



PennState



INTERNATIONAL
CENTRE *for*
THEORETICAL
SCIENCES

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Adventures in Perturbation Theory

Jacob Bourjaily
Penn State University
& The Niels Bohr Institute

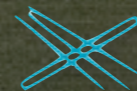
based on work in collaboration with

*Herrmann, Langer, Trnka;
McLeod, von Hippel, Vergu, Volk, Wilhelm; ...*



The Niels Bohr
International Academy

VILLUM FONDEN



Organization & Outline

- ◆ ***Spiritus Movens***: the *surprising* simplicity of QFT
- ◆ **Loop Integrands**
 - ▶ *generalized unitarity* (generally speaking)
 - ▶ *building bases big-enough* (for *e.g.* the Standard Model)
 - ▶ *non-planar power-counting* (a modest proposal)
- ◆ **Loop Integration**: what makes an integral easy?
 - ▶ *integration polemics* (what constitutes *being integrated*?)
 - ▶ *direct integration* (made easy)
- ◆ **Loop Integrals**: their generic analytic structure
 - ▶ *a bestiary of Feynman integral Calabi-Yau geometries*

Spiritus Movenens

*the Surprising Simplicity
of Scattering Amplitudes*

Traditional Description of QFT

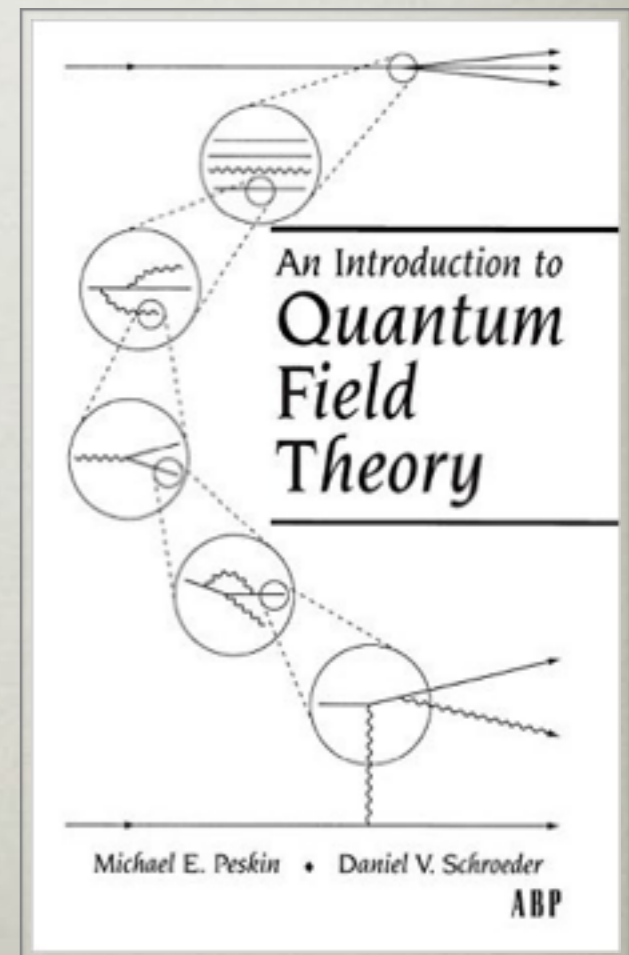
- ◆ **Quantum Field Theory:** the marriage of (special) *relativity* with *quantum mechanics*
- ◆ Theories (can be) specified by Lagrangians—or equivalently, by *Feynman rules* for virtual particles

$$\mathcal{L} \equiv -\frac{1}{4} \sum_i (F_{i\mu\nu}^a)^2 + \sum_J \bar{\psi}_J (i\not{D}) \psi_J$$

- ◆ Predicted probability (*amplitudes*) from path integrals (over virtual ‘histories’):



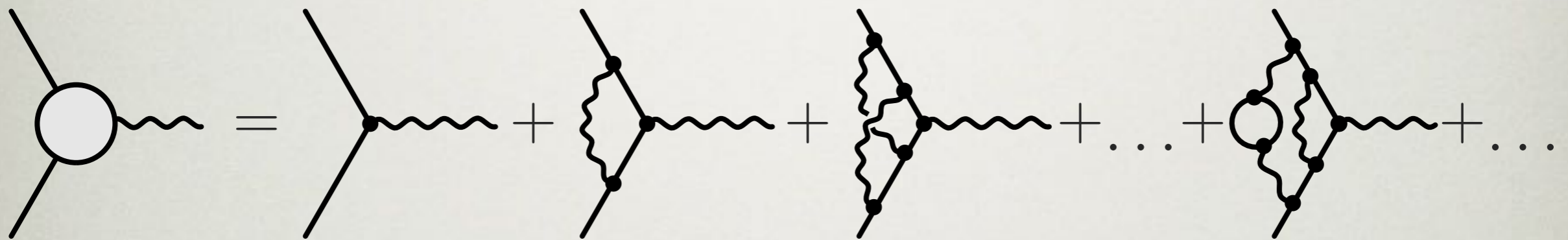
$$\int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \mathcal{L}}$$



Perturbation Theory and Loops

- ◆ Predictions (often) made perturbatively, according to the **loop** expansion:

$$\alpha \approx 1/137.036$$



$$g_e^{\text{thy}} = 2 + \frac{\alpha}{\pi} (1) + \frac{\alpha^2}{\pi^2} \left(\frac{3}{2} \zeta_3 - \pi^2 \log(2) + \zeta_2 + \frac{197}{72} \right) + \dots$$

[Dirac (1933)]

$$= 2.00231930435801\dots$$

[Feynman; Schwinger; Tomanaga (1947)]

$$g_e^{\text{exp}} = 2.00231930436146\dots$$

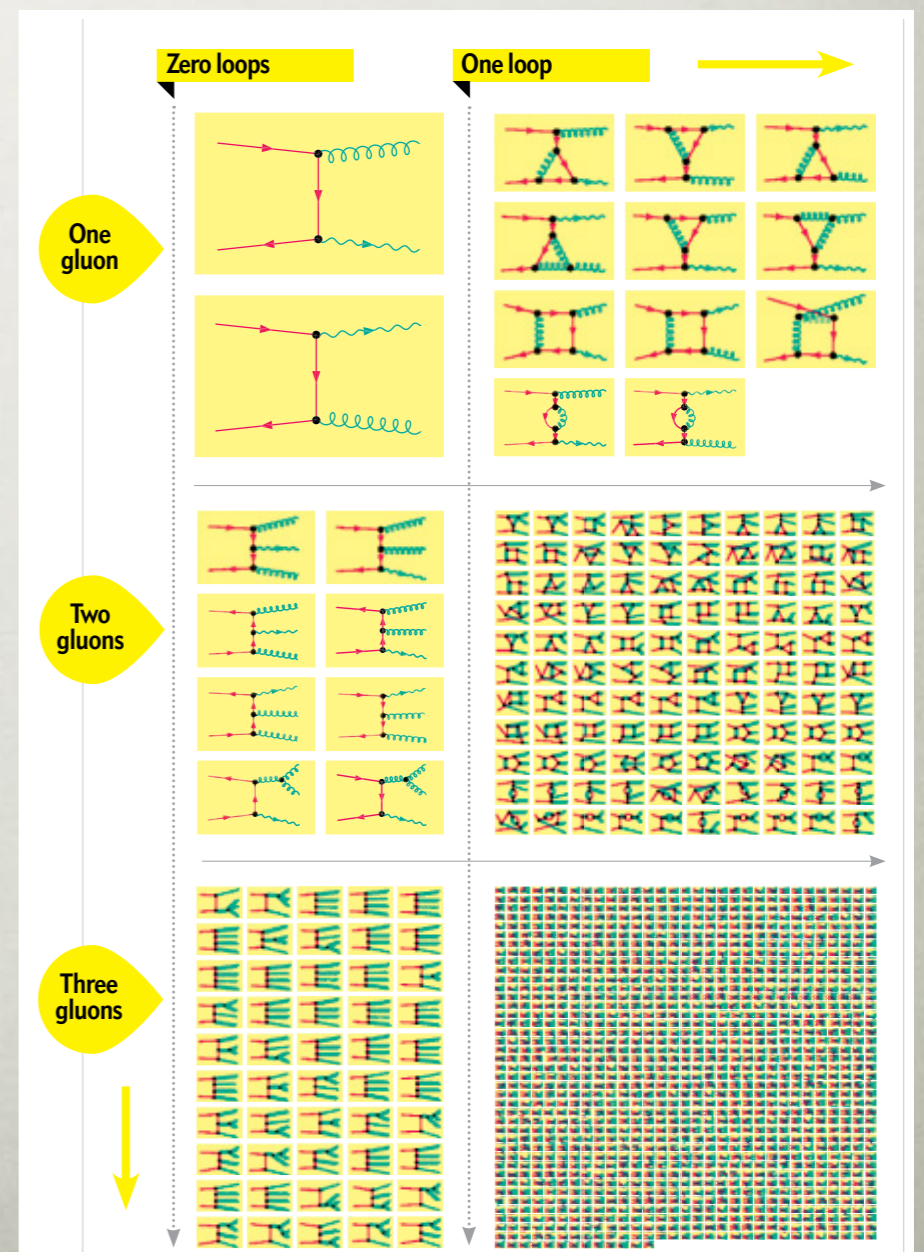
[Petermann (1957)]

