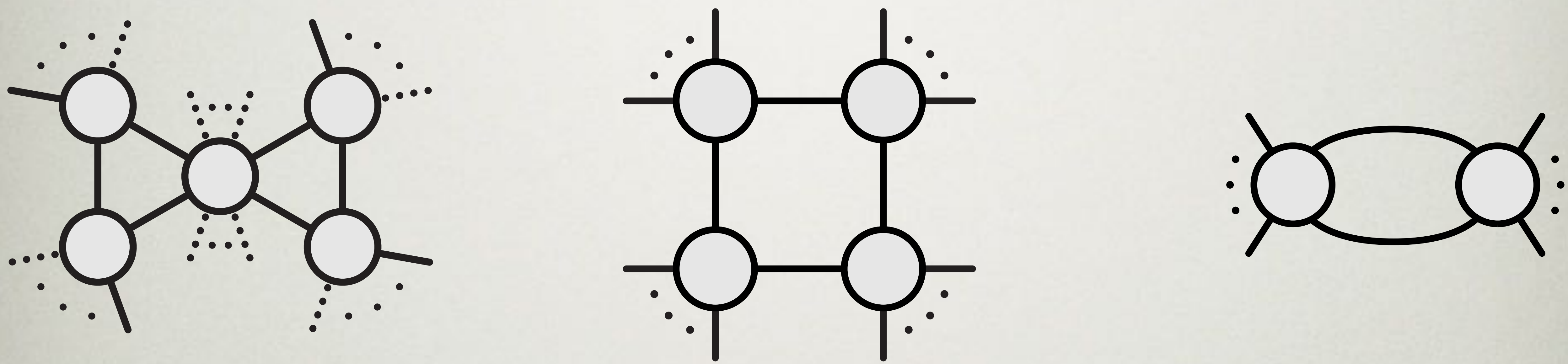


- defined for all ***all*** quantum field theories—*exclusively* in terms of physical (observable) states
- can be used to reconstruct ***all*** loop amplitudes

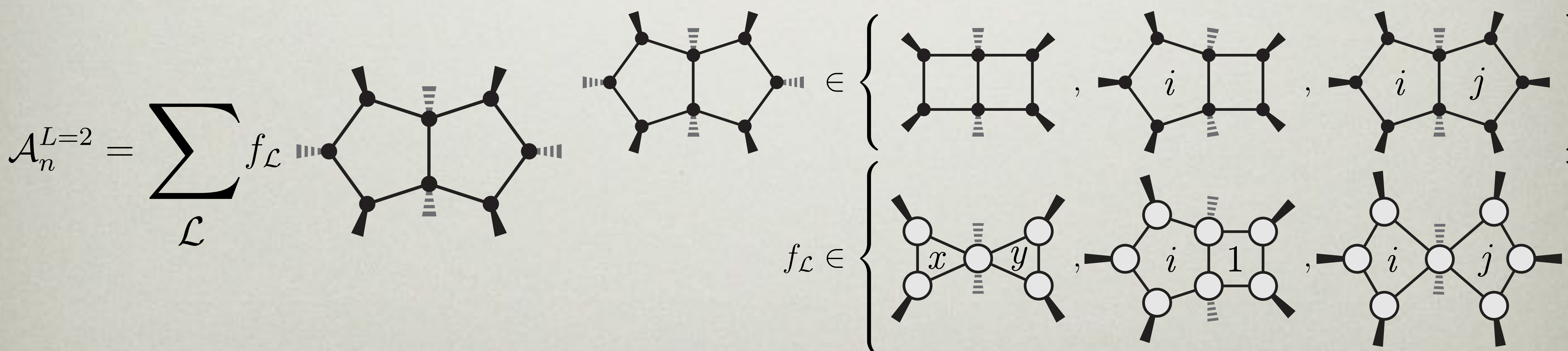
$$f_\Gamma \equiv \prod_{i\in I}\Big(\sum_{h_i,c_i}\int d^{d-1}\mathrm{LIPS}_i\Big)\prod_{v\in V}\mathcal{A}_v$$

General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; …]

- **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum a_i\, \mathcal{I}_i^L \qquad \mathcal{I}_i \in \mathfrak{B}$$



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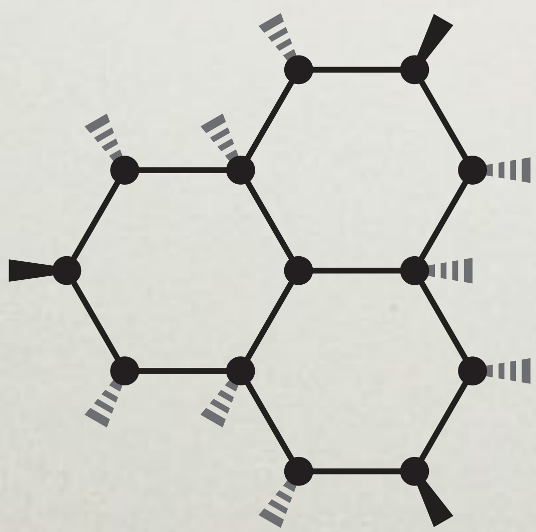
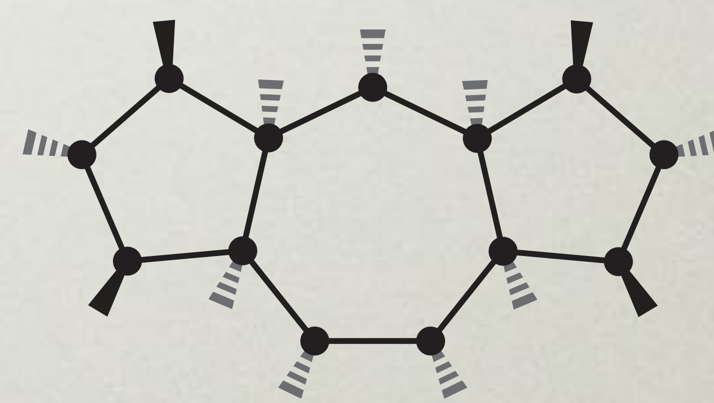
$$\mathcal{A}_n^{L=3} = \sum_{\mathcal{W}} f_{\mathcal{W}}$$  $$+ \sum_{\mathcal{L}} f_{\mathcal{L}}$$ 

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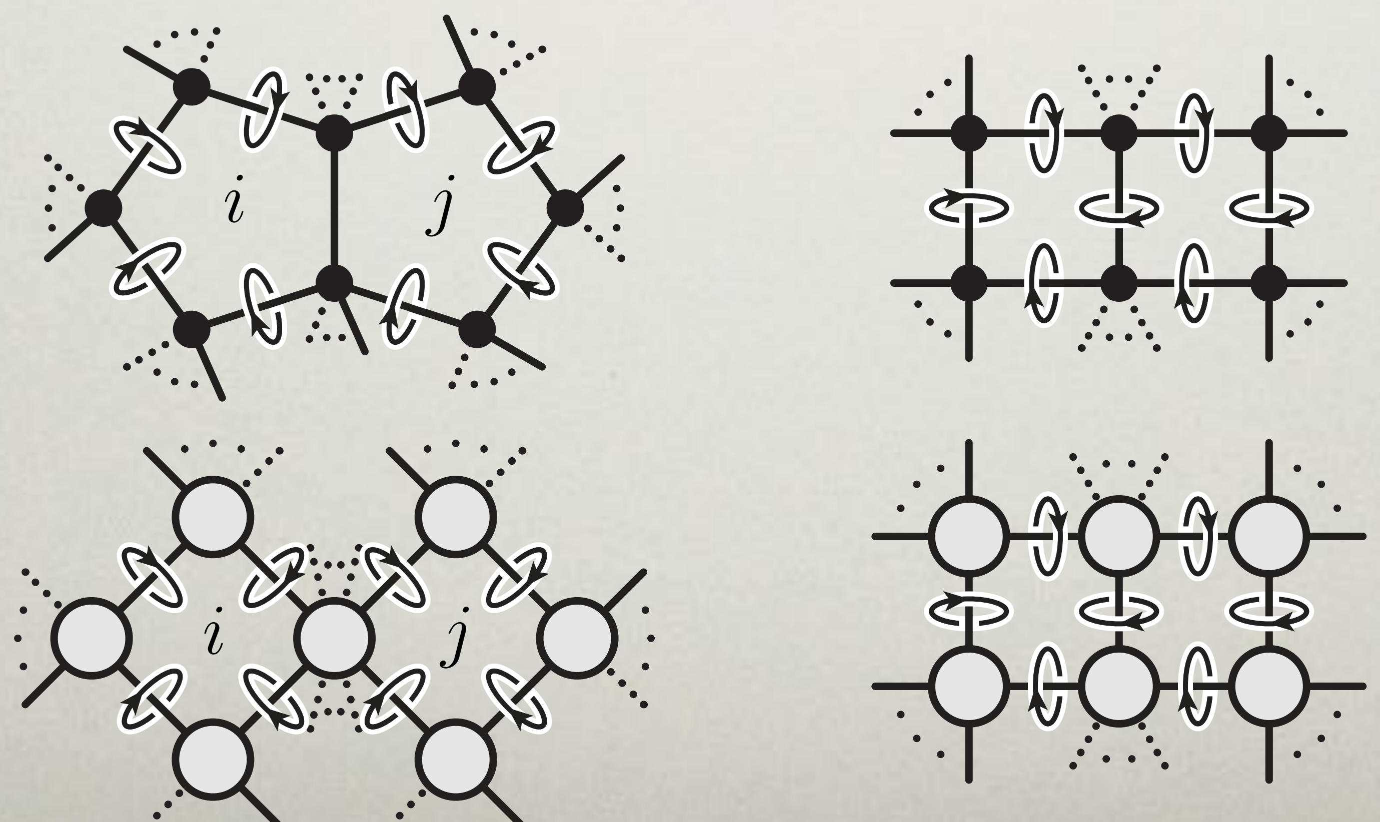

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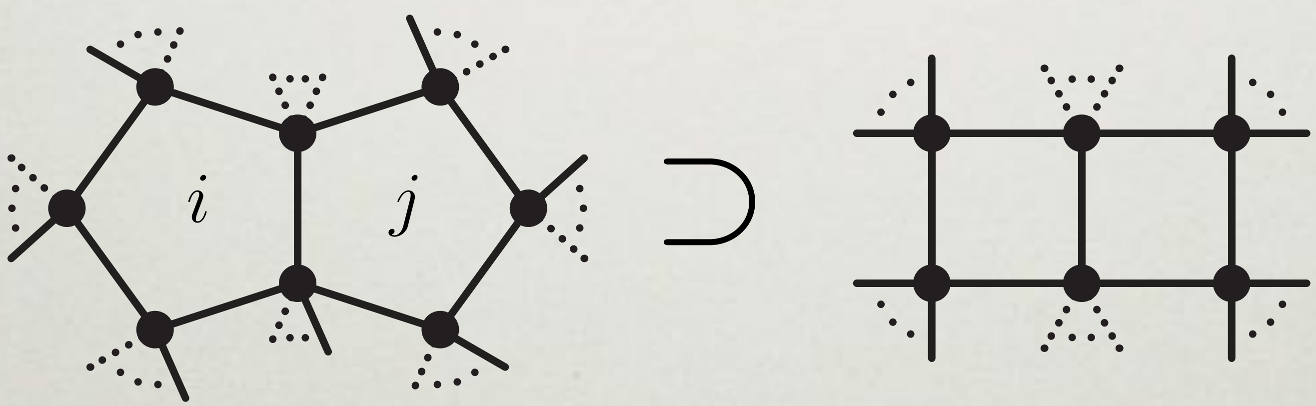

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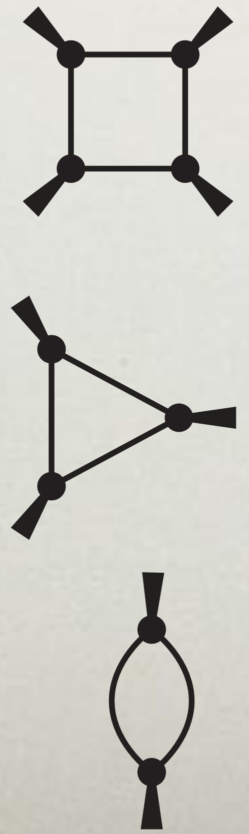

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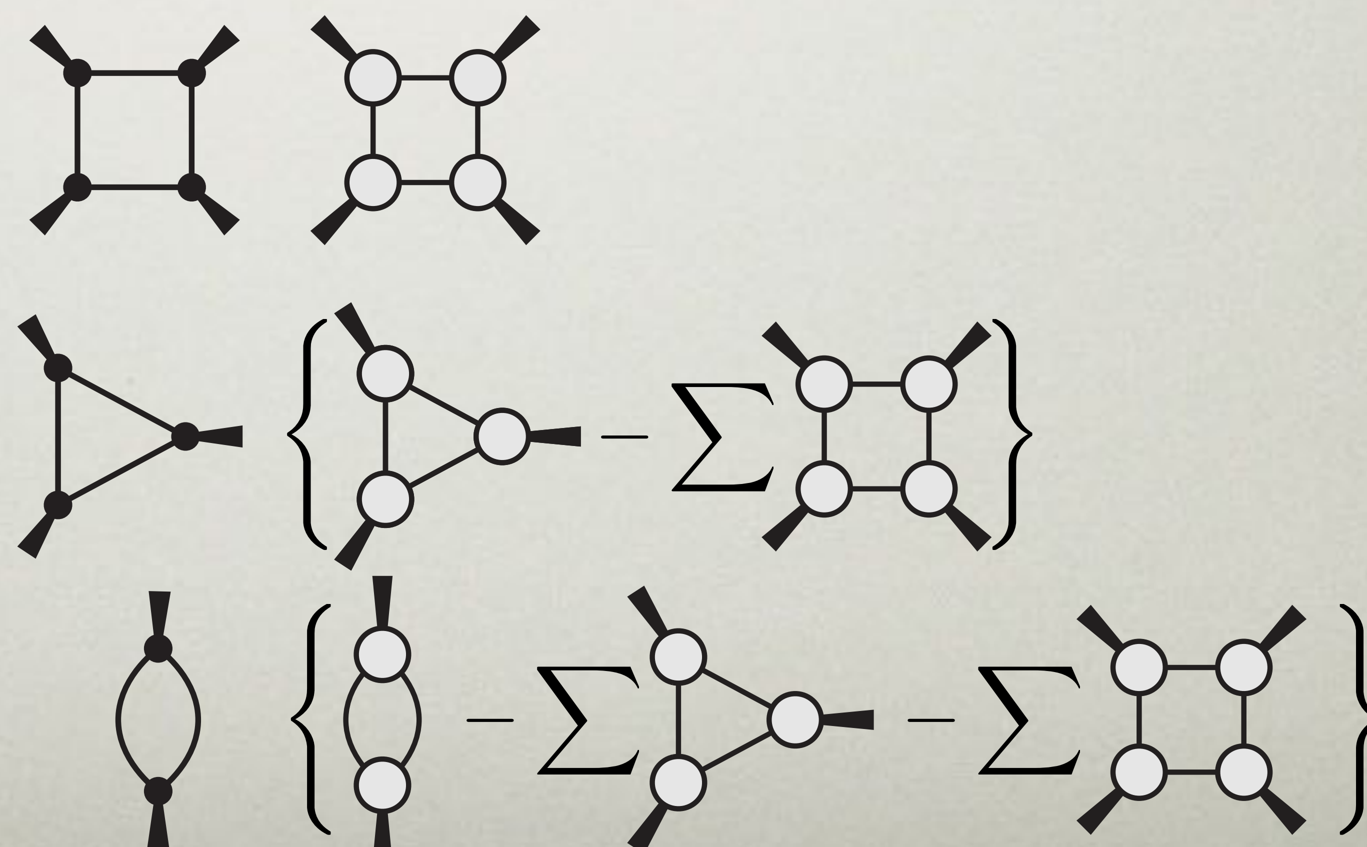

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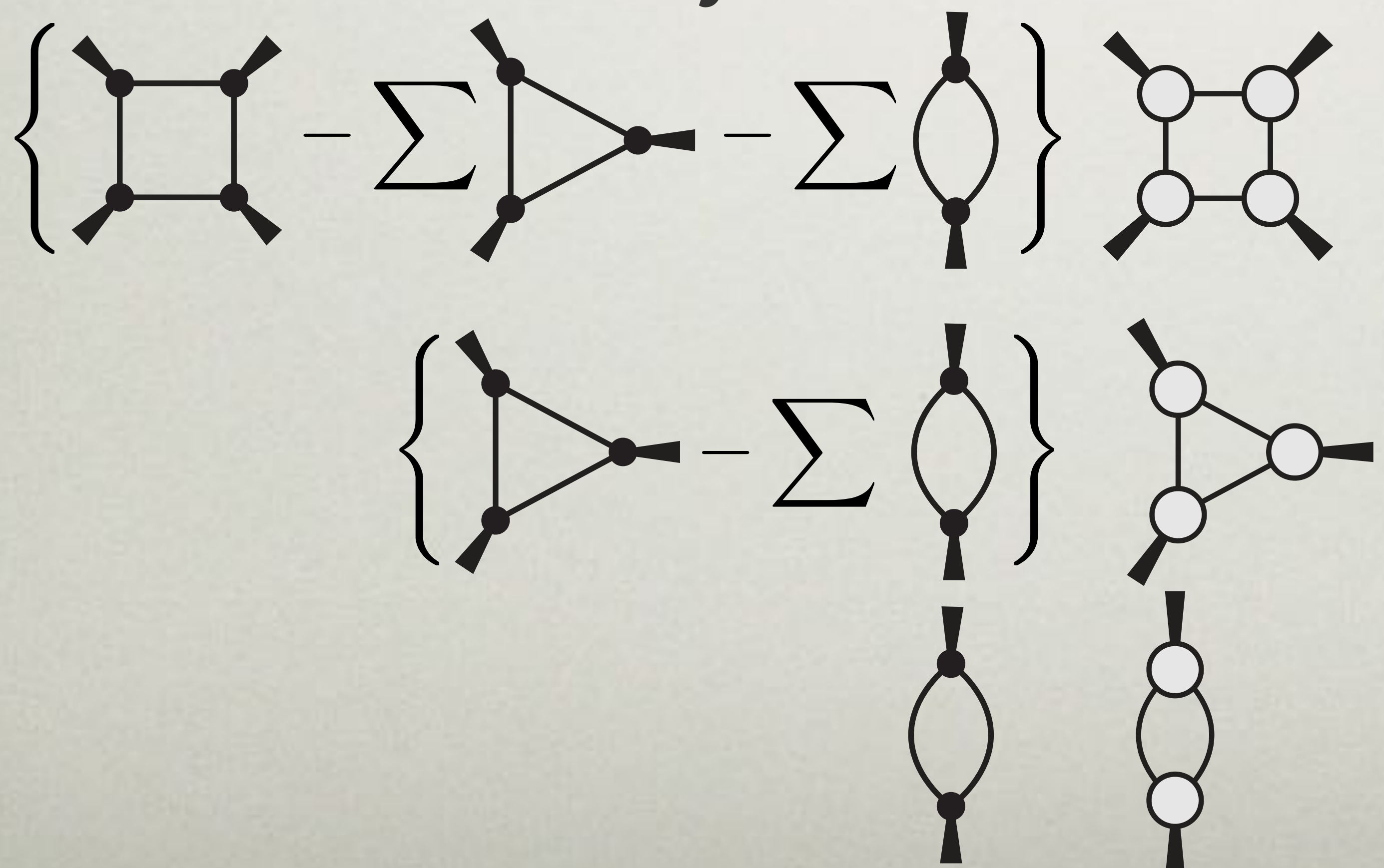

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¿What makes a basis a ***good basis***?

A Basis "Big Enough" for Integrands in the Standard Model

Building a Basis 'Big Enough'

- WLOG: write loop-dependent numerators as **sums** of **products** of (**translates** of) ***inverse propagators***:

$$(\ell|Q)_m := (\ell+Q)^2 - m^2 + i\epsilon \qquad (\ell|Q) := (\ell|Q)_{m=0}$$

$$[\ell] := \operatorname{span}_{Q}\{(\ell|Q)\} \qquad \operatorname{rank}\big([\ell]\big) = (d+2)$$

$$= \operatorname{span}\{\ell^2, \ell\cdot k_i, 1\} \quad \operatorname{rank}\big([\ell]^k\big) = \binom{d+k}{d} + \binom{d+k-1}{d}$$

$$[\ell]^k := \operatorname{span}_{Q_i}\left\{\prod_{i=1}^{k}(\ell|Q_i)\right\} \qquad [\ell]^0 \subset [\ell]^1 \subset [\ell]^2 \subset \cdots \subset [\ell]^q$$

$$\widehat{[\ell]^q} := [\ell]^q \backslash [\ell]^{q-1} \qquad [\ell]^q = 1 \oplus \widehat{[\ell]^1} \oplus \widehat{[\ell]^2} \oplus \cdots \oplus \widehat{[\ell]^q}$$

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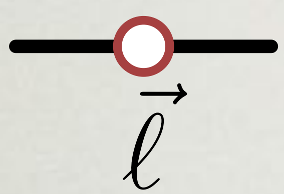
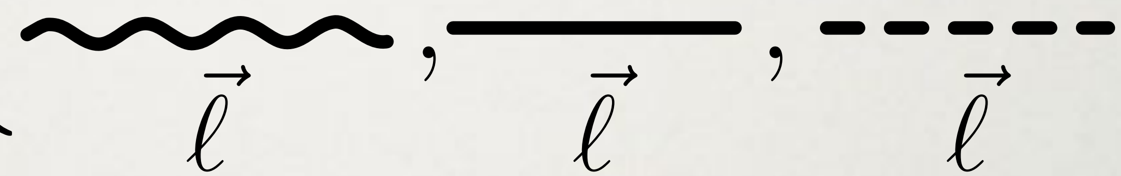
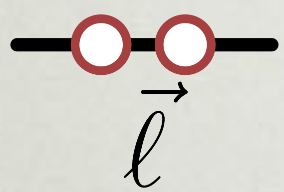
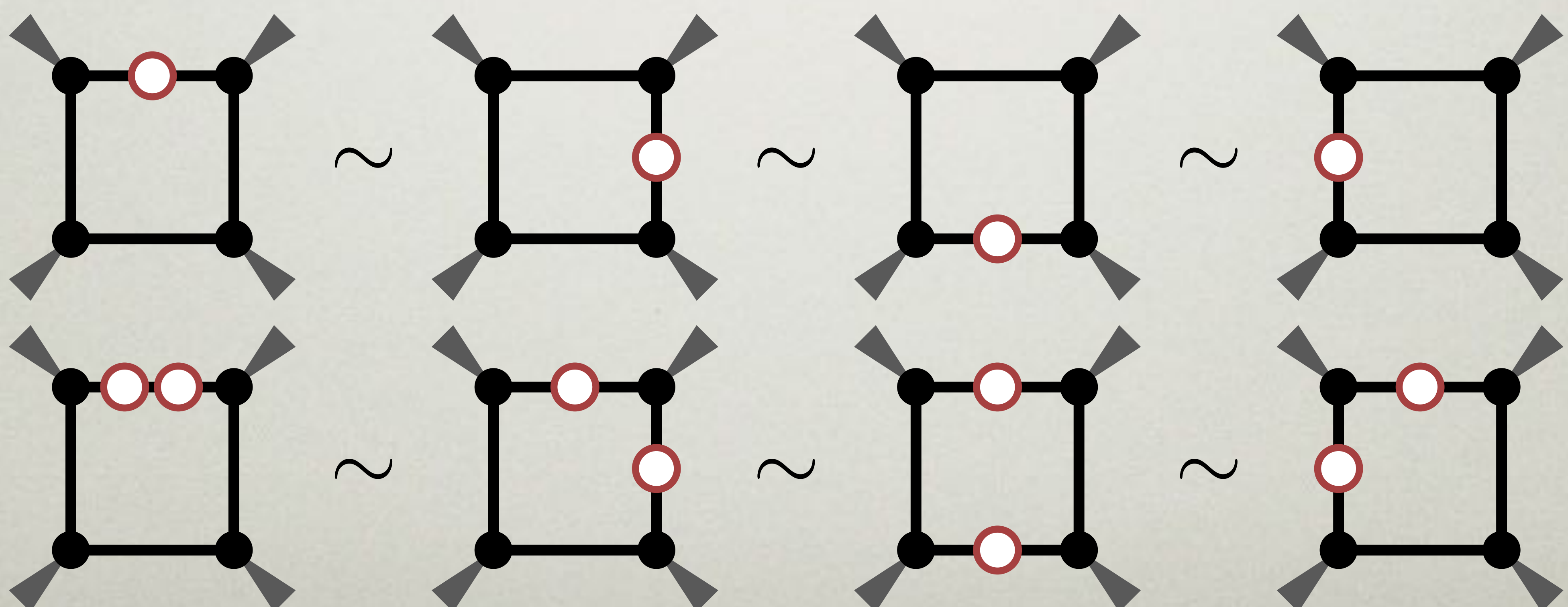
Building a Basis 'Big Enough'

- In terms of these, define a generalized propagator:

 $:= \dfrac{[\ell]}{\ell^2} \supset \Big\{$  $, 1 \Big\}$

 $:= \dfrac{[\ell]^2}{\ell^2}$ (would include gravitons)



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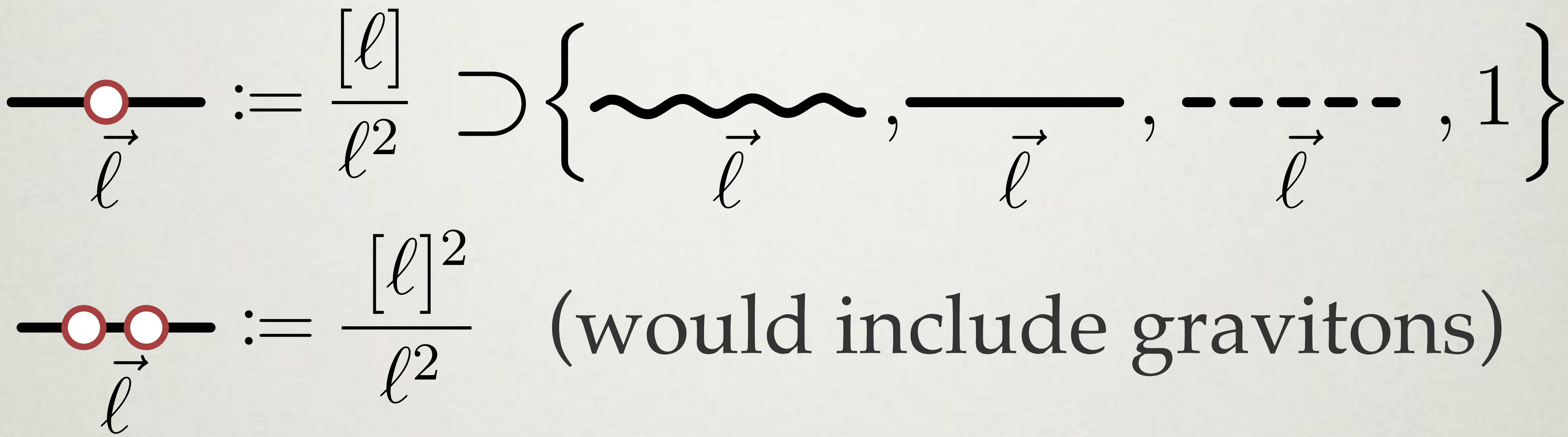

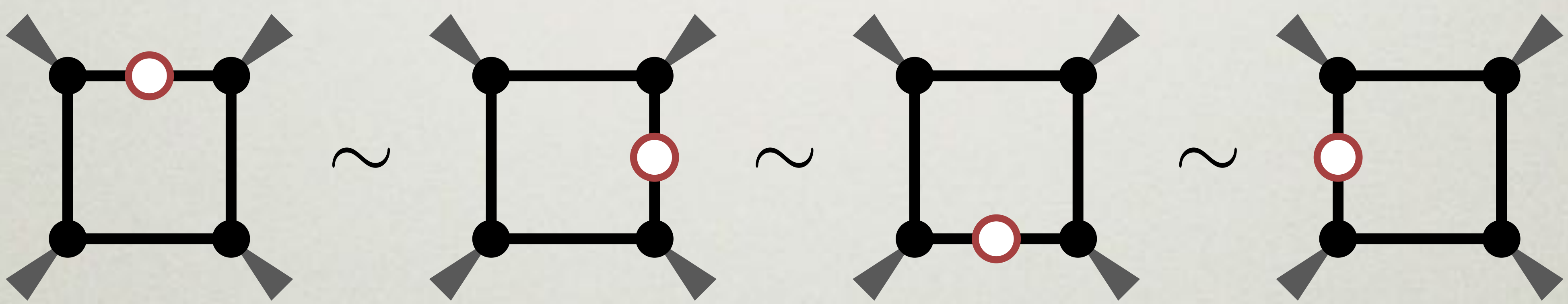

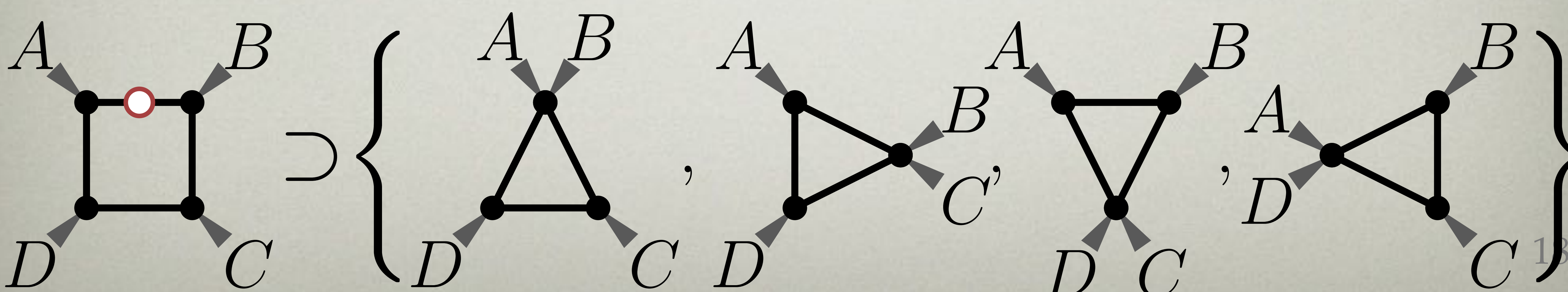

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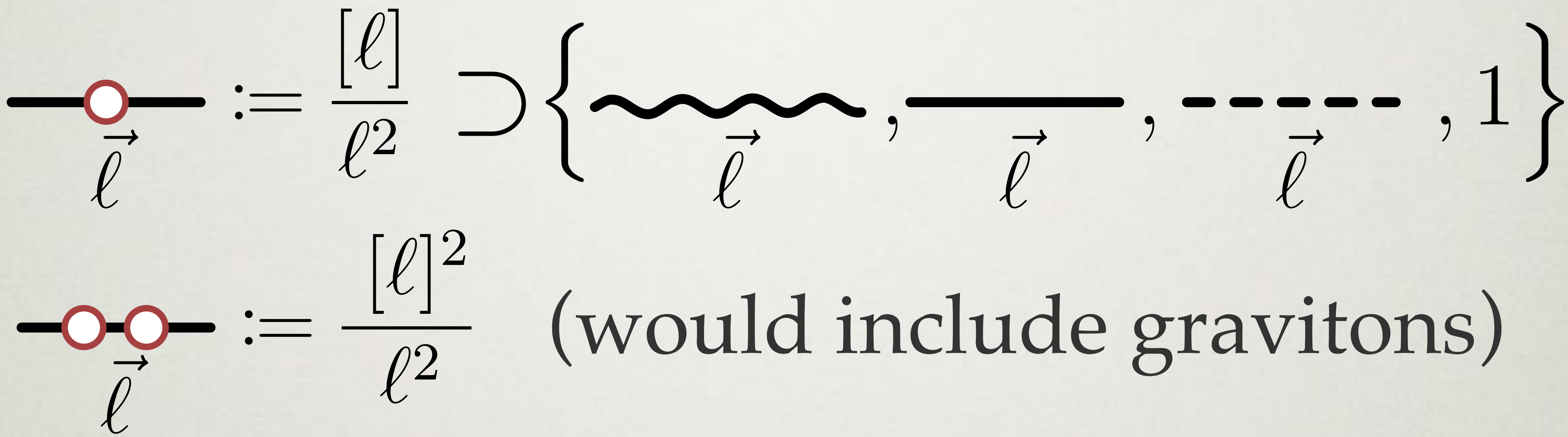
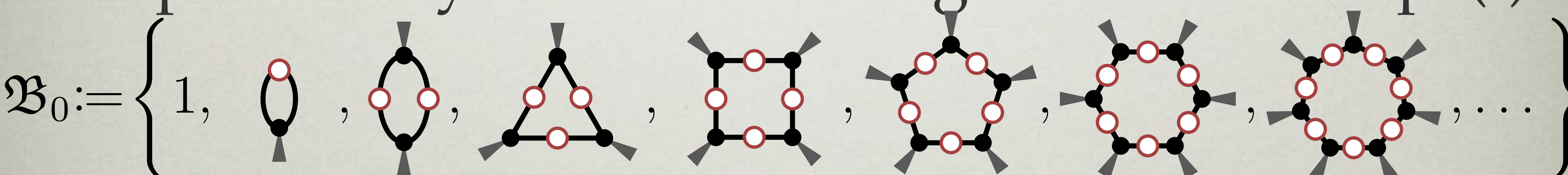
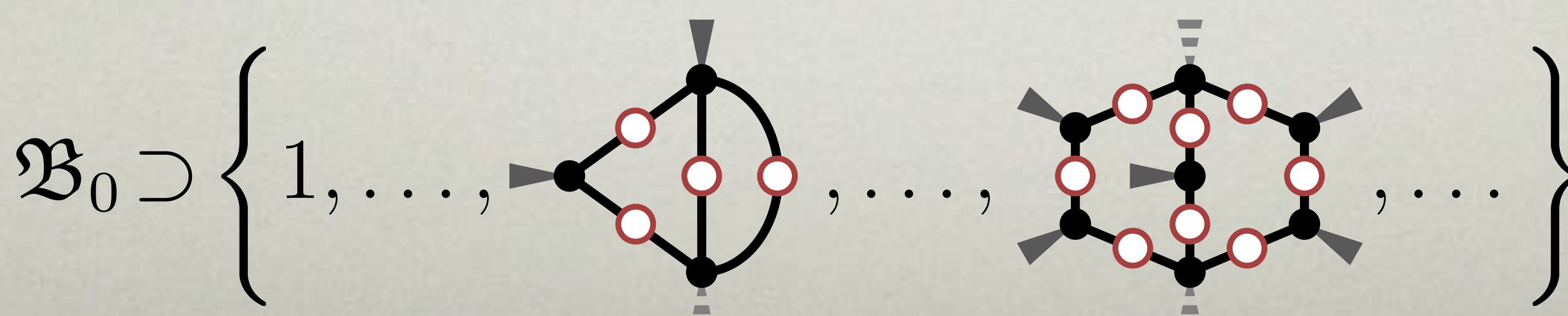


- The loop-dependent part of any SM integrand will be spanned by the basis of "0-gons"—at L loops(!)