[Kinoshita (1990)]

- ◆ *the most precisely tested idea in all of science!*

Explosions of Complexity

- ◆ While ultimately correct, the Feynman expansion renders *all but the most trivial* predictions—
involving the **fewest particles**, at the **lowest orders** of perturbation—
computationally *intractable* or theoretically *inscrutable*



[Bern, Dixon, Kosower, *Scientific American* (2012)]

Needs (Once) Beyond Our Reach

- ◆ Background amplitudes crucial for *e.g.* colliders

Supercollider physics [Rev.Mod.Phys. 56 (1984)]

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

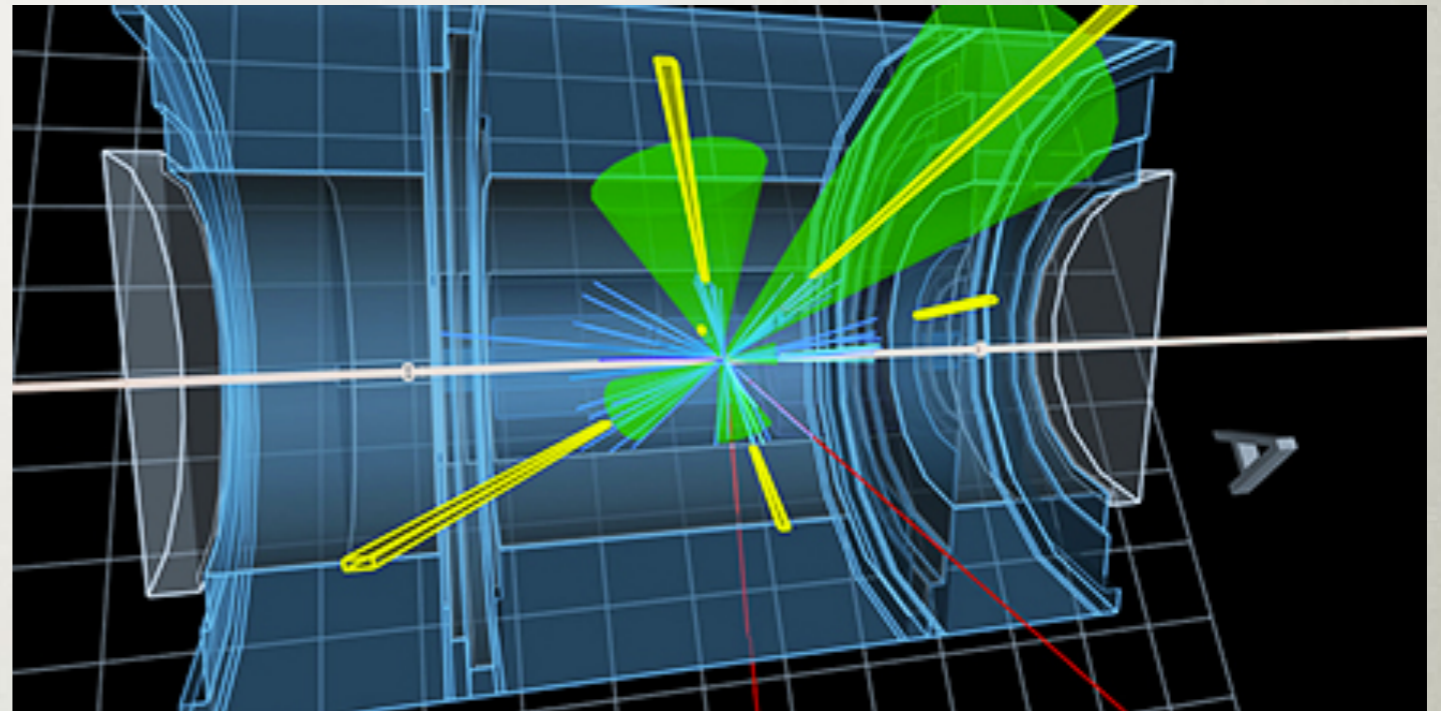
K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.

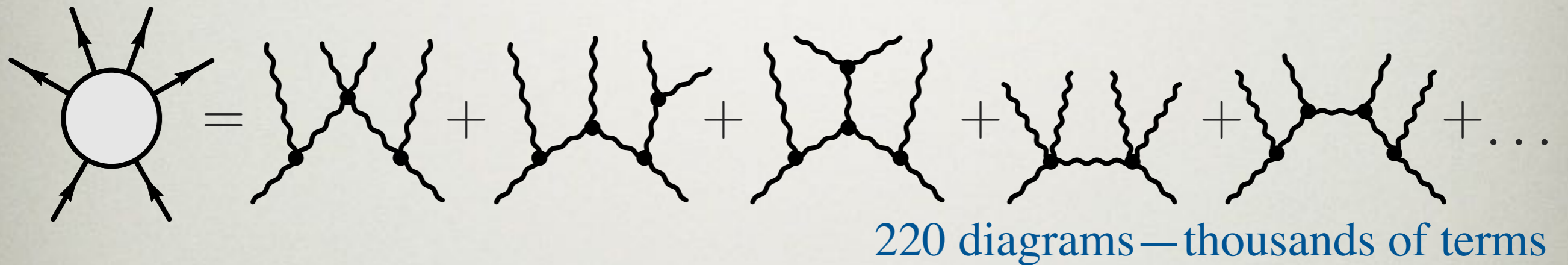


- ◆ Once considered *computationally intractable*

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Needs (Once) Beyond Our Reach

- ◆ Background amplitudes **crucial** for *e.g.* colliders



- ◆ Once considered *computationally intractable*

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Needs (Once) Beyond Our Reach

- ◆ Background amplitudes **crucial** for *e.g.* colliders

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON-GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

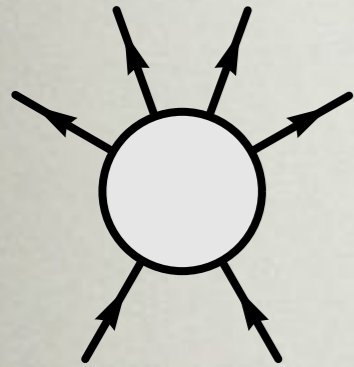
Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[Nucl.Phys. B269 (1985)]

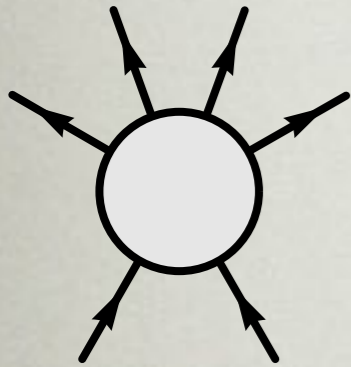
- ◆ Once considered *computationally intractable*

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.



Needs (Once) Beyond Our Reach

◆ Background amplitudes crucial for e.g. colliders



412 S.J. Parke, T.R. Taylor / Four gluon production
gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $I=2$ and replacing the momenta p_1, p_2, p_3, p_4 by $-p_1, -p_2, -p_3, -p_4$.
As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$A_{(2)}^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = (\mathcal{D}^0, \mathcal{D}^1, \mathcal{D}^2, \mathcal{D}^3, \mathcal{D}^4) \cdot \begin{pmatrix} K & K_p & K_s & K_t \\ K_p & K & K_s & K_t \\ K_s & K_s & K & K_t \\ K_t & K_t & K_t & K \end{pmatrix} \cdot \begin{pmatrix} \mathcal{D}_p \\ \mathcal{D}_s \\ \mathcal{D}_t \\ \mathcal{D}_u \end{pmatrix} \quad (6)$$

where $\mathcal{D}_p, \mathcal{D}_s, \mathcal{D}_t$ and \mathcal{D}_u are 11-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5 and p_6 , and K, K_p, K_s and K_t are constant 11×11 symmetric matrices. The vectors $\mathcal{D}_p, \mathcal{D}_s$ and \mathcal{D}_t are obtained from the vector \mathcal{D} by the permutations $(p_1 \leftrightarrow p_2), (p_3 \leftrightarrow p_4)$ and $(p_3 \leftrightarrow p_5, p_4 \leftrightarrow p_6)$, respectively, of the momentum variables in \mathcal{D} . The individual components of the vector \mathcal{D} represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K , which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N^2(N^2-1)$ and $N^2(N^2-1)$, respectively (N is the number of colors, $N=3$ for QCD).