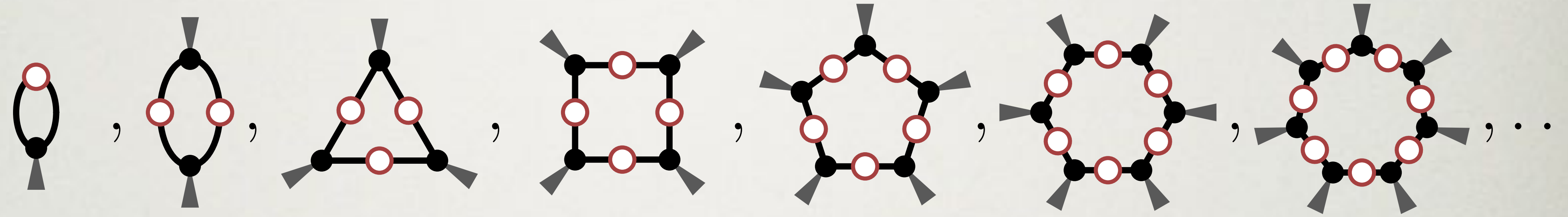


[Feng, Huang (2012)]

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

- In any dimension, the 0-gons reduce to finite size:

$\mathfrak{B}_0 := \Big\{ 1,$  $, \ldots \Big\}$

$$\mathcal{I}_p^q := \text{span}\left\{ \frac{[\ell]^q}{(\ell|P_1)\cdots(\ell|P_p)} \right\}$$

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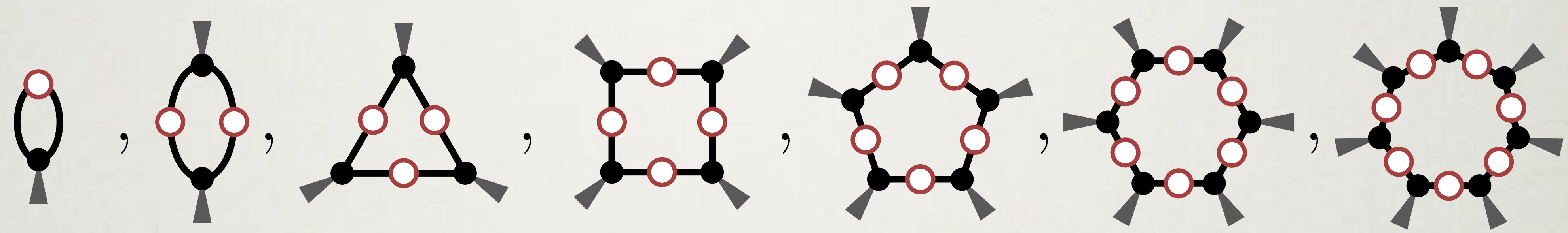

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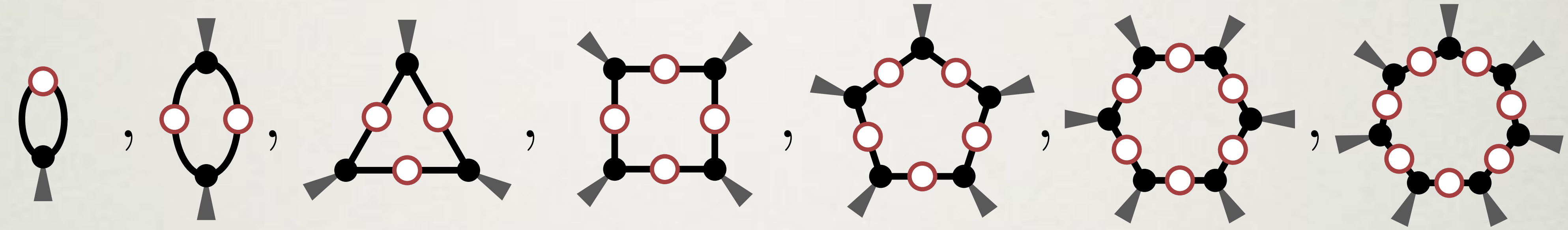

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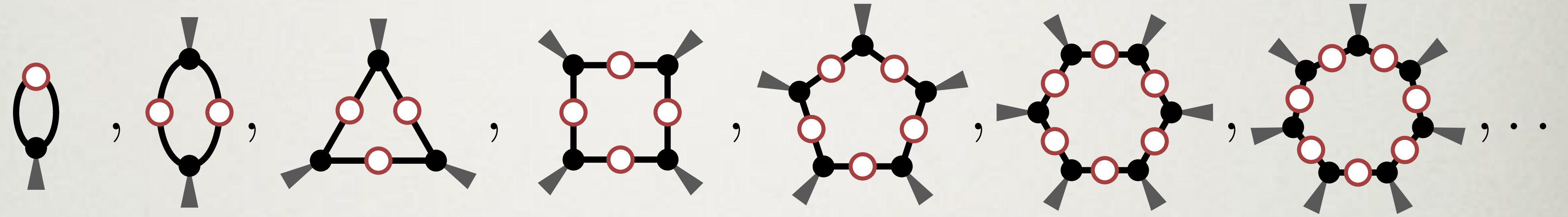

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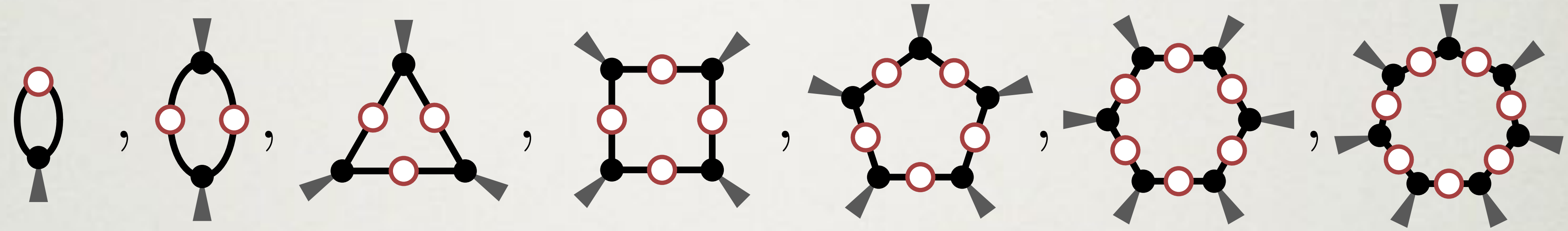
$\mathfrak{B}_0 := \Big\{ 1,$  $, \ldots \Big\}$

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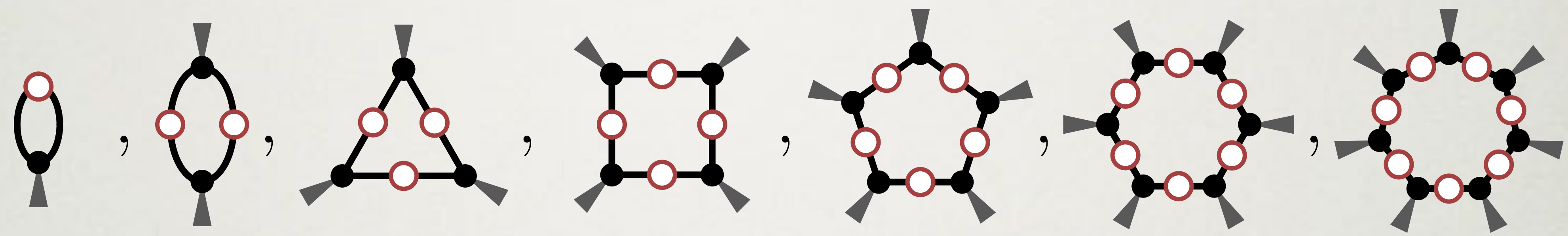
$\mathfrak{B}_0 := \{ 1,$  $, \ldots \}$

d=2	1	4	9	16	25	36	49	64
d=3	1	5	14	30	55	91	140	204
d=4	1	6	20	50	105	196	336	540

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total rank=top rank+contact terms

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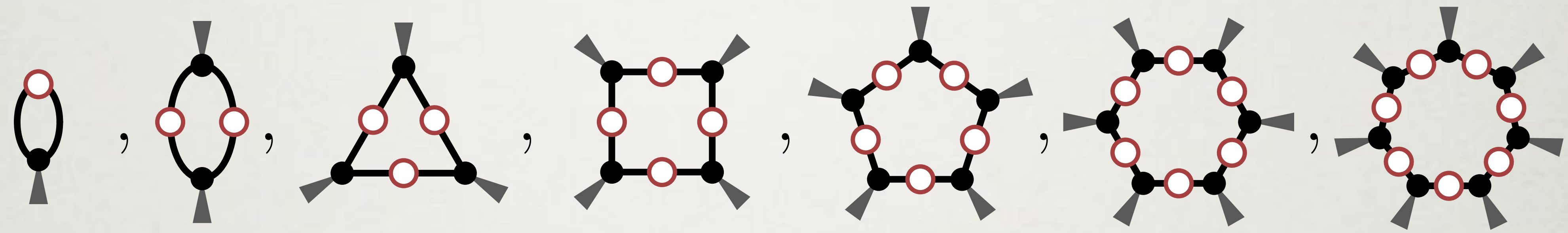
$\mathfrak{B}_0 := \{ 1,$  $, \ldots \}$

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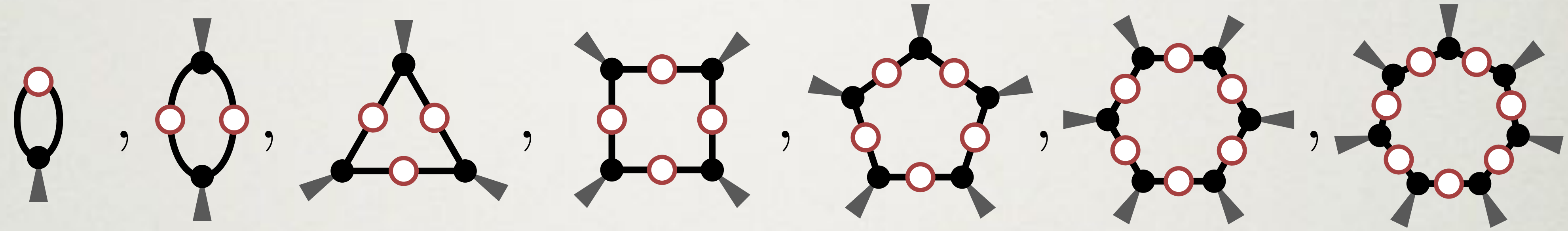
$\mathfrak{B}_0 := \Big\{ 1,$  $, \ldots \Big\}$

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	1+0	4+1	5+9	2+28	0+55	0+91	0+140	0+204
d=4	1	6	20	50	105	196	336	540
	1+0	5+1	9+11	7+41	2+103	0+196	0+336	0+540

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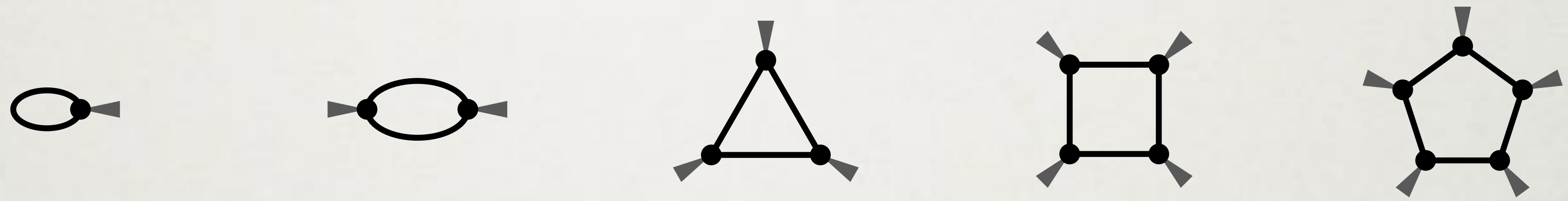
Integrand Reduction at 1 Loop

- Re-considering one-loop bases in four dimensions

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

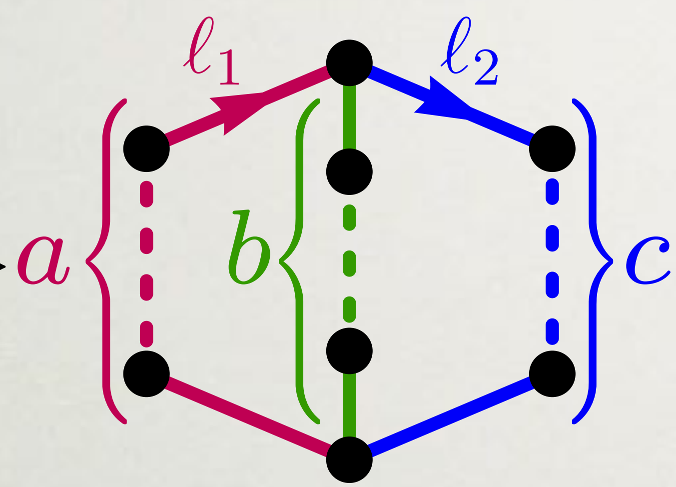


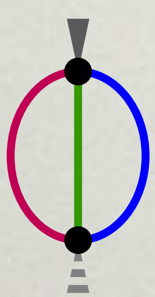
	1						…
$\mathfrak{B}_0^{(4)}$	$[\ell]^0 = \widehat{\mathfrak{b}}_0^0$	$[\ell]^1 = \widehat{\mathfrak{b}}_1^0 \oplus \cdots$	$[\ell]^2 = \widehat{\mathfrak{b}}_2^0 \oplus \cdots$	$[\ell]^3 = \widehat{\mathfrak{b}}_3^0 \oplus \cdots$	$[\ell]^4 = \widehat{\mathfrak{b}}_4^0 \oplus \cdots$	$[\ell]^5 = \widehat{\mathfrak{b}}_5^0 \oplus \cdots$	
	$1 = \mathbf{1}$	$6 = \mathbf{5} + 1$	$20 = \mathbf{9} + 11$	$50 = \mathbf{7} + 41$	$105 = \mathbf{2} + 103$	$196 = \mathbf{0} + 196$	
$\mathfrak{B}_1^{(4)}$		$[\ell]^0 = \widehat{\mathfrak{b}}_1^1$	$[\ell]^1 = \widehat{\mathfrak{b}}_2^1 \oplus \cdots$	$[\ell]^2 = \widehat{\mathfrak{b}}_3^1 \oplus \cdots$	$[\ell]^3 = \widehat{\mathfrak{b}}_4^1 \oplus \cdots$	$[\ell]^4 = \widehat{\mathfrak{b}}_5^1 \oplus \cdots$	
		$1 = \mathbf{1}$	$6 = \mathbf{4} + 2$	$20 = \mathbf{5} + 15$	$50 = \mathbf{2} + 48$	$105 = \mathbf{0} + 105$	
$\mathfrak{B}_2^{(4)}$			$[\ell]^0 = \widehat{\mathfrak{b}}_2^2$	$[\ell]^1 = \widehat{\mathfrak{b}}_3^2 \oplus \cdots$	$[\ell]^2 = \widehat{\mathfrak{b}}_4^2 \oplus \cdots$	$[\ell]^3 = \widehat{\mathfrak{b}}_5^2 \oplus \cdots$	
			$1 = \mathbf{1}$	$6 = \mathbf{3} + 3$	$20 = \mathbf{2} + 18$	$50 = \mathbf{0} + 50$	
$\mathfrak{B}_3^{(4)}$				$[\ell]^0 = \widehat{\mathfrak{b}}_3^3$	$[\ell]^1 = \widehat{\mathfrak{b}}_4^3 \oplus \cdots$	$[\ell]^2 = \widehat{\mathfrak{b}}_5^3 \oplus \cdots$	
				$1 = \mathbf{1}$	$6 = \mathbf{2} + 4$	$20 = \mathbf{0} + 20$	
$\mathfrak{B}_4^{(4)}$					$[\ell]^0 = \widehat{\mathfrak{b}}_4^4$	$[\ell]^1 = \widehat{\mathfrak{b}}_5^4 \oplus \cdots$	
					$1 = \mathbf{1}$	$6 = \mathbf{1} + 5$	

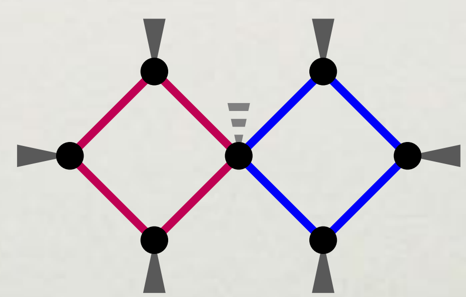
Integrand Reduction at 2 Loops

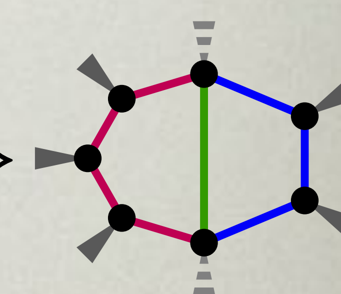
- At two loops, all loop integrands can be labeled by:

[Gluza, Kajda, Kosower]
[**JB**, Herrmann, Langer, Trnka (2020)]

$\Gamma_{[a,b,c]} \Leftrightarrow$ 

$\Gamma_{[1,1,1]} \Leftrightarrow$ 

$\Gamma_{[4,0,4]} \Leftrightarrow$ 

$\Gamma_{[4,1,3]} \Leftrightarrow$ 

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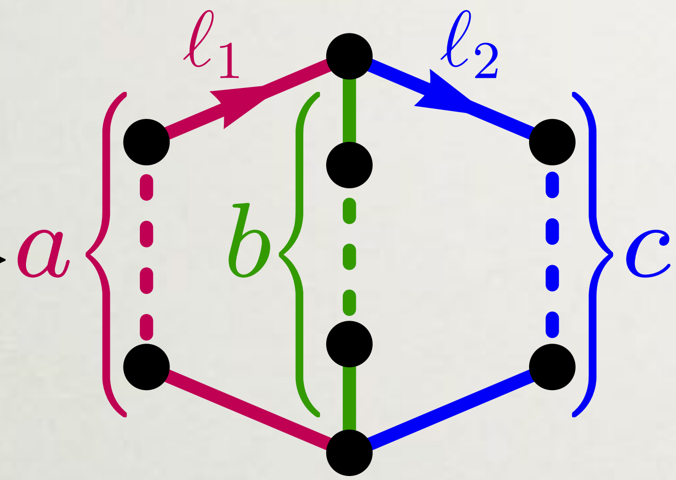
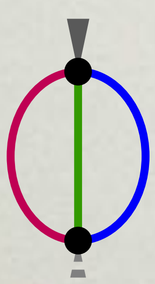
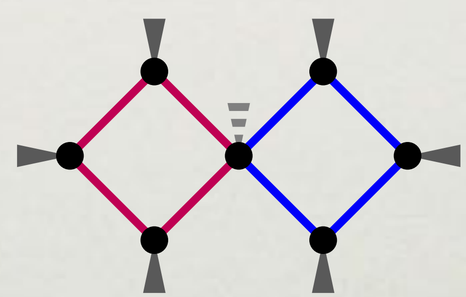
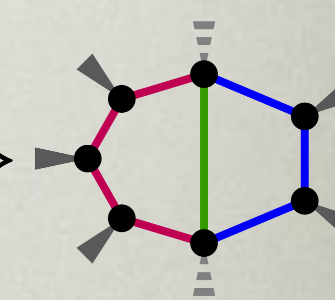
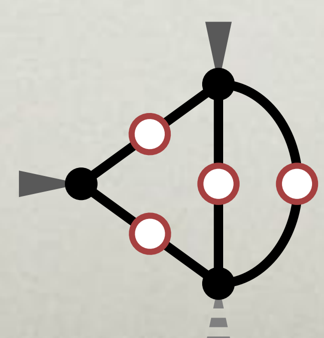
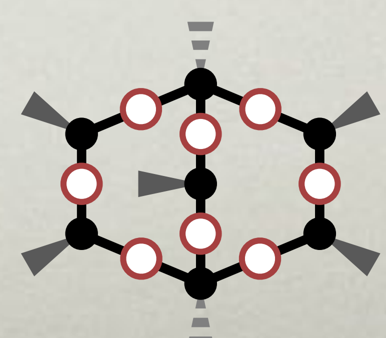
$\Gamma_{[a,b,c]} \Leftrightarrow$ 

$$\mathfrak{N}[a,b,c] := [\ell_1]^a [\ell_1 - \ell_2]^b [\ell_2]^c$$

$\Gamma_{[1,1,1]} \Leftrightarrow$ 

$\Gamma_{[4,0,4]} \Leftrightarrow$ 

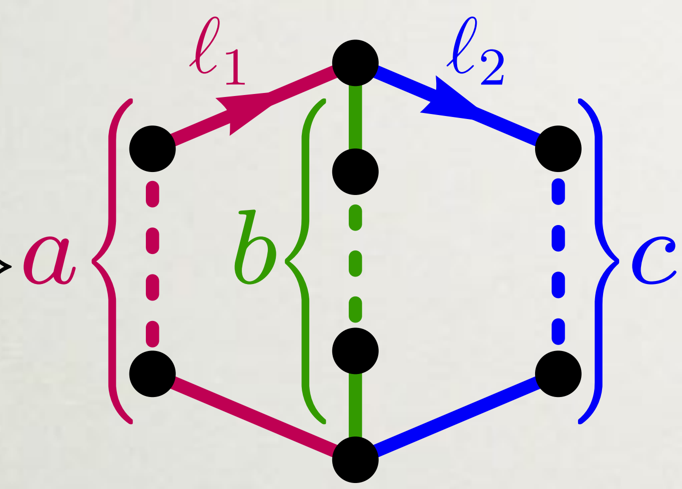
$\Gamma_{[4,1,3]} \Leftrightarrow$ 

$\mathfrak{B}_0 \supset \Big\{ 1, \ldots,$  $, \ldots,$  $, \ldots \Big\}$

Integrand Reduction at 2 Loops

- At two loops, all loop integrands can be labeled by:

[Gluza, Kajda, Kosower]
[**JB**, Herrmann, Langer, Trnka (2020)]

$\Gamma_{[a,b,c]} \Leftrightarrow$ 

$$\mathfrak{N}[a,b,c] := [\ell_1]^a[\ell_1 - \ell_2]^b[\ell_2]^c$$

$$\mathfrak{d}_d[a,b,c] := \text{rank}\left(\mathfrak{N}[a,b,c]\right) = \text{rank}\left([\ell_1]_d^a[\ell_1 - \ell_2]_d^b[\ell_2]_d^c\right)$$

$$=: \widehat{\mathfrak{d}}_d[a,b,c] + \sum_{(i,j,k)>(0,0,0)}^{(a,b,c)} \binom{a}{i}\binom{b}{j}\binom{c}{k}\widehat{\mathfrak{d}}_d[a-i, b-j, c-k]$$

total rank=top rank+contact terms

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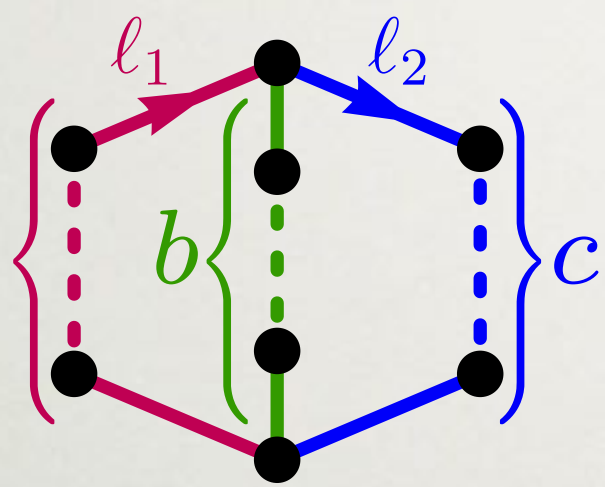
$\Gamma_{[a,b,c]} \Leftrightarrow$ 

$$\mathfrak{N}[a,b,c] := [\ell_1]^a[\ell_1 - \ell_2]^b[\ell_2]^c$$

$$\mathfrak{N}_p(\Gamma_{[a,b,c]}) =: \underbrace{\widehat{\mathfrak{N}}_p(\Gamma_{[a,b,c]})}_{\text{top-level numerators}}$$

$$\underbrace{\bigoplus_{(i,j,k)>\vec{0}}[(\ell_A|Q_{a_1})\cdots(\ell_A|Q_{a_i})][(\ell_B|Q_{b_1})\cdots(\ell_B|Q_{b_j})][(\ell_C|S_{c_1})\cdots(\ell_C|S_{c_k})]\widehat{\mathfrak{N}}_p(\Gamma_{[a-i,b-j,c-k]})}_{\text{contact-term numerators}}$$

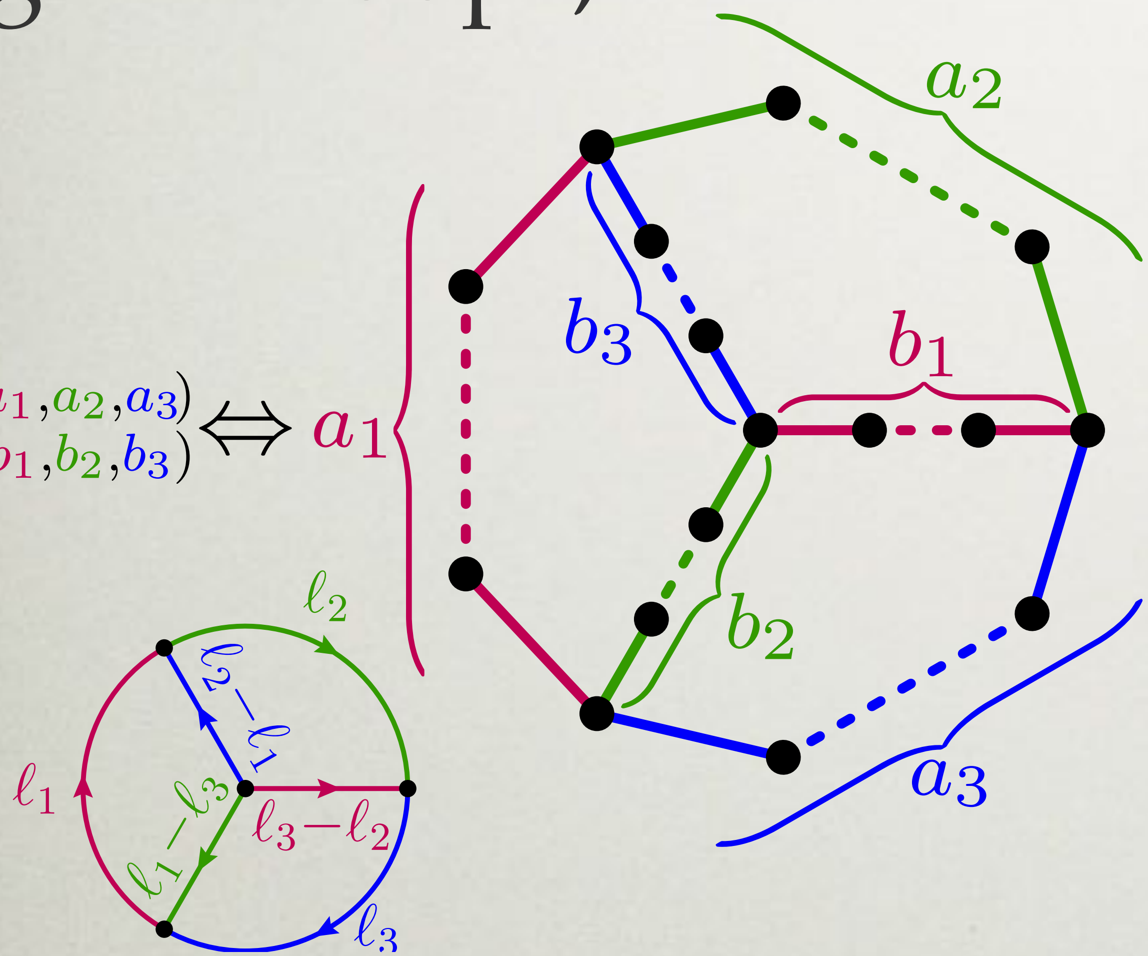
Integrand Reduction at 3 Loops

[**JB**, Herrmann, Langer, Trnka (2020)]

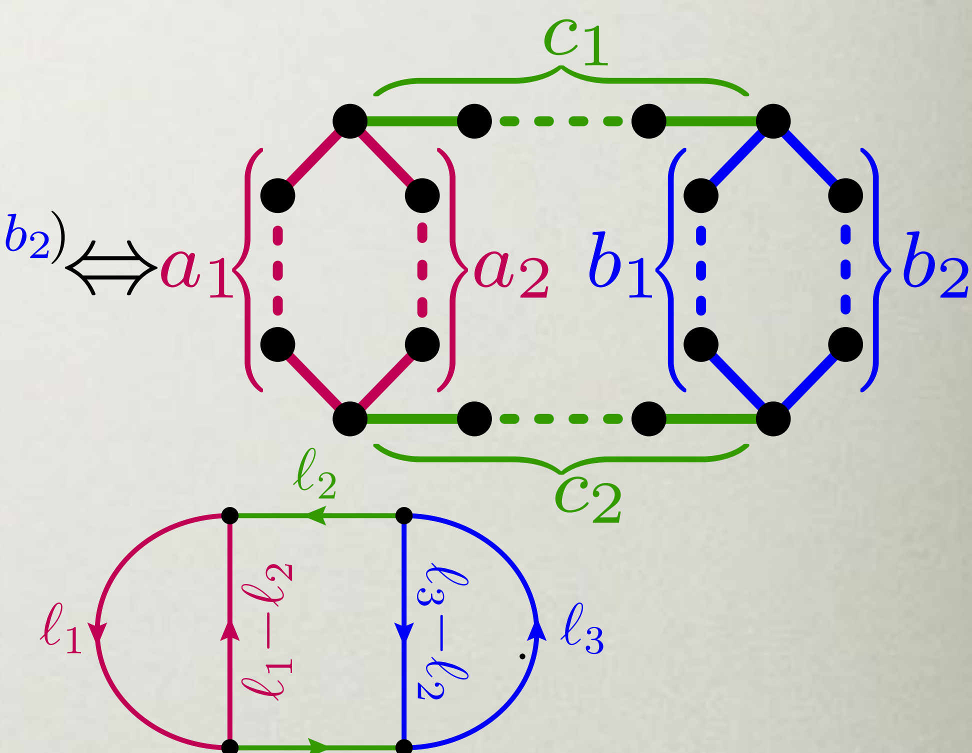
- We can obviously continue this to higher loops—e.g. at 3 loops, we have the integrand topologies:

[Gluza, Kajda, Kosower]

$\mathcal{W}^{(a_1,a_2,a_3)}_{(b_1,b_2,b_3)} \Leftrightarrow$



$\mathcal{L}^{(a_1,a_2)(b_1,b_2)}_{(c_1,c_2)} \Leftrightarrow$



$$\mathfrak{w}_d^0(a_1,\ldots,b_3)=\text{rank}\Big([\ell_1]^{a_1}[\ell_2]^{a_2}[\ell_3]^{a_3}[\ell_3-\ell_2]^{b_1}[\ell_1-\ell_3]^{b_2}[\ell_2-\ell_1]^{b_3}\Big),$$

$$\mathfrak{l}_d^0(a_1,\ldots,c_2)=\text{rank}\Big([\ell_1]^{a_1}[\ell_1-\ell_2]^{a_2}[\ell_2]^{c_1+c_2}[\ell_3]^{b_1}[\ell_3-\ell_2]^{b_2}\Big).$$

¿Can someone derive these formulae?

A (modest) Proposal for non-Planar Power-Counting

Stratifying Theories by Unitarity

[**JB**, Herrmann, Langer, Trnka (2020)]

- QFTs can be (partially) **ordered** by the scope of the integrands needed to represent amplitudes

$$\mathcal{A}^L = \sum a_i \, \mathcal{I}_i^L \qquad \mathcal{I}_i \in \mathfrak{B}$$

$$\text{(Standard Model)} \succ \text{(SM} \setminus \text{Higgs)} \succ \text{(QCD)} \succ \text{(Yang-Mills)}$$
$$\succ (\mathcal{N}=2 \text{ sYM}) \succ (\mathcal{N}=4 \text{ sYM}) \succ (\text{planar } \mathcal{N}=4 \text{ sYM})$$
$$\succ (\text{fishnet theory}) \succ \cdots$$

$$\mathfrak{B}^{\text{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$$

This reflects UV behavior (“***power-counting***”) of theories; can be used to ***stratify integrand bases***

¿Can we define $\mathfrak{B}^{\mathcal{N}=4}$?

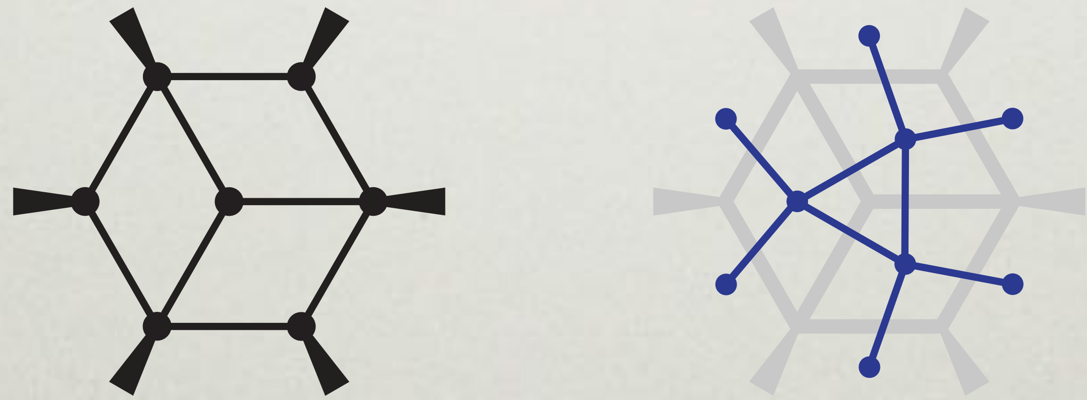
—a *basis of **just the best** UV-behaved amplitudes?*

Power-Counting when Planar

- For a ***planar*** graph, there is a natural ***routing*** of the loop momenta associated with its dual graph.
- A **planar *integrand*** $\mathcal{I}$ has "***p-gon power-counting***" if

$$\lim_{\ell_i\to\infty}\big(\mathcal{I}\big) = \frac{1}{(\ell_i^2)^{q\geq p}}\big(1+\mathcal{O}(1/\ell_i^2)\big) \quad \text{for all } \ell_i$$

- Let $\mathfrak{B}_p$ denote the complete basis of integrands with p-gon power-counting.



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$$\mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \mathfrak{B}_3 \supset \mathfrak{B}_4 \supset \mathfrak{B}_5 \supset \cdots$$

$$\widehat{\mathfrak{B}_p} := \mathfrak{B}_p \backslash \mathfrak{B}_{p+1} \qquad \mathfrak{B}_p = \widehat{\mathfrak{B}_p} \bigoplus \widehat{\mathfrak{B}}_{p+1} \bigoplus \cdots$$

- An **amplitude** is "***p-gon constructible***" if $\mathcal{A} \subset \mathfrak{B}_p$

$$\mathcal{A}_p := \mathcal{A} \cap \widehat{\mathfrak{B}_p} \qquad \mathcal{A} = \mathcal{A}_d \bigoplus \mathcal{A}_{d-1} \bigoplus \cdots$$

(Optimality of Dual-Conformality?)

- For ***planar*** $\mathcal{N}=4$ sYM, we know that amplitude integrands are ***dual-conformally invariant***

[Drummond, Henn, Smirnov, Sokatchev;
Drummond, Korchemsky, Henn;
Alday, Maldacena;…]

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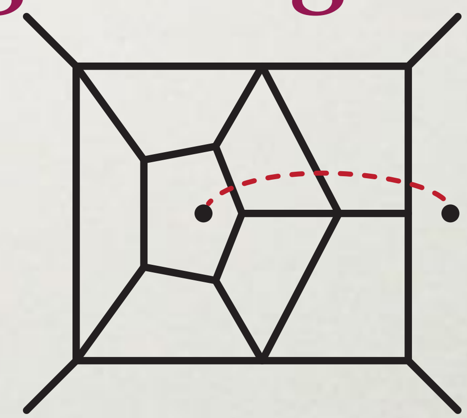
$$\mathfrak{B}_4 \supset \mathfrak{B}^{\mathrm{DCI}} \supset \mathfrak{B}^{\mathcal{N}=4}$$

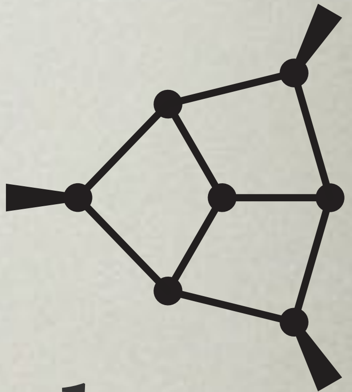
[Drummond, Henn, Smirnov, Sokatchev; Drummond, Korchemsky, Henn; Alday, Maldacena;…]

But even DCI is far from strong enough!