413 S.J. Parke, T.R. Taylor / Four gluon production

TABLE 1
Matrices $K(I, J, K, J=1-11, J=1-11)$

Matrix $K^{(1)}$	Matrix $K^{(2)}$
8 4 -2 2 -1 2 0 1 0 0 -1 4 8 -1 1 -1 0 2 2 0 1 1 -2 -1 8 4 4 1 -1 2 2 1 2 -1 4 8 2 -1 -1 4 1 1 1 -1 -1 4 2 8 1 2 4 -2 -1 4 2 0 -1 -1 18 4 -1 0 1 0 0 -2 1 -1 -2 8 1 -1 0 0 0 1 1 -1 4 -1 -2 8 -1 -1 2 0 0 0 2 1 -2 0 0 -1 8 4 -2 -1 1 1 -1 -1 -1 -1 -4 8 -1 -1 -1 2 -1 4 0 0 -2 -1 8	0 0 0 0 0 0 0 0 0 3 3 -3 0 0 0 0 0 0 0 0 0 3 3 0
Matrix $K^{(3)}$	Matrix $K^{(4)}$
0 0 0 0 2 1 1 0 1 0 -1 0 0 0 0 1 0 0 2 0 1 1 0 0 0 0 0 1 1 1 0 1 1 0 0 0 0 0 1 0 0 2 0 0 1 2 0 0 1 0 2 2 0 0 2 1 0 1 0 1 2 2 4 0 0 -2 1 1 0 0 0 4 2 0 0 0 -1 1 1 0 0 0 0 4 0 0 0 0 1 2 0 0 0 0 0 4 0 0 0 1 2 0 0 0 0 0 0 4 0 0 0 1 1 0 0 0 0 0 2 4 0 -1 -2 0 0 2 -1 -2 0 -1 0 4	3 3 0 0 0 0 0 0 3 0 0 0 3 3 0 0 0 0 0 0 3 0 0 0 3 0 0 0 0 0 0 0 3 0 0 0 3 0 0 0 0 0 0 0 3 0 0 0 0 0 0 0 0 0 0 0 0 3 3 0 0 0 0 0 0 0 0 0 0 3 3 0

414 S.J. Parke, T.R. Taylor / Four gluon production

TABLE 2
Matrices $C(I, J, K, J=1-11, J=1-11)$

Matrix $C^{(1)}$	Matrix $C^{(2)}$
0 0	0 0

415 S.J. Parke, T.R. Taylor / Four gluon production
where ϵ is the totally antisymmetric tensor, $\epsilon_{123456}=1$. For the future use, we define one more function,

$$F(p, p) = ((p_1, p_2)(p_1, p_2) + (p_1, p_3)(p_1, p_3) - (p_1, p_4)(p_1, p_4)) / (p_1, p_4) \quad (10)$$

Note that when evaluating A_3 and A_4 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_2, p_3, p_4 .
The diagrams D_i^2 are listed below:

$$D_1^2(1) = \frac{\delta_2}{s_{12}s_{34}} [((p_1 - p_2)(p_3 - p_4))((p_1 - p_2)(p_5 + p_6)) - ((p_1 - p_2)(p_3 + p_4)) \times ((p_1 - p_2)(p_5 - p_6)) + ((p_3 + p_4)(p_5 - p_6))((p_1 - p_2)(p_3 - p_4))]$$
$$D_2^2(2) = \frac{1}{s_{23}s_{36}} (2E(p_2 - p_3, p_3 - p_4) - 2E(p_3 - p_4, p_2 - p_3) + \delta_2((p_2 - p_3)(p_3 - p_4)))$$
$$D_3^2(3) = \frac{4}{s_{23}s_{34}s_{125}} [((p_1 + p_2 - p_3)(p_4 + p_5 - p_6))E(p_2, p_3) - ((p_1 + p_2 - p_3)(p_4 - p_5 + p_6))E(p_2, p_4) - ((p_1 - p_2 + p_3)(p_4 + p_5 - p_6))E(p_2, p_5)]$$

[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has successfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed.

$$D_1^2(16) = \frac{-4}{s_{12}s_{34}} [s_{23} - s_{34}]E(p_2, p_3)$$
$$D_1^2(17) = \frac{4}{s_{34}s_{45}} [s_{23} - s_{34}]E(p_2, p_3)$$
$$D_1^2(18) = \frac{-4}{s_{12}s_{34}s_{45}} [2(p_1 + p_2)(p_3 - p_4) - s_{34}]E(p_2, p_3)$$
$$D_1^2(19) = \frac{-2}{s_{12}s_{36}} E(p_2, p_3 - p_4)$$
$$D_1^2(20) = \frac{2}{s_{34}s_{45}} E(p_2, p_3 - p_4)$$
$$D_1^2(21) = \frac{-4}{s_{12}s_{34}s_{125}} [s_{23} - s_{34} + s_{23}]E(p_2, p_3)$$
$$D_1^2(22) = \frac{4}{s_{14}s_{23}s_{146}} [s_{23} - s_{34} - s_{23}]E(p_2, p_3)$$
$$D_1^2(23) = \frac{4}{s_{14}s_{23}s_{34}} [2(p_1 + p_4)(p_2 - p_3) + s_{23}]E(p_2, p_3)$$

$$D_2^2(32) = \frac{-4}{s_{12}s_{34}s_{125}} [(p_1 - p_2 + p_3)(p_4 - p_5 - p_6) + s_{125}]E(p_2, p_3)$$
$$D_2^2(33) = \frac{4}{s_{12}s_{34}s_{125}} [(p_1 - p_2 + p_3)(p_4 - p_5 + p_6) - s_{125}]E(p_2, p_3) \quad (11)$$

where $\delta_2 = 1$.
The diagrams D_i^2 are obtained from D_i^0 by replacing δ_2 by $\delta_2 = 0$ and the functions $E(p, p)$ by $G(p, p)$.
The diagrams D_i^0 are listed below:

$$D_2^0(1) = \frac{4}{s_{12}s_{34}s_{125}} (F(p_2, p_3)E(p_2, p_3) - F(p_2, p_4)E(p_2, p_4)) + [F(p_2, p_3) + s_{34}]E(p_2, p_3)$$
$$D_2^0(2) = \frac{-4}{s_{14}s_{23}s_{34}} [(F(p_2, p_3) + \frac{1}{2}s_{34})E(p_2, p_3) + [F(p_2, p_3) + \frac{1}{2}s_{34}]E(p_2, p_3) - F(p_2, p_4)E(p_2, p_4)]$$
$$D_2^0(3) = \frac{4}{s_{12}s_{34}s_{125}} (F(p_2, p_3)E(p_2, p_3) - F(p_2, p_4)E(p_2, p_4) - [F(p_2, p_3) - \frac{1}{2}s_{34} - \frac{1}{2}s_{34}]E(p_2, p_3))$$

$$D_2^0(11) = \frac{1}{2s_{12}s_{25}s_{36}} [(s_{23} + s_{34} - s_{24} - s_{34})E(p_2 - p_3, p_4) - (s_{23} + s_{34} - s_{34} - s_{24})E(p_2 - p_4, p_3) - (s_{23} + s_{34} - s_{34} - s_{24})E(p_2 + p_3, p_4)] \quad (12)$$

The diagrams D_i^0 are listed below:

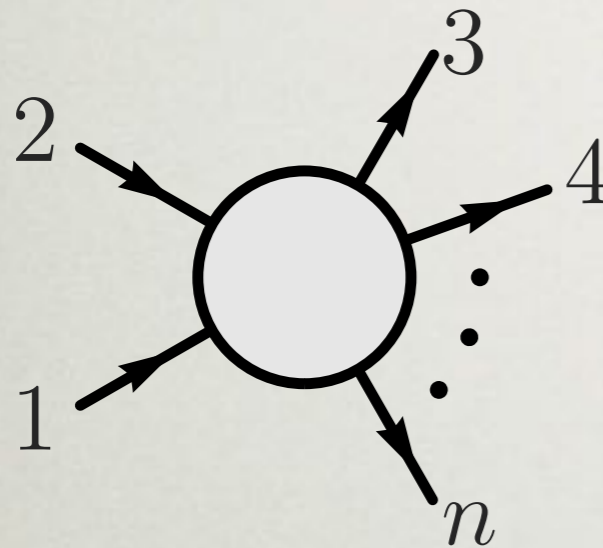
$$D_2^0(1) = \frac{1}{s_{23}s_{34}s_{125}} [s_{12} - s_{24} + s_{34}]E(p_1 - p_2, p_3)$$
$$D_2^0(2) = \frac{1}{s_{14}s_{34}s_{124}} [s_{12} - s_{24} - s_{34}]E(p_1 - p_2, p_3) + s_{34}$$
$$D_2^0(3) = \frac{1}{s_{14}s_{34}s_{145}} [s_{12} - s_{24} + s_{34}]E(p_1 - p_2, p_3) + s_{34}$$
$$D_2^0(4) = \frac{1}{s_{12}s_{34}s_{125}} [s_{12} + s_{23} - s_{12}]E(p_1 - p_2, p_3) + s_{34}$$
$$D_2^0(5) = \frac{1}{s_{12}s_{34}s_{156}} [s_{12} - s_{13} - s_{34}]E(p_1 - p_2, p_3) + s_{34}$$
$$D_2^0(6) = \frac{1}{s_{12}s_{34}s_{125}} [s_{12} - s_{24} - s_{34}]E(p_1 - p_2, p_3)$$