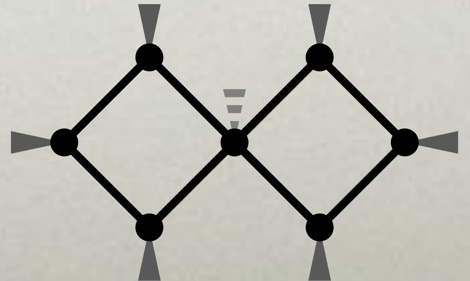
- doesn't ensure UV finiteness



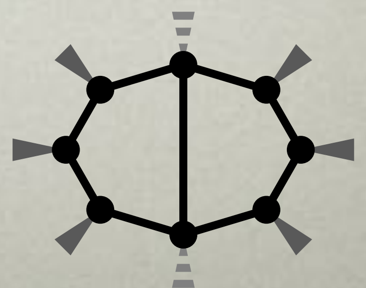
- doesn't ensure maximal transcendentality



- it *forces* a topological over-completeness **and** non-triangularity of bases



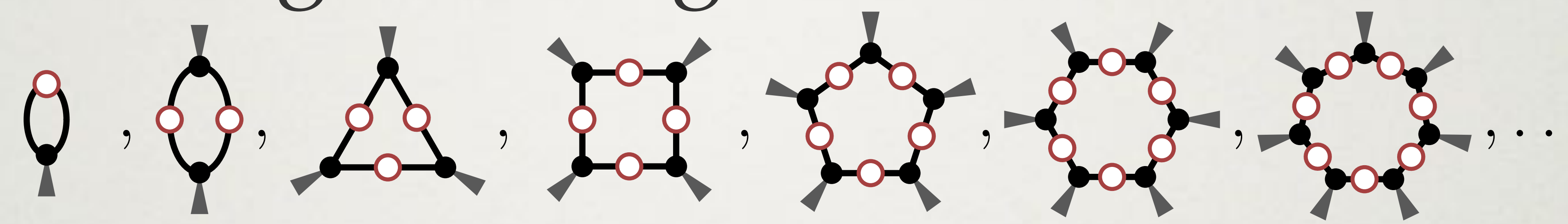
$\subset$



Power-Counting Strata at 1-Loop

[**JB**, Herrmann, Langer, Trnka (2020)]

- Recall how "0-gon" integrands could be defined:

$$\mathfrak{B}_0 := \Big\{ 1, $$

$$, \ldots \Big\}$$

$$\mathcal{I}_p^q := \text{span}\left\{ \frac{[\ell]^q}{(\ell|P_1)\cdots(\ell|P_p)} \right\} \in \mathfrak{B}_{p-q}$$

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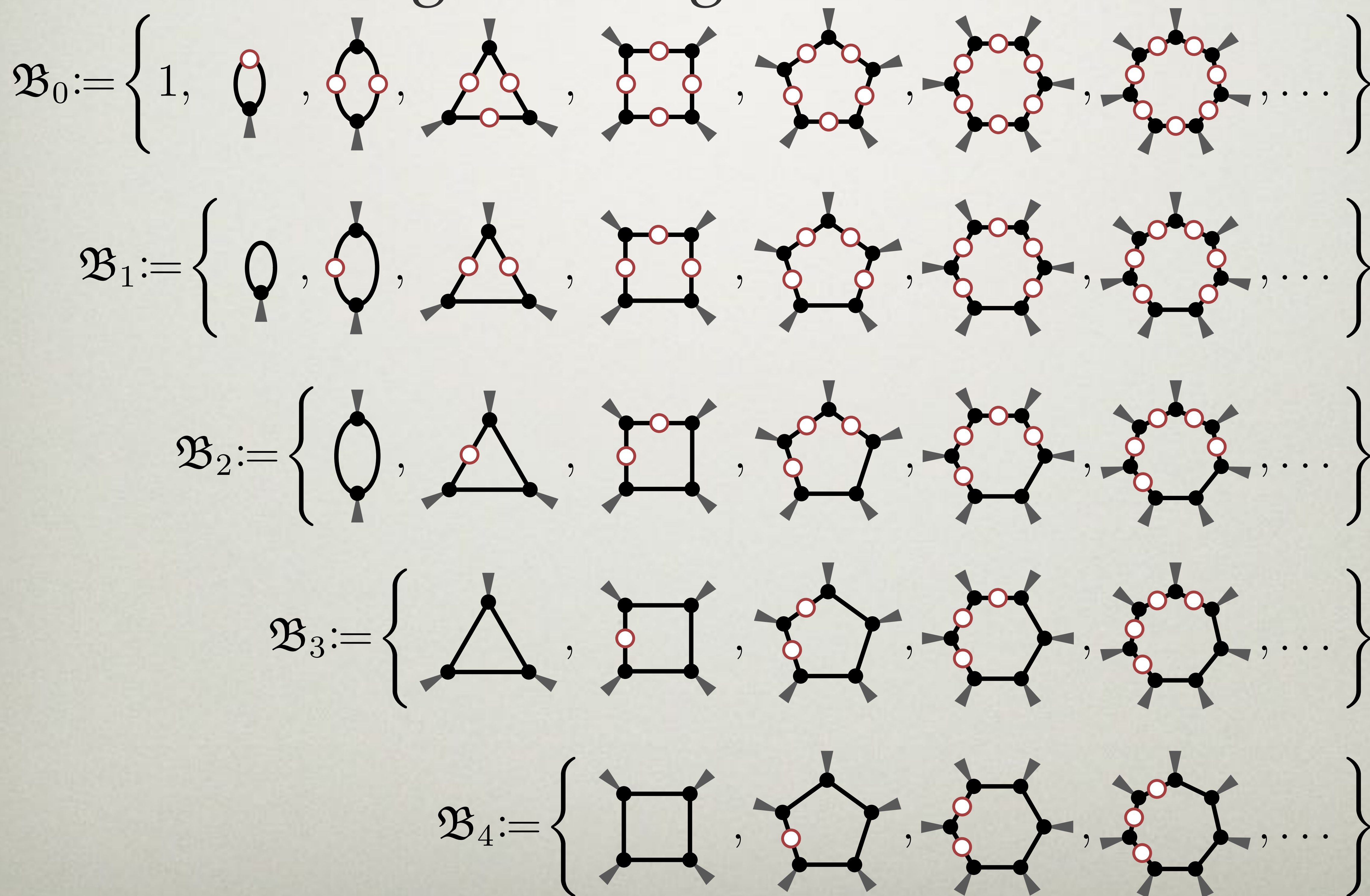

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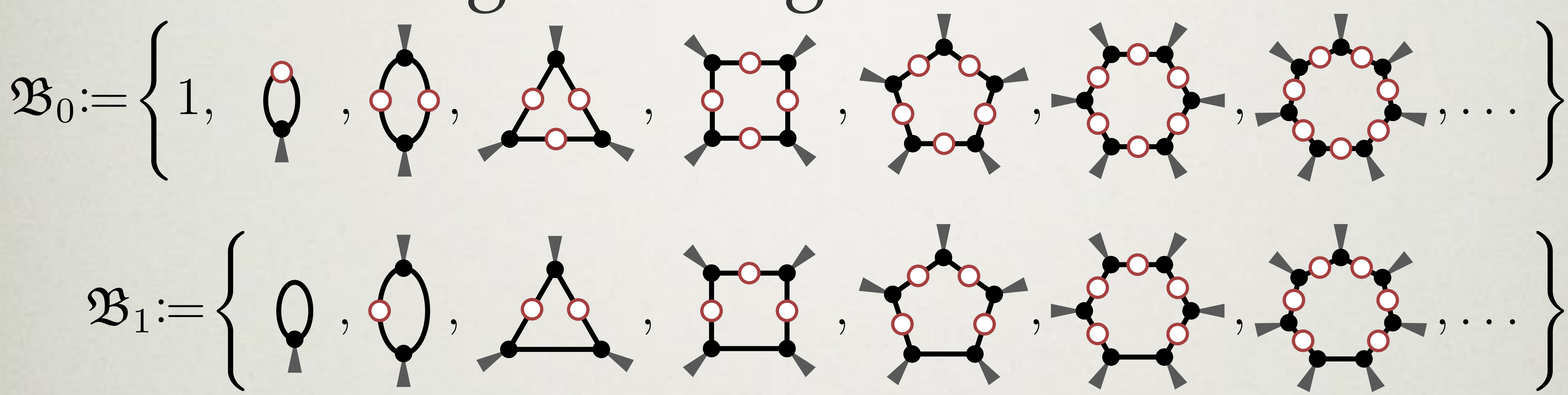

$$\mathfrak{d}_d^q[p] := \operatorname{rank}\left([\ell]_d^{p-q}\right)$$

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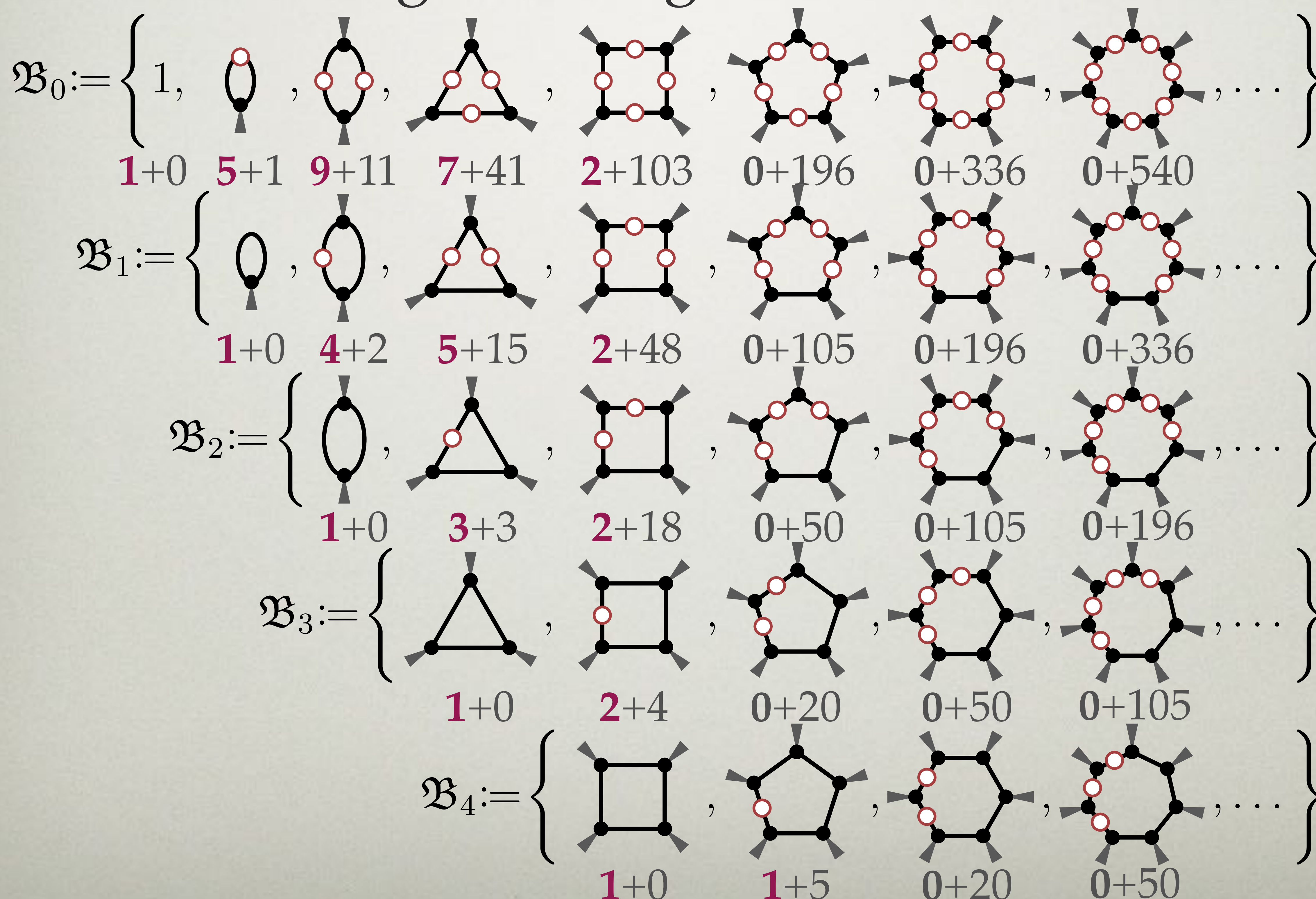

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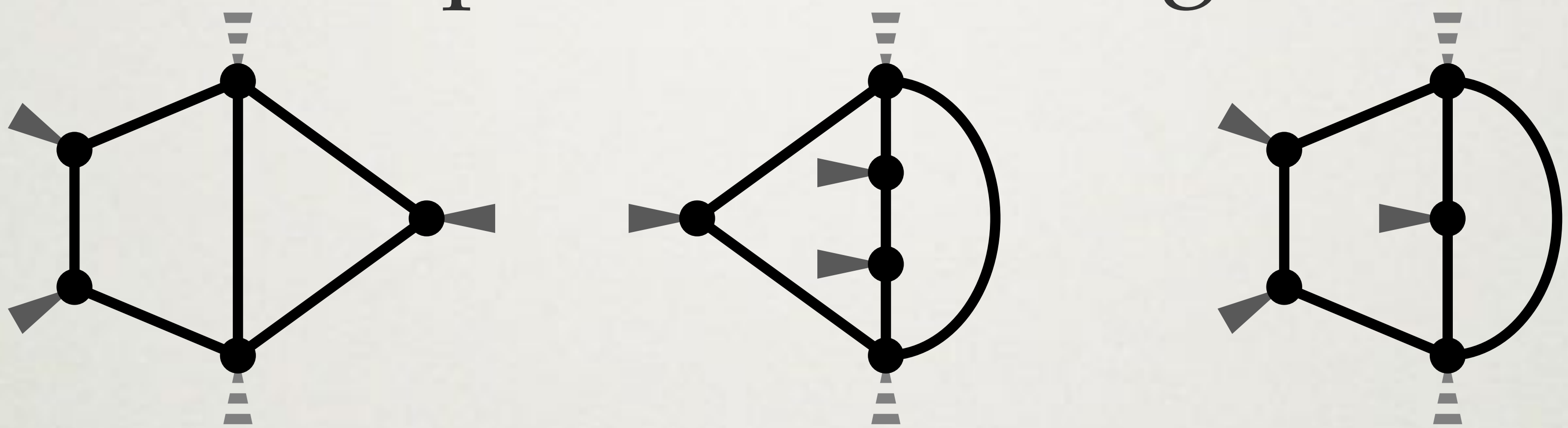
- With no preferred ***routing*** of loop momenta, the earlier notion of "power-counting" is ill-defined



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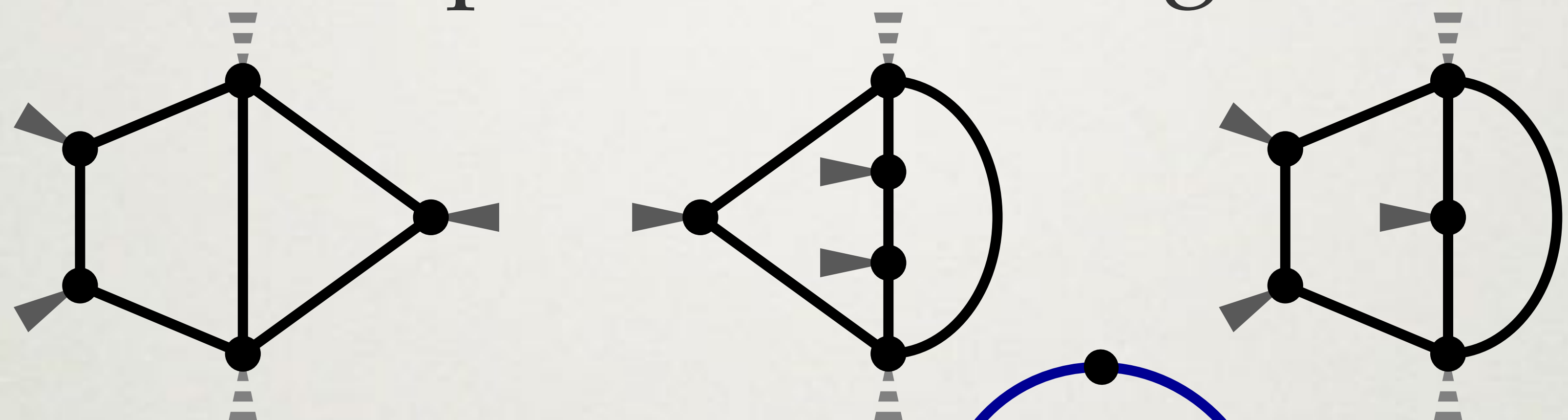

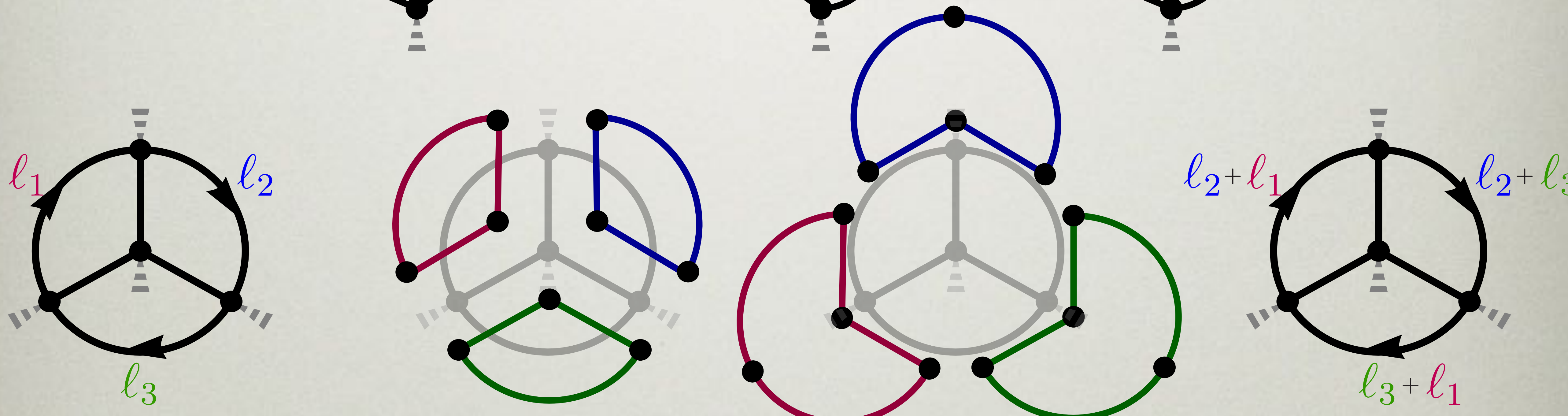

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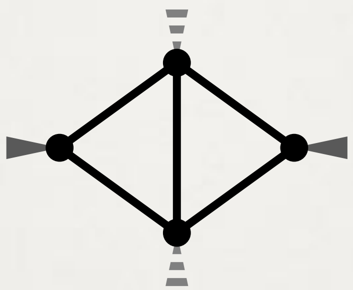
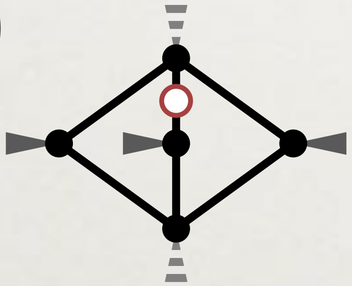
Proposal: Graph Power-Counting

- What ***would*** make sense ***independent of routing*** would be: to define the **power-counting** ***relative to*** some *graph* (or ***graphs***)

[**JB**, Herrmann, Langer, Trnka (2020)]





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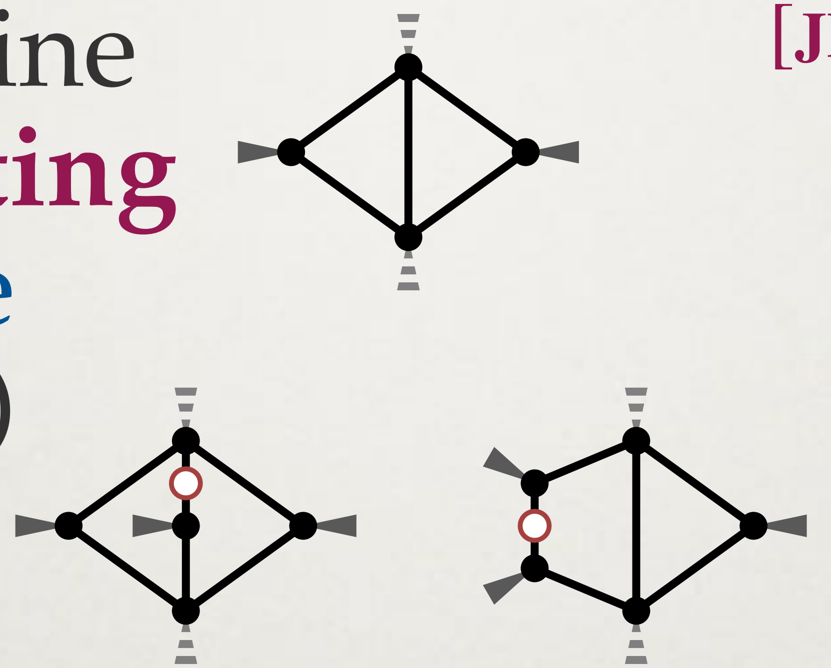