$$D_2^0(15) = \frac{1}{s_{14}s_{23}s_{36}} [((p_2 + p_3)(p_3 - p_4))((p_1 - p_4)(p_2 - p_3)) + ((p_2 - p_3)(p_3 - p_4))((p_1 - p_4)(p_3 + p_4)) + ((p_1 + p_2)(p_3 - p_4))((p_1 - p_4)(p_3 - p_4))] \quad (13)$$
$$D_2^0(16) = \frac{2}{s_{14}s_{34}s_{23}} [((p_2 - p_3)(p_3 + p_4))((p_1 - p_4)(p_2 - p_3)) + ((p_1 + p_2)(p_3 - p_4))((p_1 - p_4)(p_2 - p_3)) + ((p_1 - p_4)(p_2 + p_3))((p_1 - p_4)(p_2 - p_3))]$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11)-(13). The result is substituted to eq. (8) to obtain the vectors \mathcal{D}_p and \mathcal{D}_s . After generating the vectors $\mathcal{D}_p, \mathcal{D}_s, \mathcal{D}_t, \mathcal{D}_u$ and \mathcal{D}_v by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_3 and A_4 . Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.
Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi-gluon amplitudes are tested by checking the gauge invariance. Due to the specifics

Discovery of Shocking Simplicity

- ◆ Within six months, Parke-Taylor stumbled on a simple guess—unquestionably a *theorist's delight*:



$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle}$$

Amplitude for n -Gluon Scattering [PRL 56 (1986)]

Stephen J. Parke and T. R. Taylor

Fermi National Accelerator Laboratory, Batavia, Illinois 60510

(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

$$p_a^\mu \equiv \sigma_{\alpha\dot{\alpha}}^\mu \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}}$$

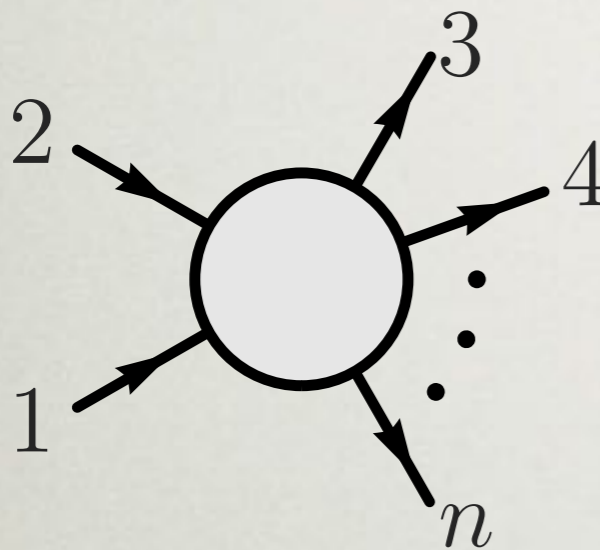
$$\langle a b \rangle \equiv \det(\lambda_a, \lambda_b)$$

$$[a b] \equiv \det(\tilde{\lambda}_a, \tilde{\lambda}_b)$$

[van der Waerden (1929)]

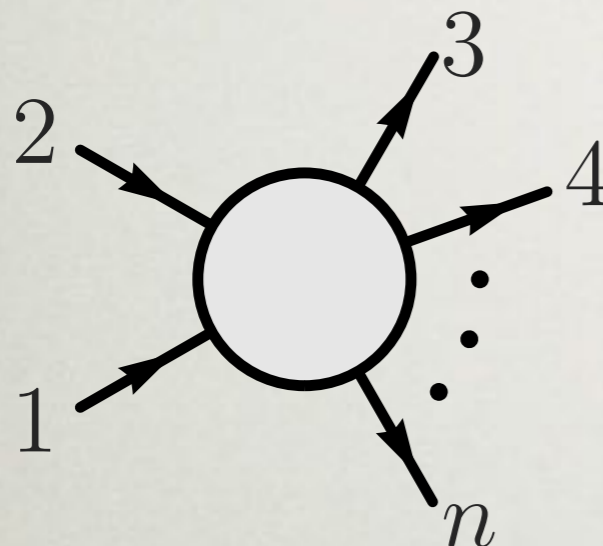
Perturbations of Parke-Taylor

- ◆ What about beyond the leading order?


$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle}$$

Perturbations of Parke-Taylor

- ◆ What about beyond the leading order?


$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times$$
$$\left\{ 1 + \cdots \right\}$$

Perturbations of Parke-Taylor

- ◆ What about beyond the leading order?

[Bern, Dixon, Dunbar, Kosower (1994)]

$$\left\{ 1 + \sum_{a < b} \text{diagram} + \dots \right\} = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times$$

Perturbations of Parke-Taylor

- ◆ What about beyond the leading order?

[Arkani-Hamed, **JB**, Cachazo, Trnka (2010)]

$$\left\{ 1 + \sum_{a < b} \text{[diagram]} + \sum_{a < b < c < d} \text{[diagram]} + \dots \right\}$$

Perturbations of Parke-Taylor

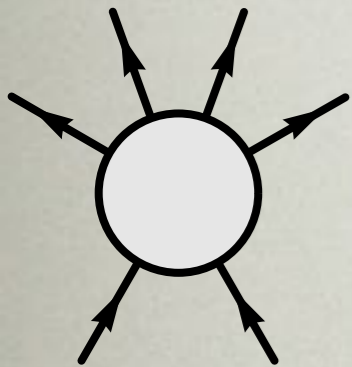
- ◆ What about beyond the leading order?

[Arkani-Hamed, **JB**, Cachazo, Trnka (2011)]

$$\begin{aligned}
 & \text{Diagram with } n \text{ external legs} = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \dots \langle n 1 \rangle} \times \\
 & \left\{ 1 + \sum_{a < b} \text{Diagram} + \sum_{a < b < c < d} \text{Diagram} + \sum_{a < b \leq c < d \leq e < f} \text{Diagram} + \sum_{a \leq b < c < d \leq e < f} \text{Diagram} + \dots \right\}
 \end{aligned}$$

What About After Integration?

- ◆ Integrate the Parke-Taylor 2-to-4 amplitude in sYM
 - ▶ divergences exponentiate, leaving a finite *remainder*
- ◆ *Heroically* computed by Del Duca, Duhr, Smirnov in 2010, in terms of ‘Goncharov’ polylogarithms



The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

Vittorio Del Duca

*PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
E-mail: vittorio.del.duca@cern.ch*

Claude Duhr

*Institute for Particle Physics Phenomenology, University of Durham
Durham, DH1 3LE, U.K.
E-mail: claude.duhr@durham.ac.uk*

Vladimir A. Smirnov

*Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru*

[Del Duca, Duhr, Smirnov (2010)]

What About After Integration?

- ◆ Integrate the Parke-Taylor 2-to-4 amplitude in sYM
 - ▶ divergences exponentiate, leaving a finite remainder

◆ Hero in 20

The grid contains a large number of mathematical expressions, likely representing the integration of the Parke-Taylor 2-to-4 amplitude in sYM. The expressions are organized into rows labeled 'h', 'K', 'i', 't', 'c' and columns labeled '1' through '6'. The equations are complex, involving many terms with denominators and numerators, and are arranged in a way that suggests a systematic derivation or expansion. A central diagram of a six-point vertex with arrows is also present on the left side of the grid.

OV
LS

What About After Integration?