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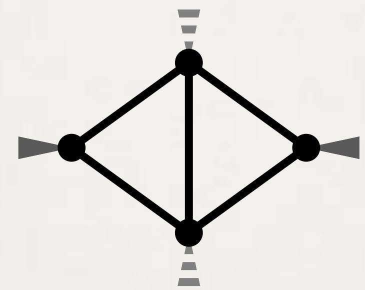

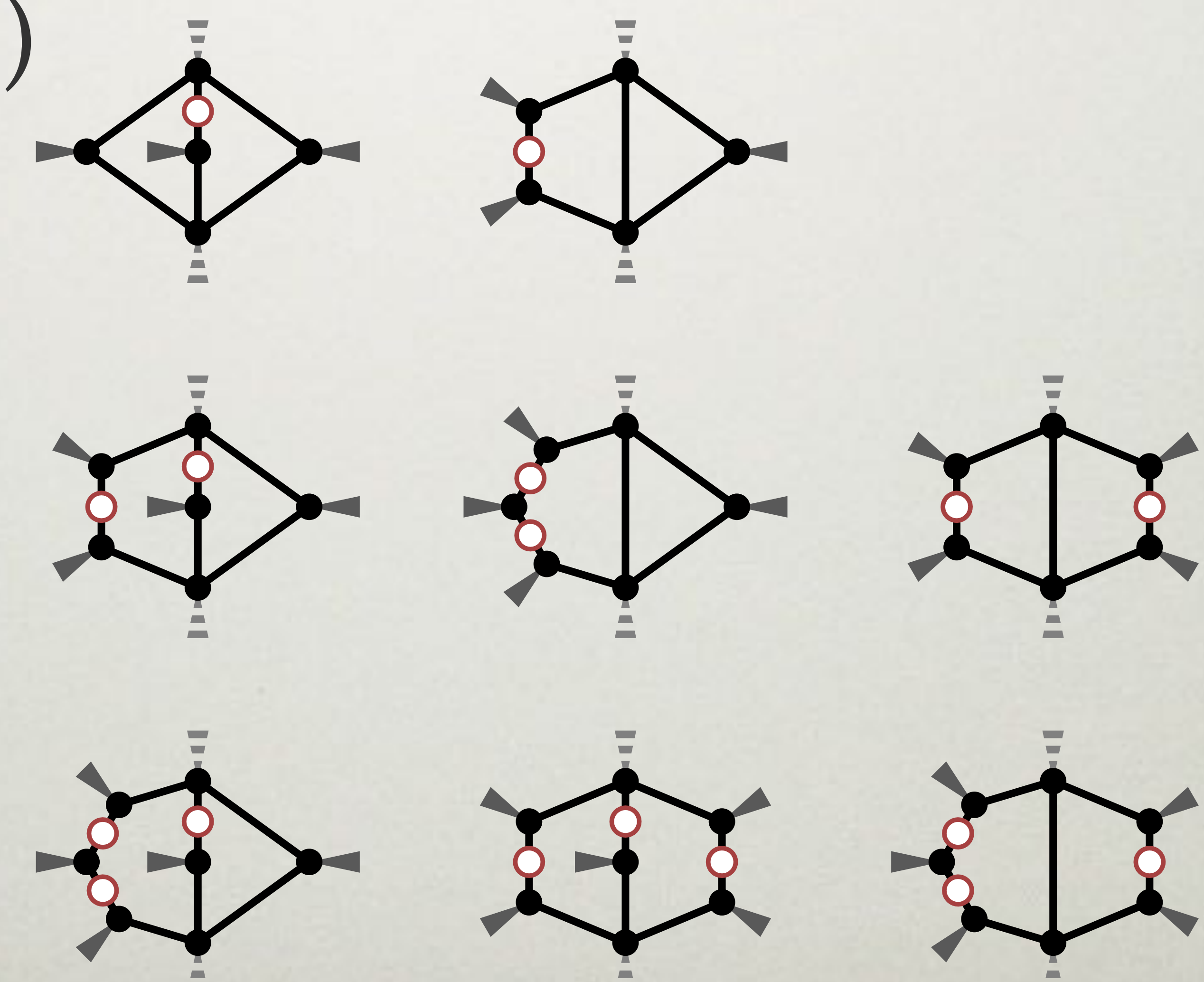

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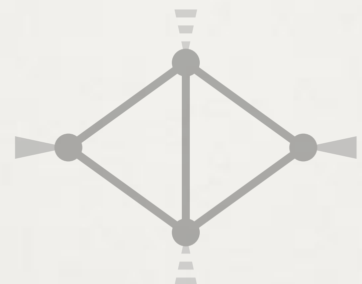

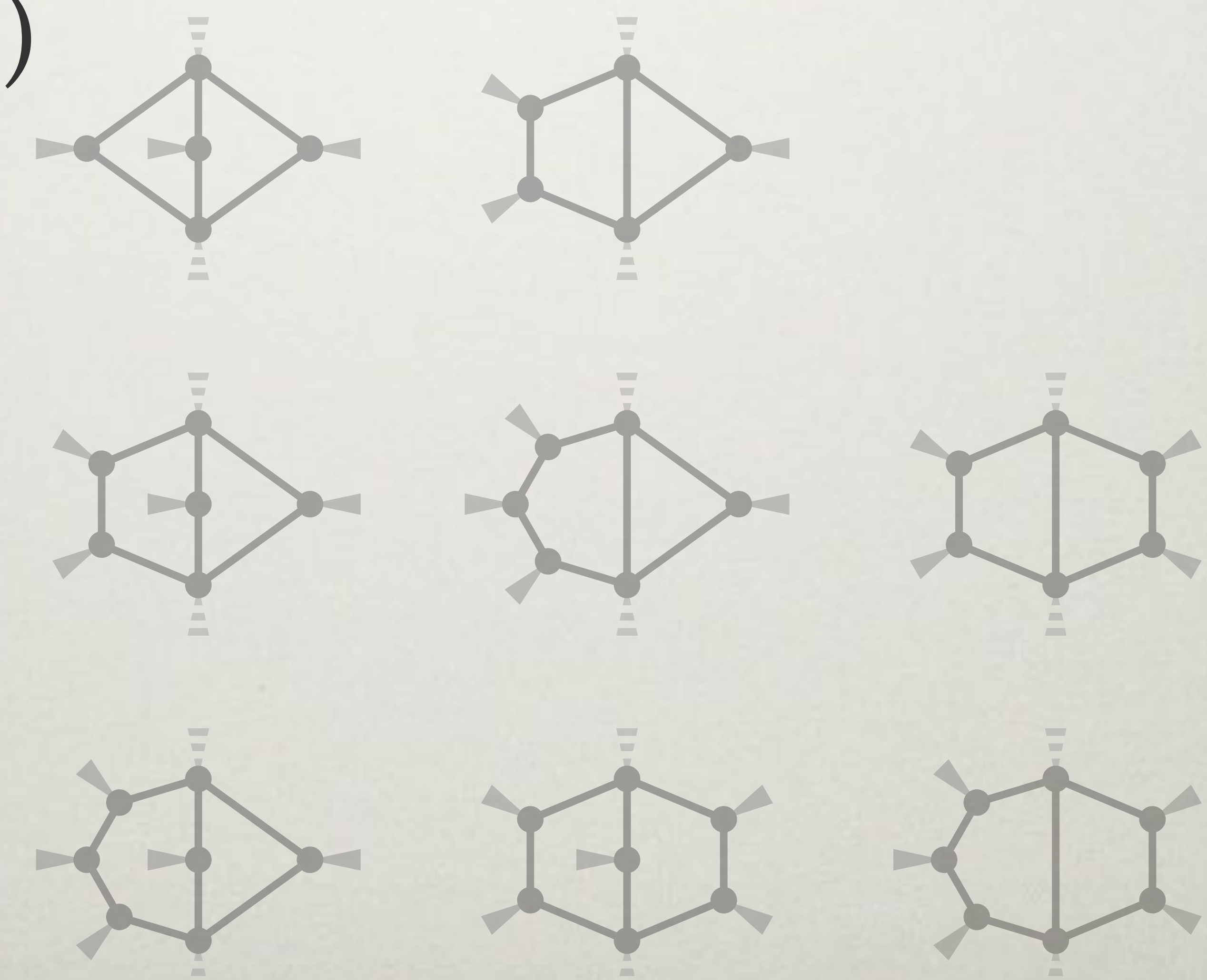

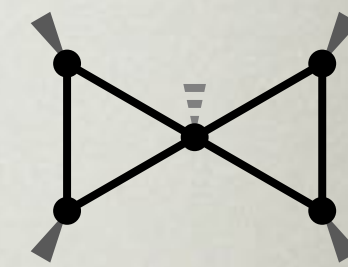

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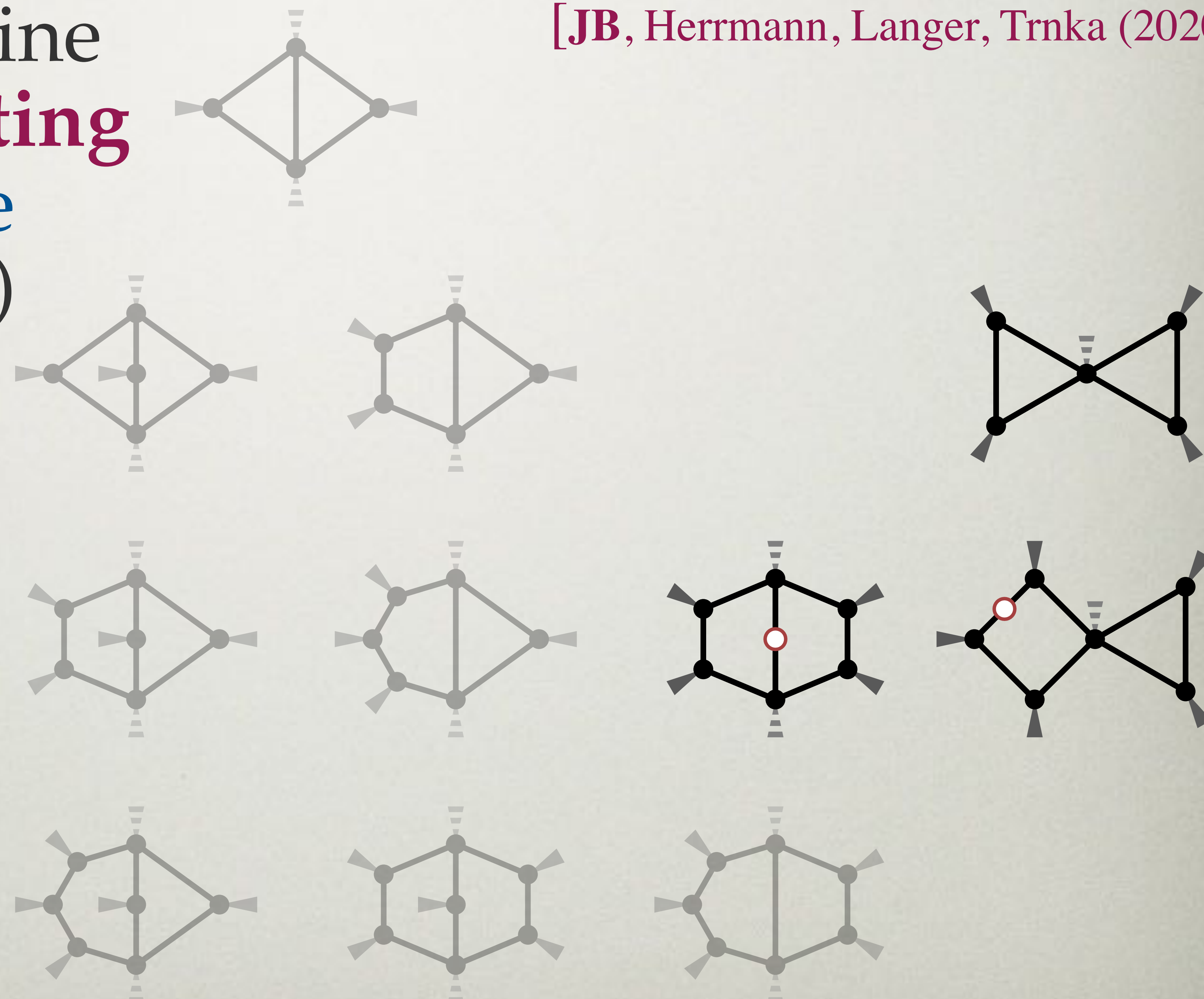

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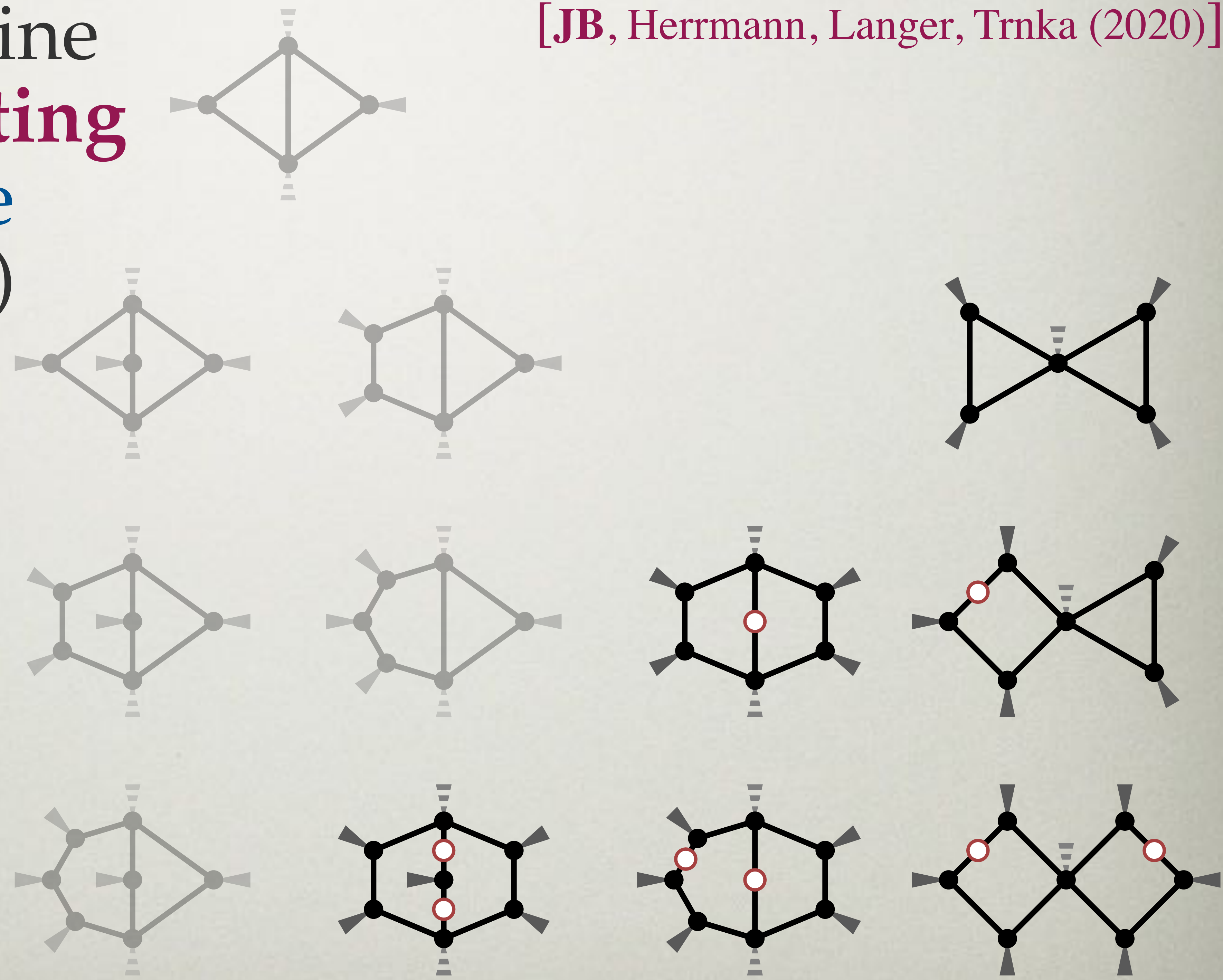

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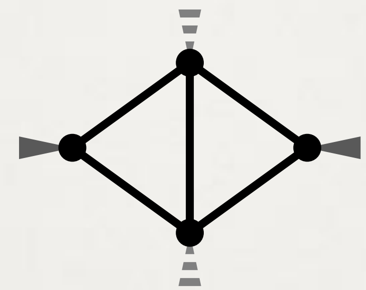

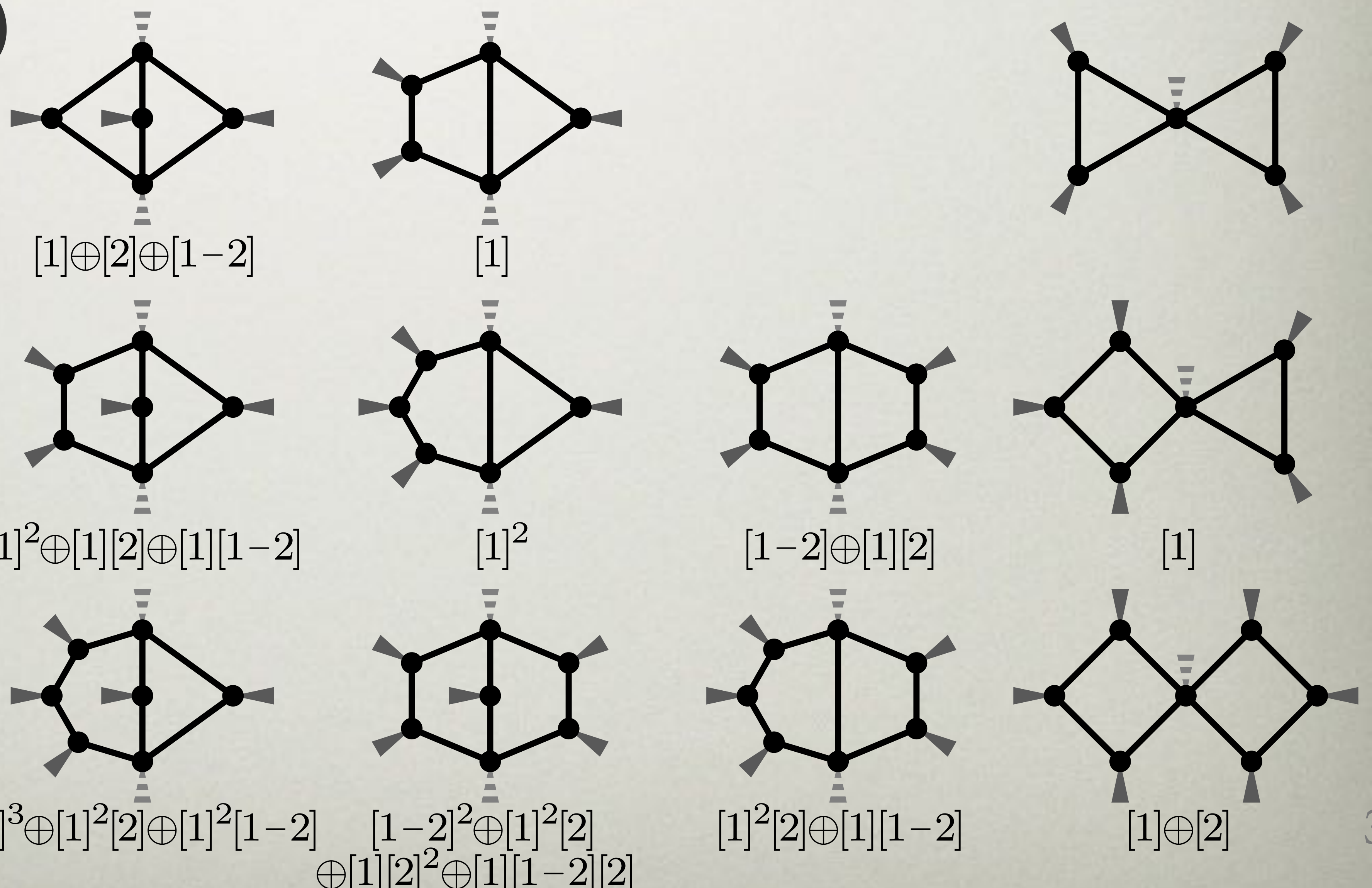

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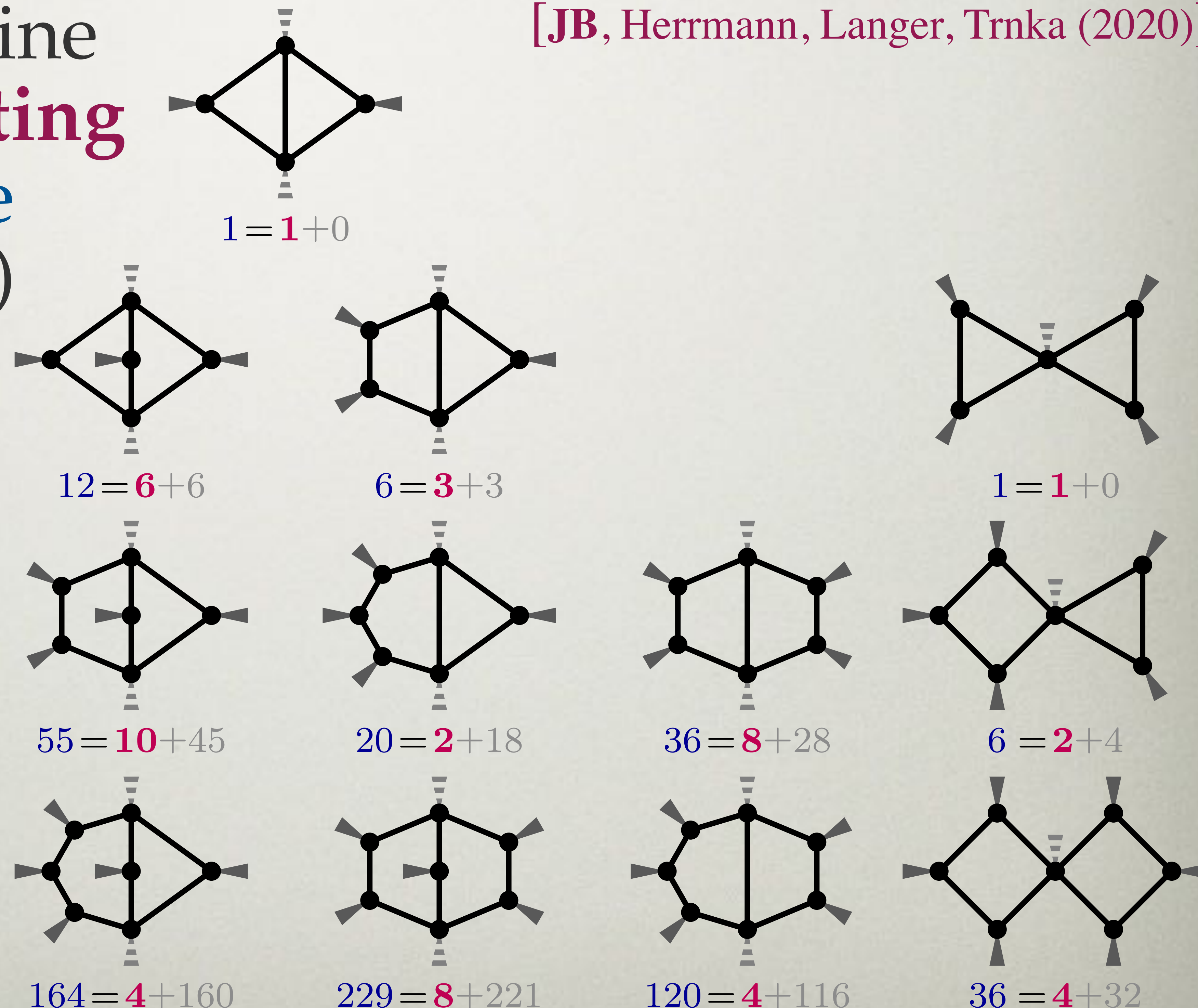

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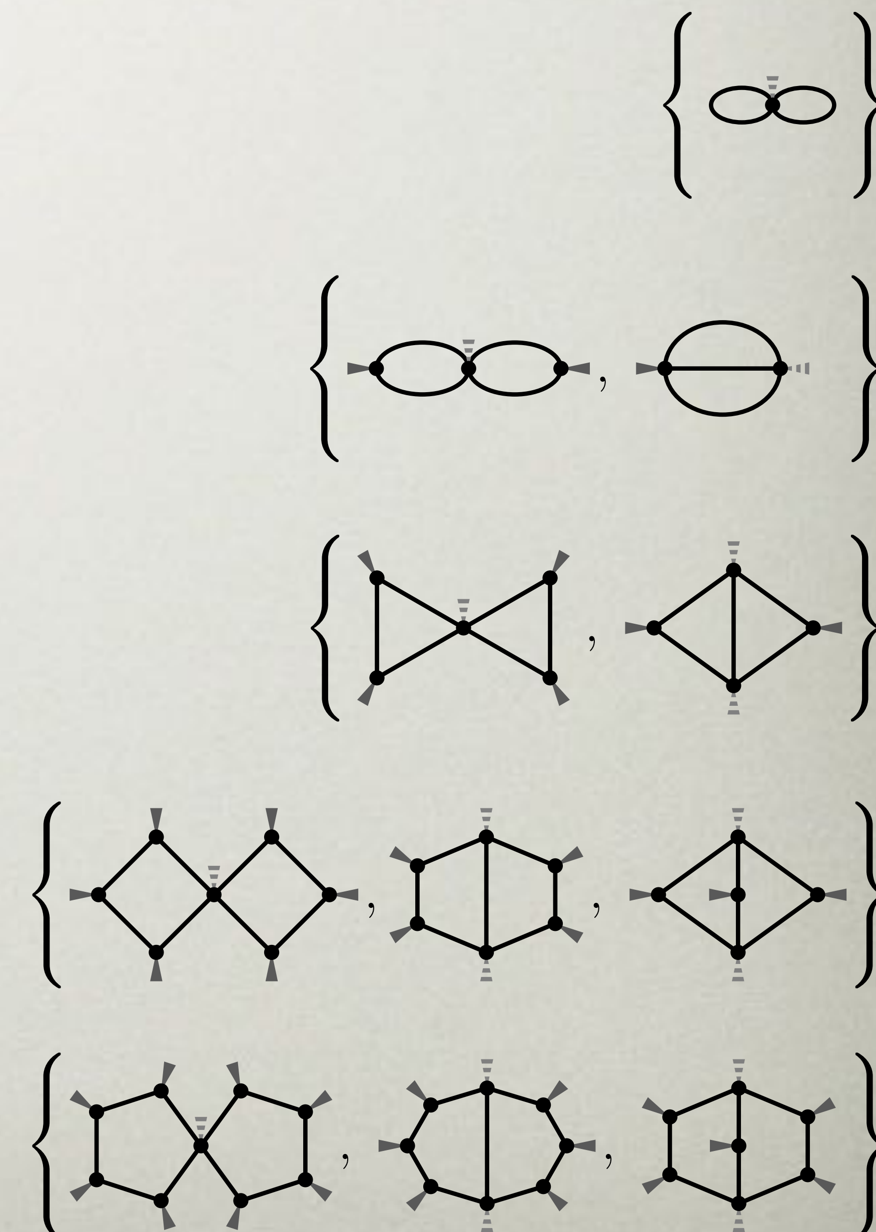

Definition

a ***scalar p-gon*** is a graph of *girth* p such that all its edge contractions have girth $<p$

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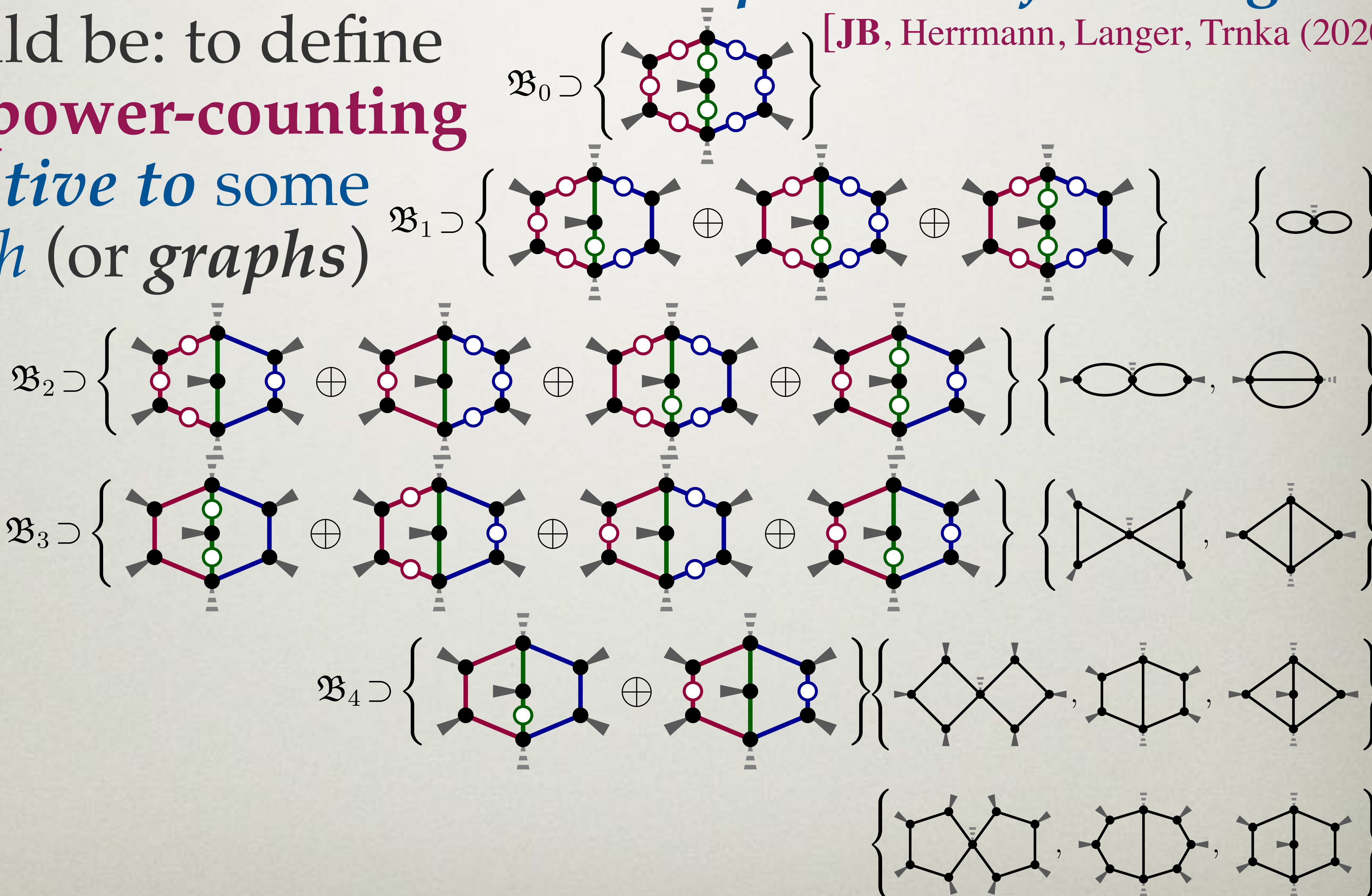

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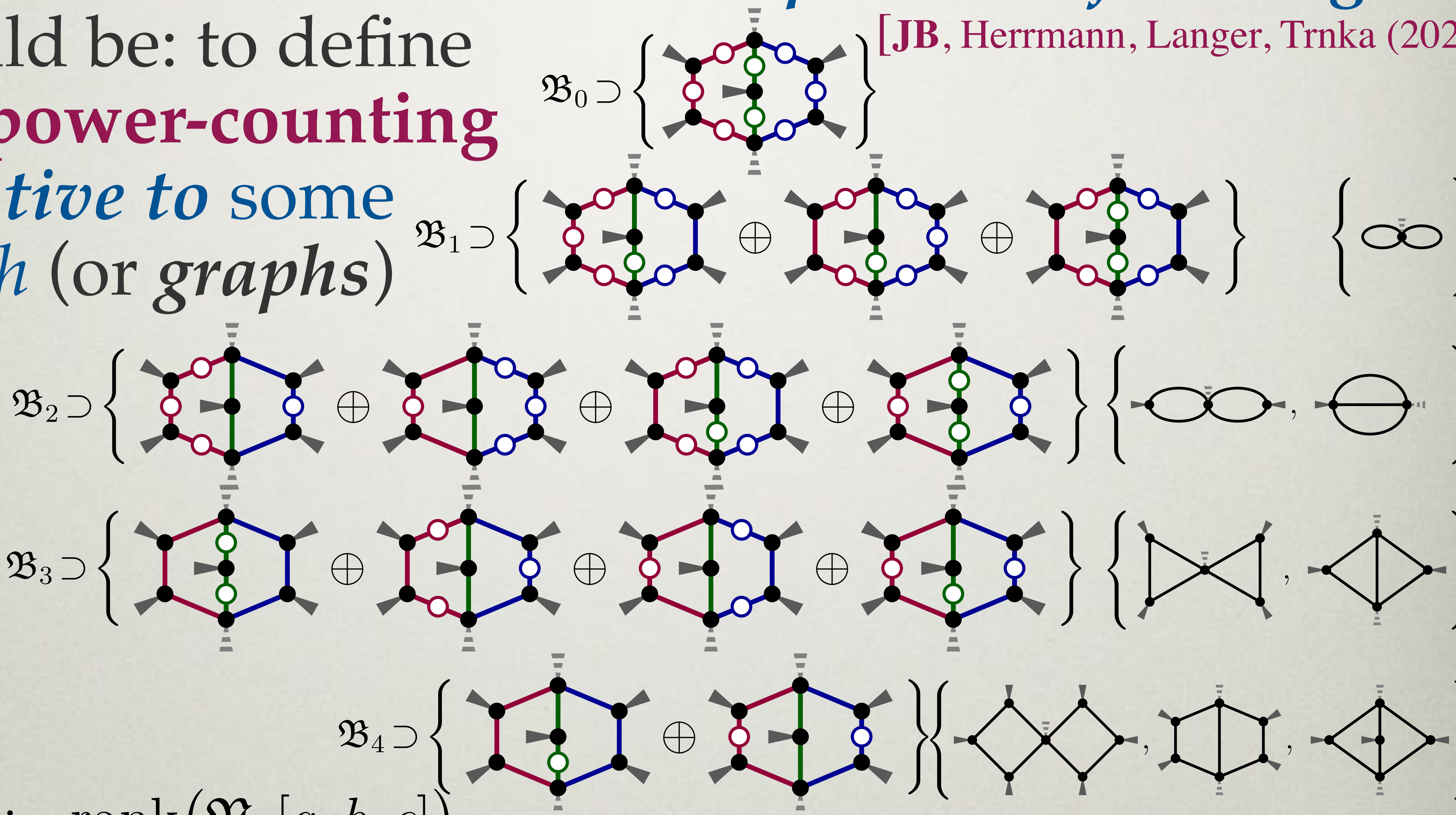

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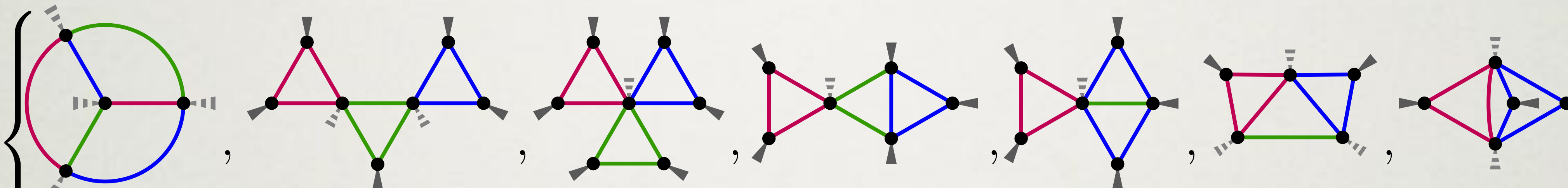
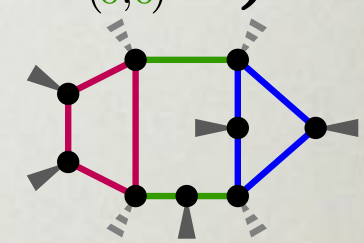
Graph Power-Counting @ L=3

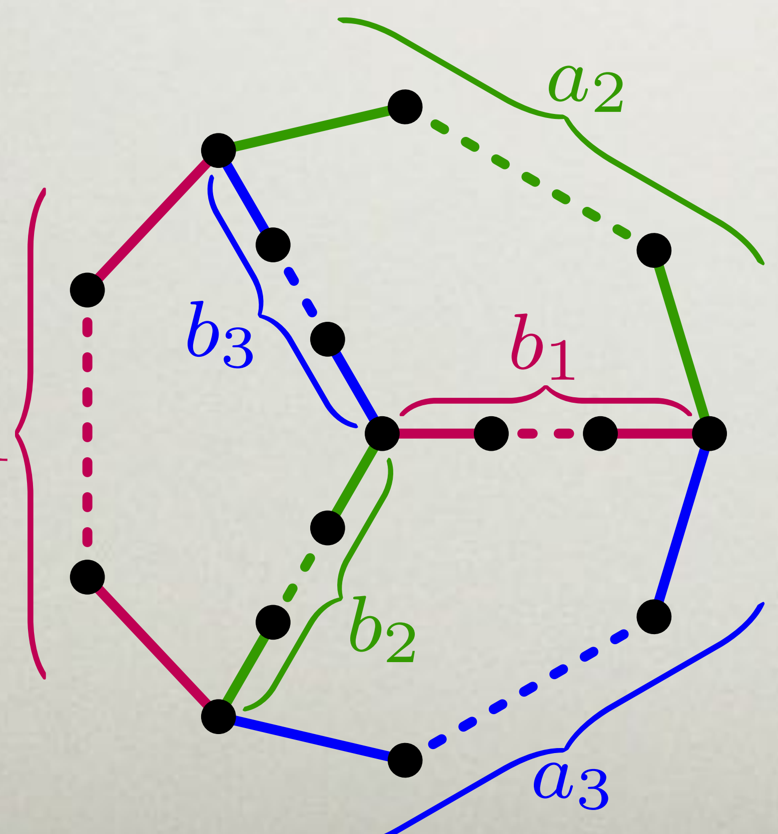
[**JB**, Herrmann, Langer, Trnka (2020)]