- ◆ Integrate the Parke-Taylor 2-to-4 amplitude in sYM
 - ▶ divergences exponentiate, leaving a finite *remainder*
- ◆ *Heroically* computed by Del Duca, Duhr, Smirnov in 2010, in terms of ‘Goncharov’ polylogarithms

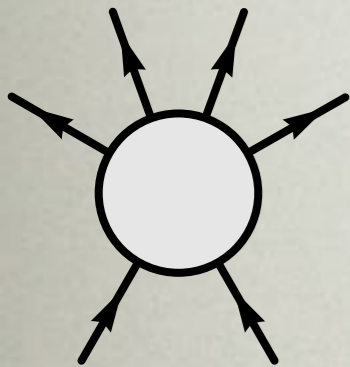
Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,¹ M. Spradlin,² C. Vergu,² and A. Volovich²

¹Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA

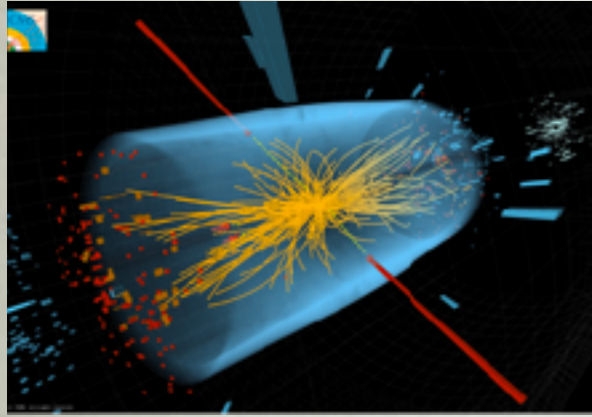
²Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives. **[Goncharov, Spradlin, Vergu, Volovich (2010)]**



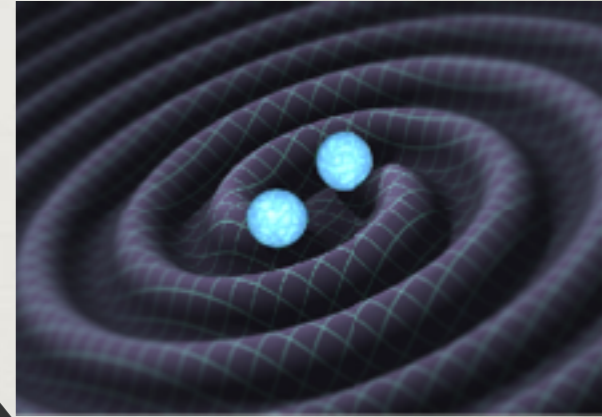
$$R(u_1, u_2, u_3) = \sum_{i=1}^3 \left(L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) - \frac{1}{8} \left(\sum_{i=1}^3 \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{J^4}{24} + \frac{1}{2} \zeta_2 (J^2 + \zeta_2)$$

Amplitudes: a Virtuous Cycle



Compute Something

beyond the reach of
recent imagination



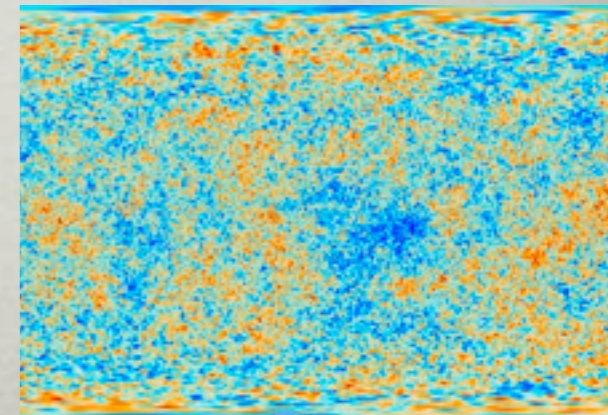
Exploit Simplicity
to build more powerful
computational technology



Discover Simplicity
beyond expectations



Understand Why
study, understand, explain it,
& explore its consequences

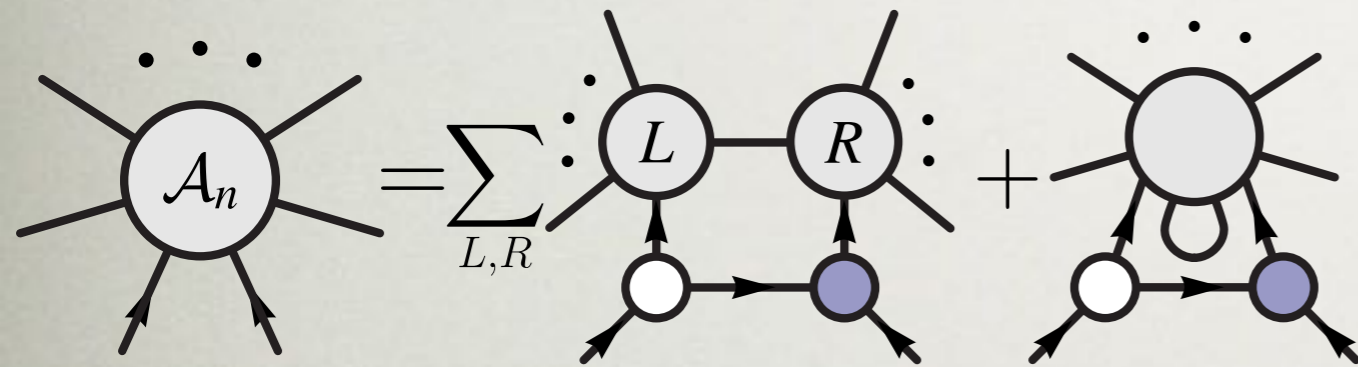


*Constructing Integrands
for Loop Amplitudes
(constructively)*

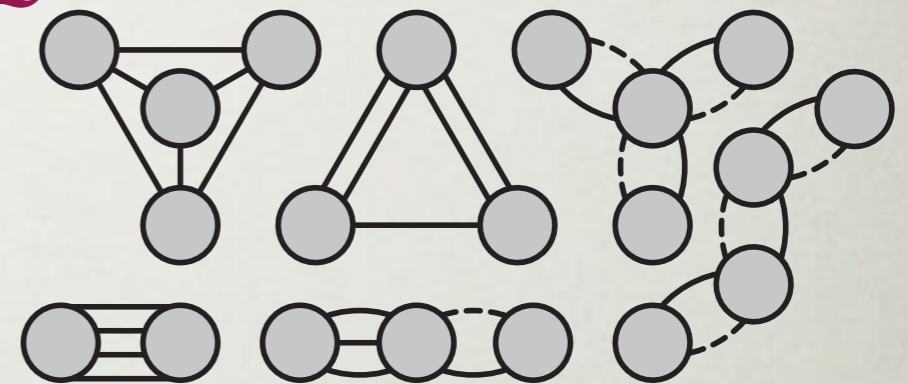
Novel Representations of Integrands

- ◆ Powerful new tools now exist for *understanding* and *computing* integrands in perturbation theory

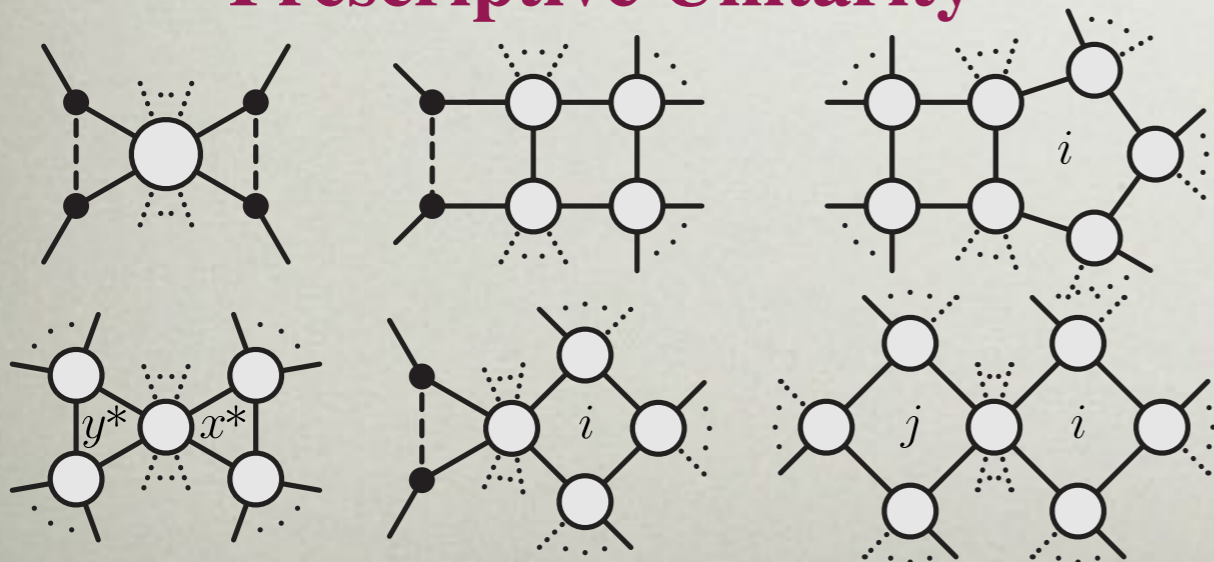
Recursion Relations



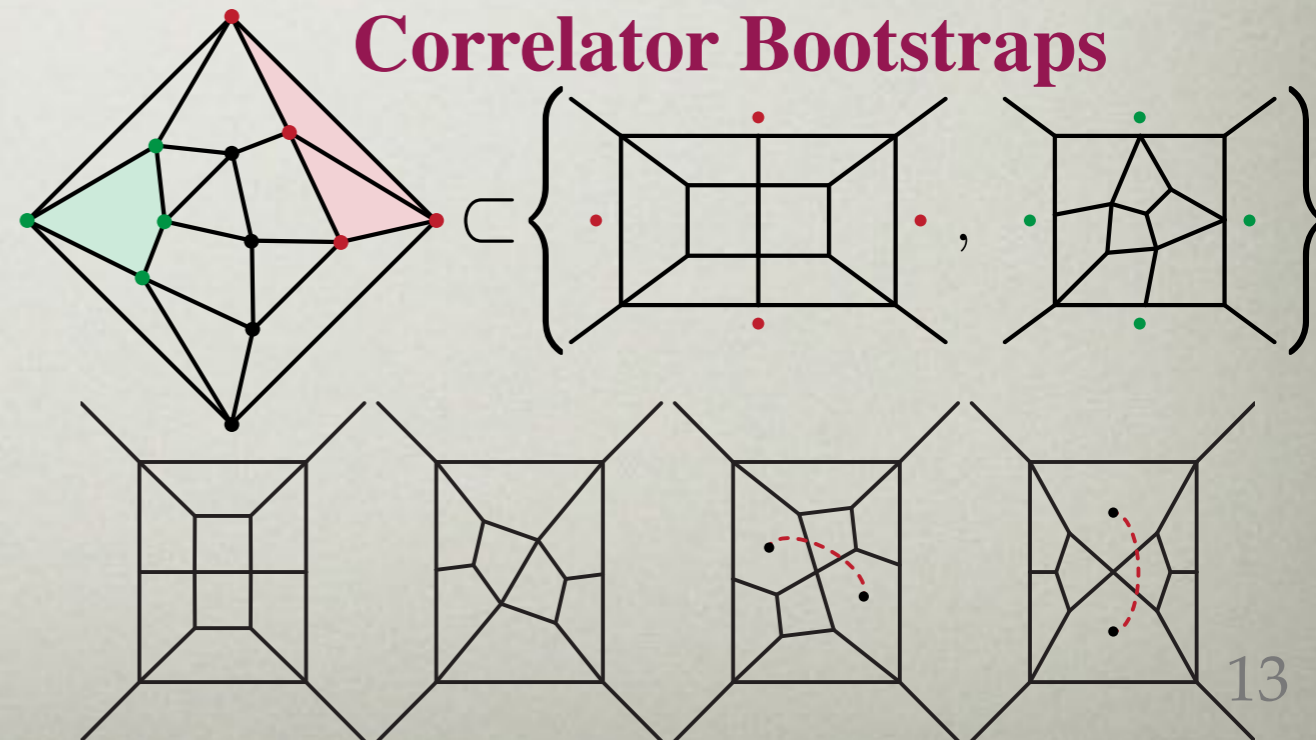
Q -cuts and Forward Limits



Prescriptive Unitarity



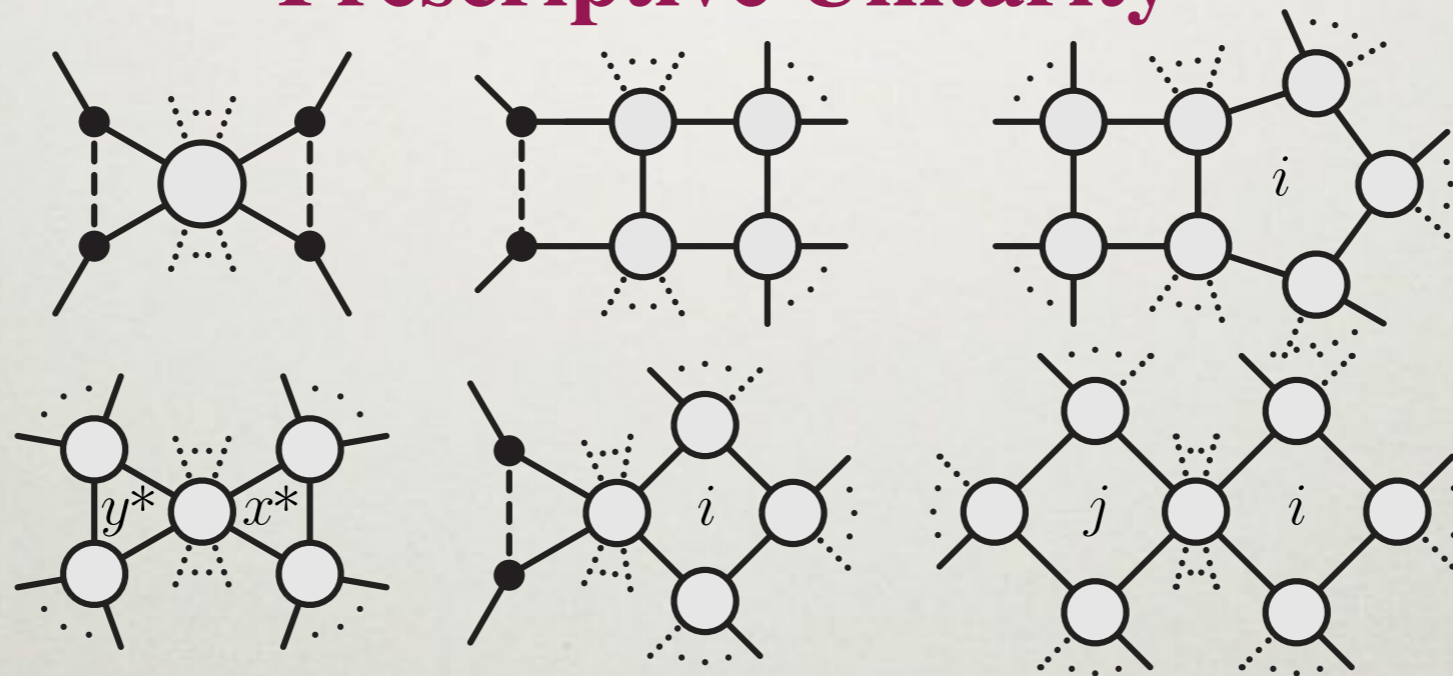
Correlator Bootstraps



Novel Representations of Integrands

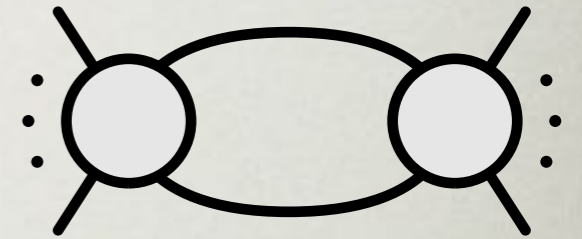
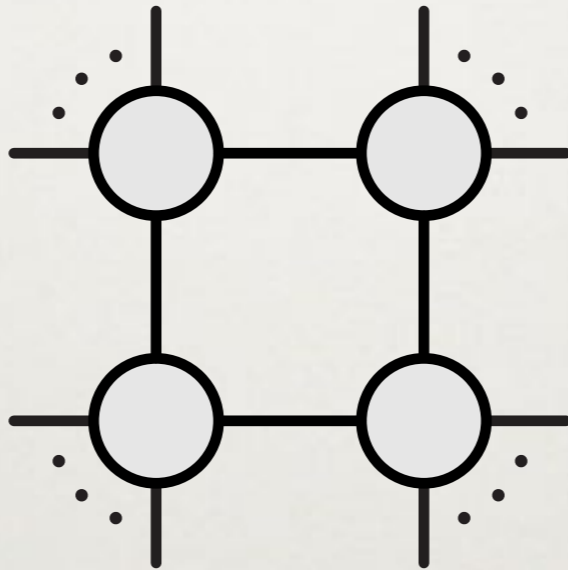
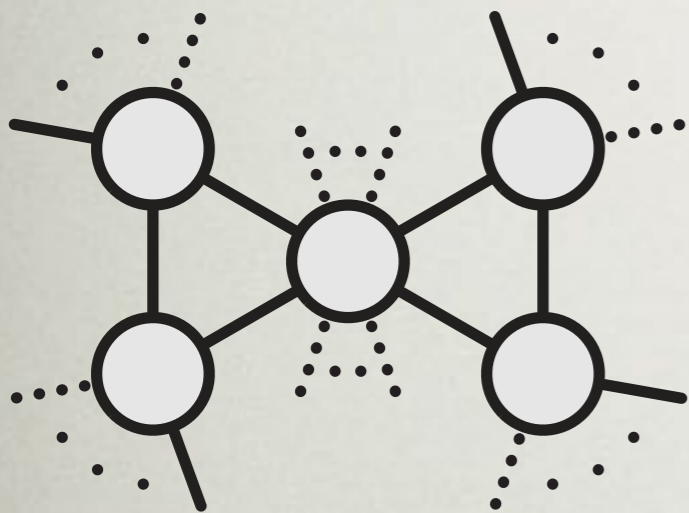
- ◆ Powerful new tools now exist for *understanding* and *computing* integrands in perturbation theory