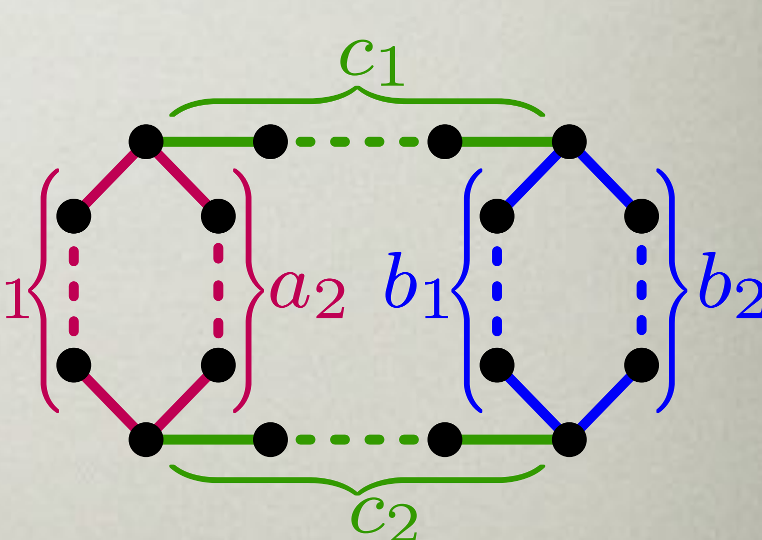
- At 3 loops, the 3-gon power-counting scalars are:



$$\left\{\mathcal{W}^{(1,1,1)}_{(1,1,1)},\ \mathcal{L}^{(3,0)(0,3)}_{(1,2)},\ \mathcal{L}^{(3,0)(0,3)}_{(0,3)},\ \mathcal{L}^{(3,0)(1,2)}_{(1,1)},\ \mathcal{L}^{(3,0)(2,2)}_{(0,1)},\ \mathcal{L}^{(2,1)(1,2)}_{(0,1)},\ \mathcal{L}^{(2,1)(2,2)}_{(0,0)}\right\}$$

- What would be the numerator for $\mathcal{L}^{(3,1)(2,2)}_{(1,2)} =$  ?

$\mathcal{W}^{(a_1,a_2,a_3)}_{(b_1,b_2,b_3)} \Leftrightarrow a_1$ 

$\mathcal{L}^{(a_1,a_2)(b_1,b_2)}_{(c_1,c_2)} \Leftrightarrow a_1$ 

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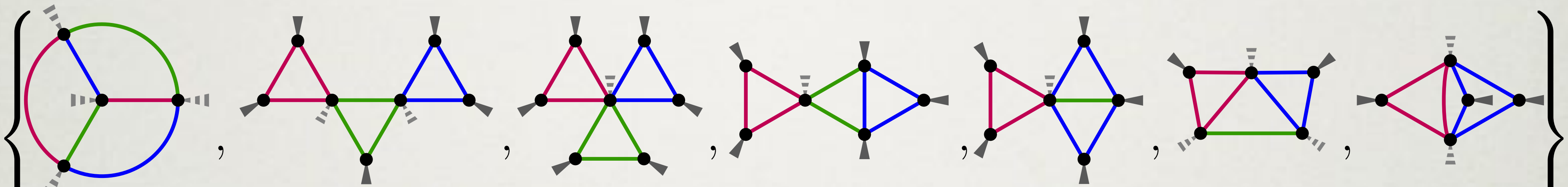
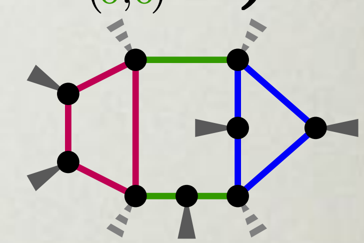
Graph Power-Counting @ L=3

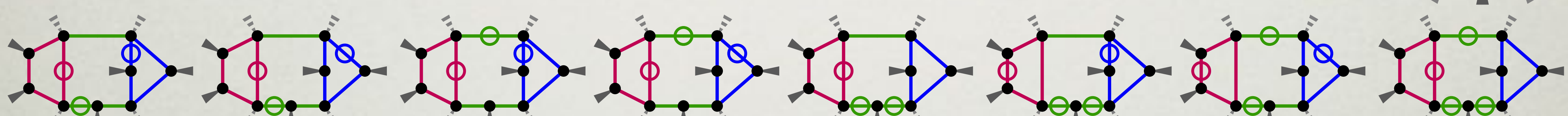
[**JB**, Herrmann, Langer, Trnka (2020)]

- At 3 loops, the 3-gon power-counting scalars are:



$$\left\{\mathcal{W}^{(1,1,1)}_{(1,1,1)},\ \mathcal{L}^{(3,0)(0,3)}_{(1,2)},\ \mathcal{L}^{(3,0)(0,3)}_{(0,3)},\ \mathcal{L}^{(3,0)(1,2)}_{(1,1)},\ \mathcal{L}^{(3,0)(2,2)}_{(0,1)},\ \mathcal{L}^{(2,1)(1,2)}_{(0,1)},\ \mathcal{L}^{(2,1)(2,2)}_{(0,0)}\right\}$$

- What would be the numerator for $\mathcal{L}^{(3,1)(2,2)}_{(1,2)} =$  ?



$$[1-2][2][2-3] \oplus [1-2][2][3] \oplus [1-2][2][2-3] \oplus [1-2][2][3] \oplus [1-2][2]^2 \oplus [1][2]^2[2-3] \oplus [1][2]^2[3] \oplus [1][2]^3$$

$$\mathfrak{N}_3\left(\mathcal{L}^{(3,1)(2,2)}_{(1,2)}\right) \coloneqq [1-2][2][2-3]\oplus[1-2][2][3]\oplus[1-2][2]^2\oplus[1][2]^2[2-3]\oplus[1][2]^2[3]\oplus[1][2]^3$$

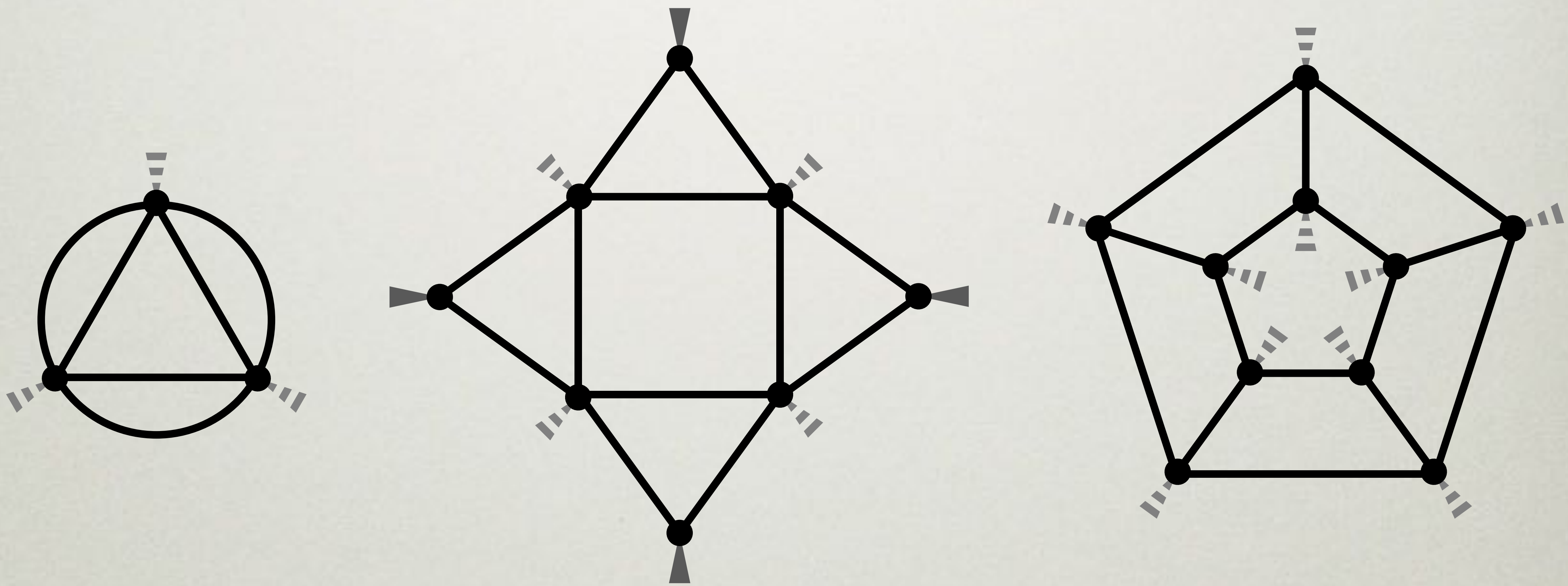
$$\left|\mathfrak{N}_3\left(\mathcal{L}^{(3,1)(2,2)}_{(1,2)}\right)\right| = \underset{d=4}{\mathrm{rank}}\left([1-2][2][2-3]\oplus[1-2][2][3]\oplus[1-2][2]^2\oplus[1][2]^2[2-3]\oplus[1][2]^2[3]\oplus[1][2]^3\right)$$

$$= 984 = 32 + 952\,. \qquad (3.7)$$

What Goes Wrong at Five Loops?

[**JB**, Herrmann, Langer, Trnka (2020)]

- Unfortunately, the "*p*-gon power-counting" basis proposed for non-planar is ***not compatible with*** *planar power-counting* (at high loops): $\mathfrak{B}_p^{\text{Pl}} \not\subset \mathfrak{B}_p^{\text{NP}}$



$$\mathfrak{B}^{\mathcal{N}=4} \subset \mathfrak{B}^{\text{DCI}} \subset \mathfrak{B}_4^{\text{Pl}} \not\subset \mathfrak{B}_4^{\text{NP}}$$

¿Can someone propose a better definition?

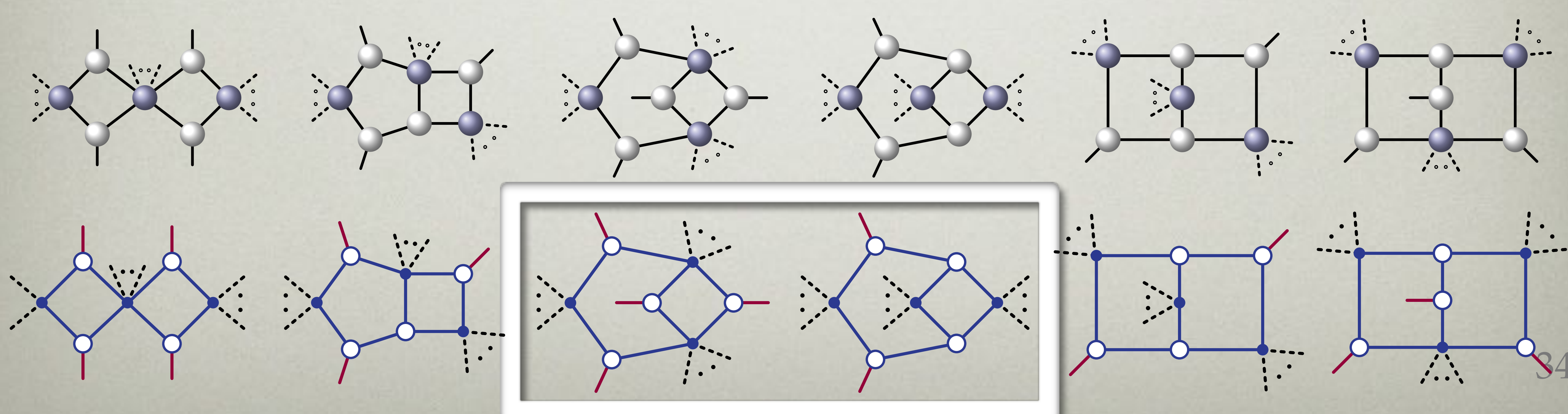
Room for Improvement: Building Better (Wiser) Bases

Normalizing Integrands Wisely

- It is often a good idea to normalize as much of the basis as possible on places in loop-momentum space where many amplitudes vanish
- This works well for ***nice*** amplitudes: those with low multiplicity, low loops, or low N^kMHV-degree (i.e. *where **polylogs** abound*)

For example, two-loop MHV amplitudes in sYM:

[**JB**, Herrmann, Langer, McLeod, Trnka (2019)]



What is 'Purity' Beyond Polylogs?

- When the basis is *d*log and all 'evaluations' are (residues) on poles, then diagonalization ensures each basis integrand is in ***canonical form*** (UT/etc.)

¿What about when no d*log *form exists?

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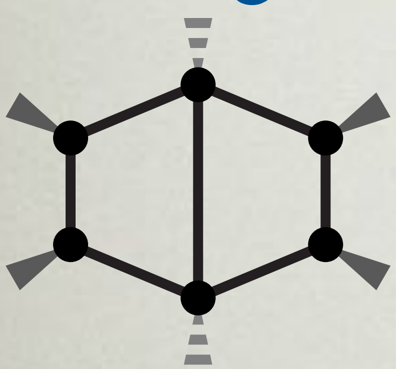
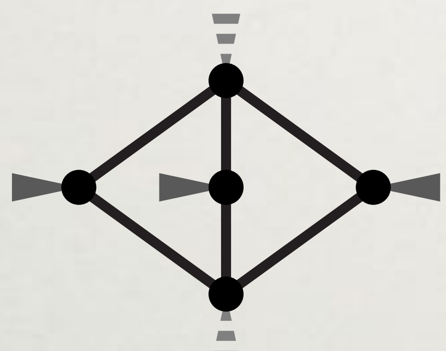


CY_1

1
1 1
1

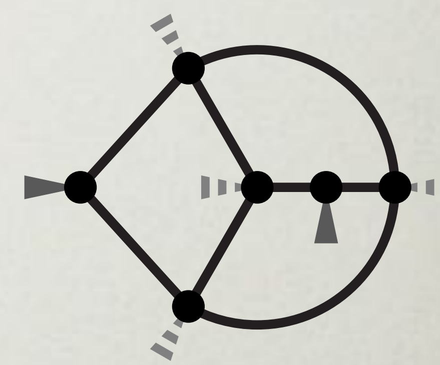


CY_2

1
0 0
1 20 1
0 0
1

CY_3

1
0 0
0 1 0
1 149 149 1
0 1 0
0 0
1



CY_4

1
0 0
0 1 0
0 0 0 0
1 976 3952 976 1
0 0 0 0
0 1 0
0 0
1

[**JB**, McLeod, von Hippel, Wilhelm (2018)]
[**JB**, He, McLeod, von Hippel, Wilhelm (2018)]
[**JB**, McLeod, Spradlin, von Hippel, Wilhelm (2018)]
[**JB**, McLeod, von Hippel, Vergu, Volk, Wilhelm (2019)]

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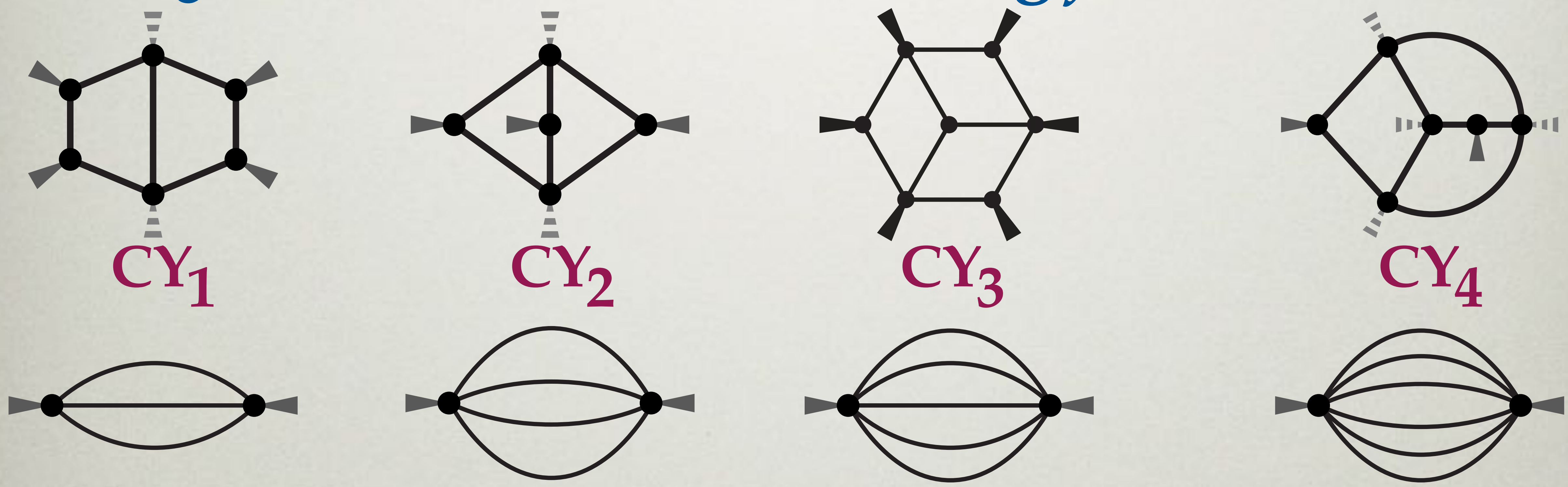

[Bloch, Kerr, Vanhove; Broadhurst;…]

[**JB**, McLeod, von Hippel, Wilhelm (2018)]
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Great Room for Improvement

- Prescriptive unitarity has made great progress, but the results raise (or sharpen) bigger questions

$$\mathcal{A} = \mathcal{A}_d \bigoplus \mathcal{A}_{d-1} \bigoplus \cdots \qquad \mathfrak{B}_p = \widehat{\mathfrak{B}_p} \bigoplus \widehat{\mathfrak{B}}_{p+1} \bigoplus \cdots$$

Better integrand bases would:

- trade *evaluations* for *periods* on all topologies (does this ensure "purity"?)
- ***stratify*** integrands by more refined criteria—e.g.
  - ***actual*** UV behavior (do finite bases exist?)
  - transcendental weight (what does this mean?)
  - dim-reg partitioning of numerator monomials
  - …