Prescriptive Unitarity



The Cuts of Loop Amplitudes

- ◆ **On-Shell Functions:** scattering amplitudes, and functions built thereof—as *networks* of amplitudes

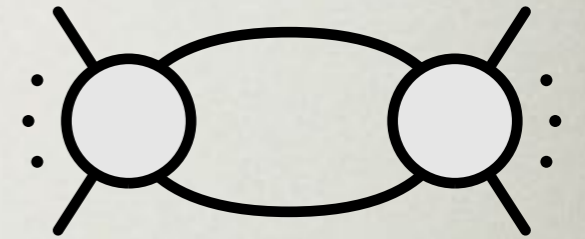
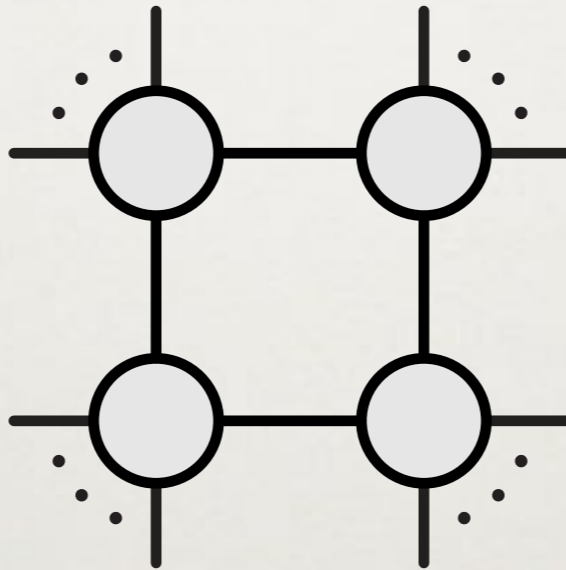
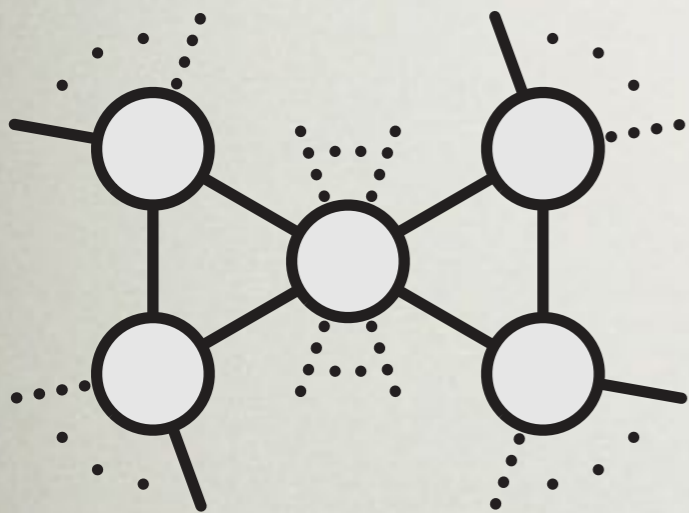


- ◆ **Locality:** amplitudes independent, so multiplied
- ◆ **Unitarity:** internal particles unseen, so summed

$$f_{\Gamma} \equiv \prod_{i \in I} \left(\sum_{h_i, c_i} \int d^{d-1} \text{LIPS}_i \right) \prod_{v \in V} \mathcal{A}_v$$

The Cuts of Loop Amplitudes

- ◆ **On-Shell Functions:** scattering amplitudes, and functions built thereof—as *networks* of amplitudes



- ◆ defined for all *all* quantum field theories—*exclusively* in terms of physical (observable) states
- ◆ can be used to reconstruct *all* loop amplitudes

$$f_{\Gamma} \equiv \prod_{i \in I} \left(\sum_{h_i, c_i} \int d^{d-1} \text{LIPS}_i \right) \prod_{v \in V} \mathcal{A}_v$$

General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; ...]

- ◆ **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum \mathcal{a}_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

$$A_n^{L=2} = \sum_{\mathcal{L}} f_{\mathcal{L}} \left[\text{Diagram 1} \right] \left[\text{Diagram 2} \right] \in \left\{ \begin{array}{l} \left[\text{Diagram 3} \right], \left[\text{Diagram 4} \right], \left[\text{Diagram 5} \right] \\ \left[\text{Diagram 6} \right], \left[\text{Diagram 7} \right], \left[\text{Diagram 8} \right] \end{array} \right\}$$

$f_{\mathcal{L}} \in \left\{ \begin{array}{l} \left[\text{Diagram 6} \right], \left[\text{Diagram 7} \right], \left[\text{Diagram 8} \right] \end{array} \right\}$

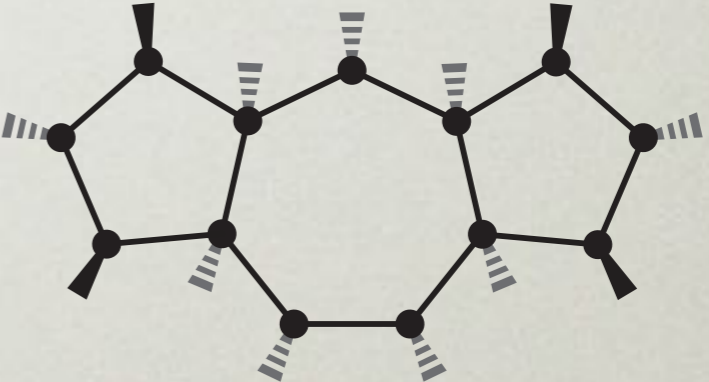
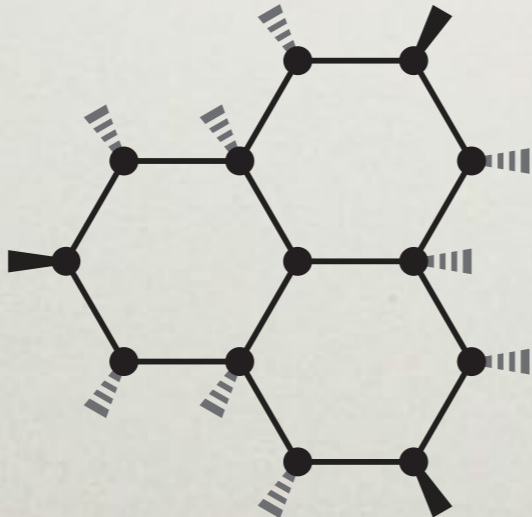
General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; ...]

- ◆ **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum a_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

- ◆ Once an independent basis is chosen, coefficients are determined by (*evaluations* / cuts on) **cuts**

$$A_n^{L=3} = \sum_{\mathcal{W}} f_{\mathcal{W}} \text{ [Diagram 1]} + \sum_{\mathcal{L}} f_{\mathcal{L}} \text{ [Diagram 2]}$$


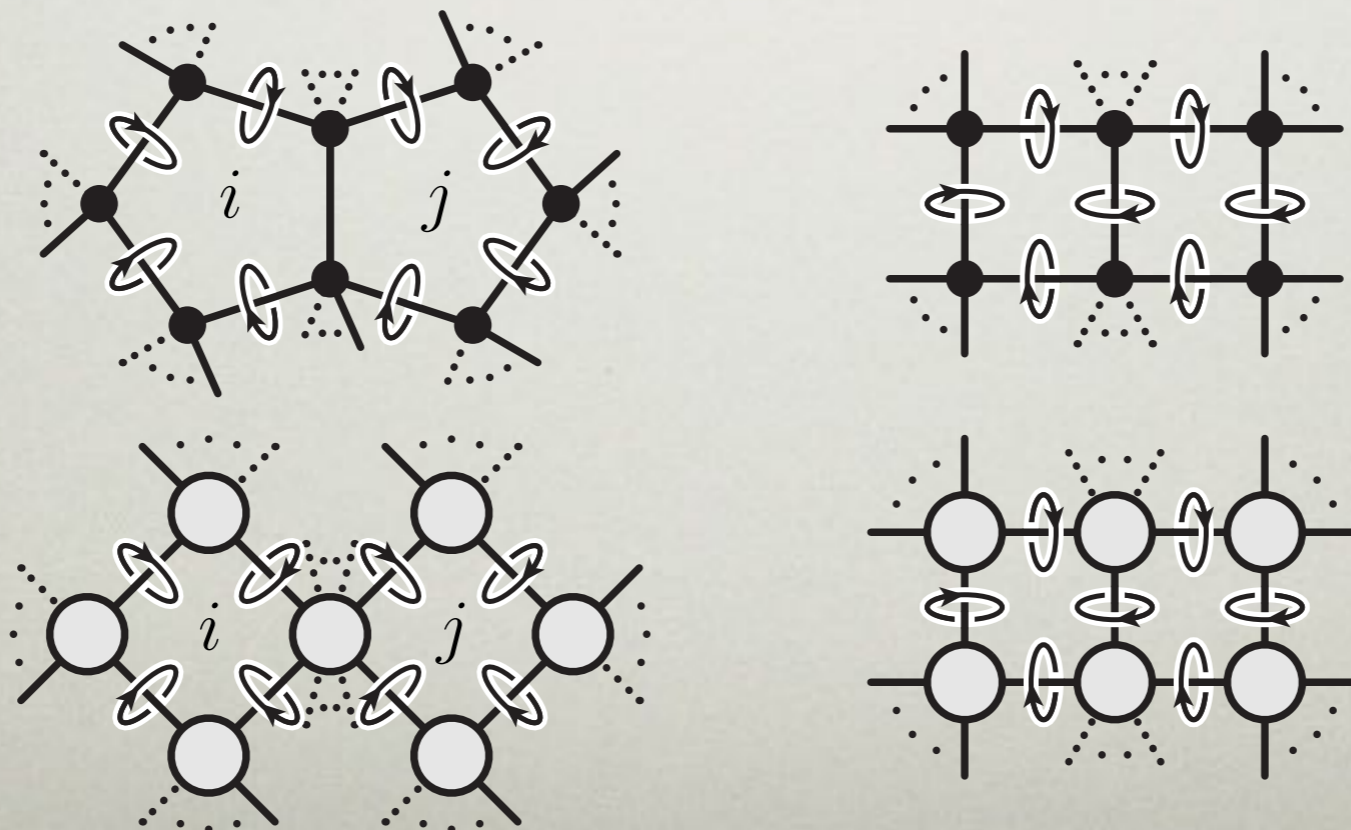
General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; ...]

- ◆ **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum a_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

- ◆ Once an independent basis is chosen, coefficients are determined by (*evaluations* / cuts on) **cuts**



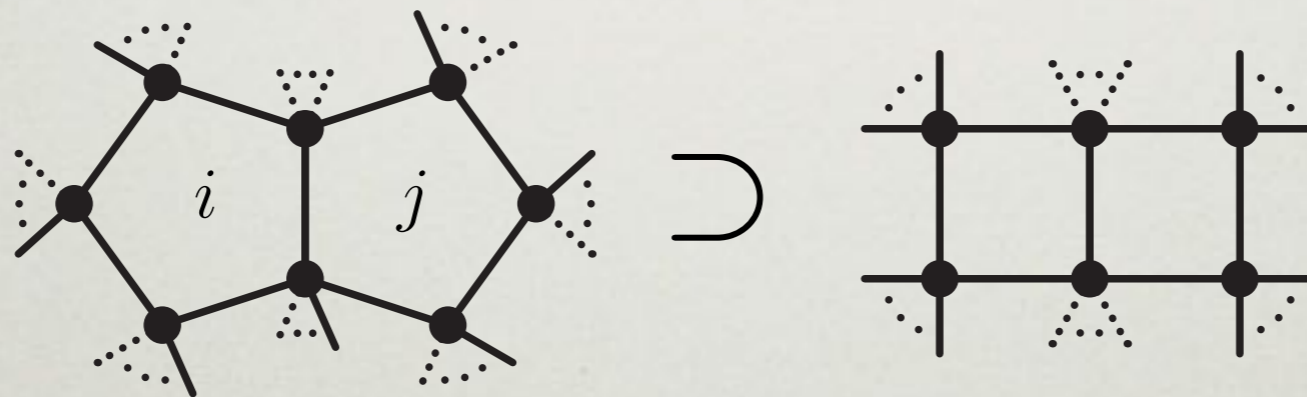
General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; ...]

- ◆ **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum a_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

- ◆ Once an independent basis is chosen, coefficients are determined by (*evaluations* / cuts on) **cuts**



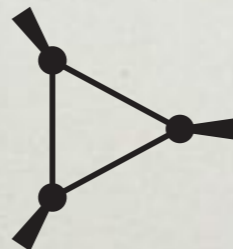
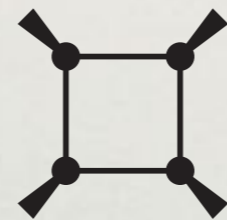
General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; ...]

- ◆ **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum a_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

- ◆ Once an independent basis is chosen, coefficients are determined by (*evaluations* / cuts on) **cuts**



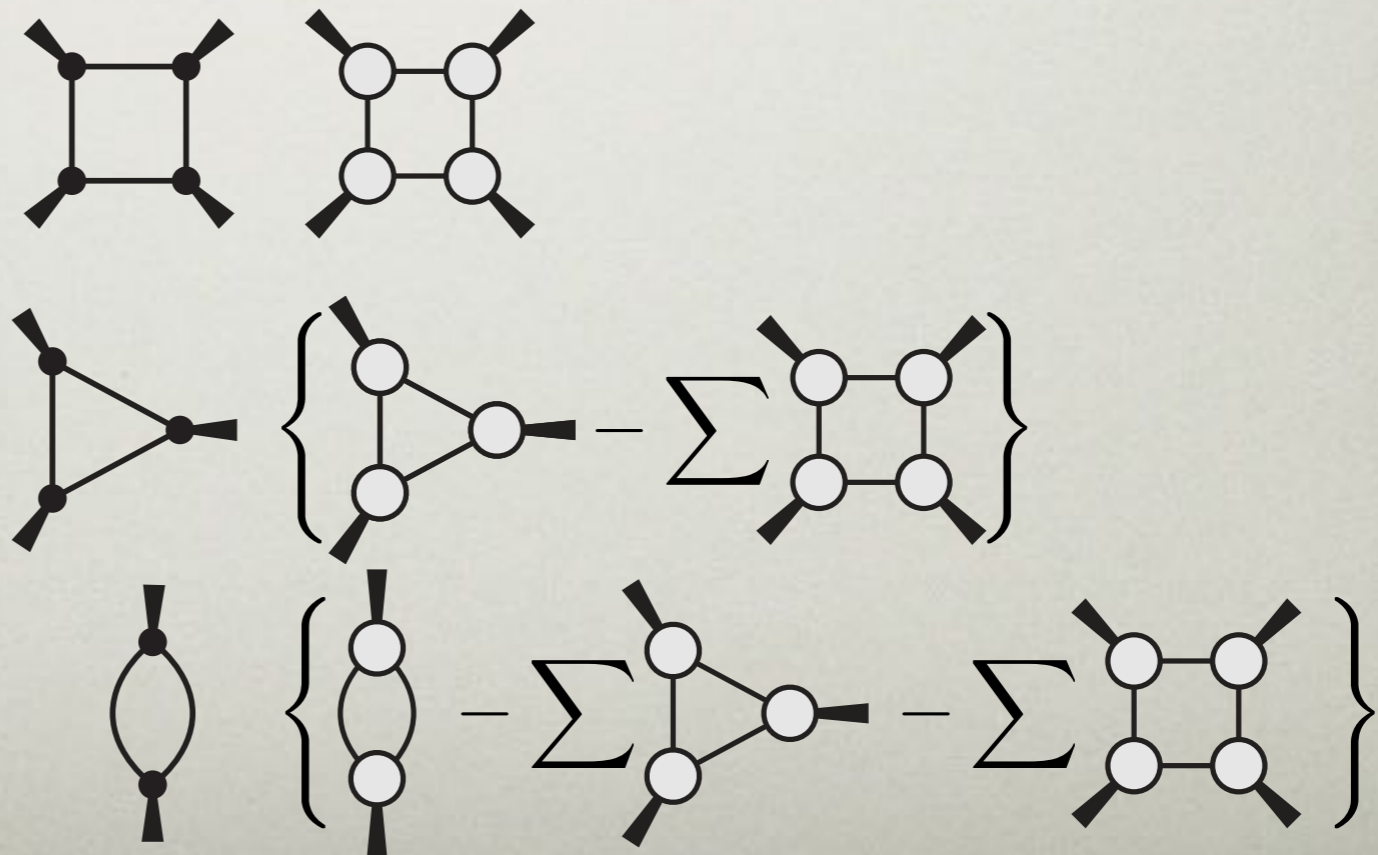
General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; ...]

- ◆ **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum a_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

- ◆ Once an independent basis is chosen, coefficients are determined by (*evaluations* / cuts on) **cuts**



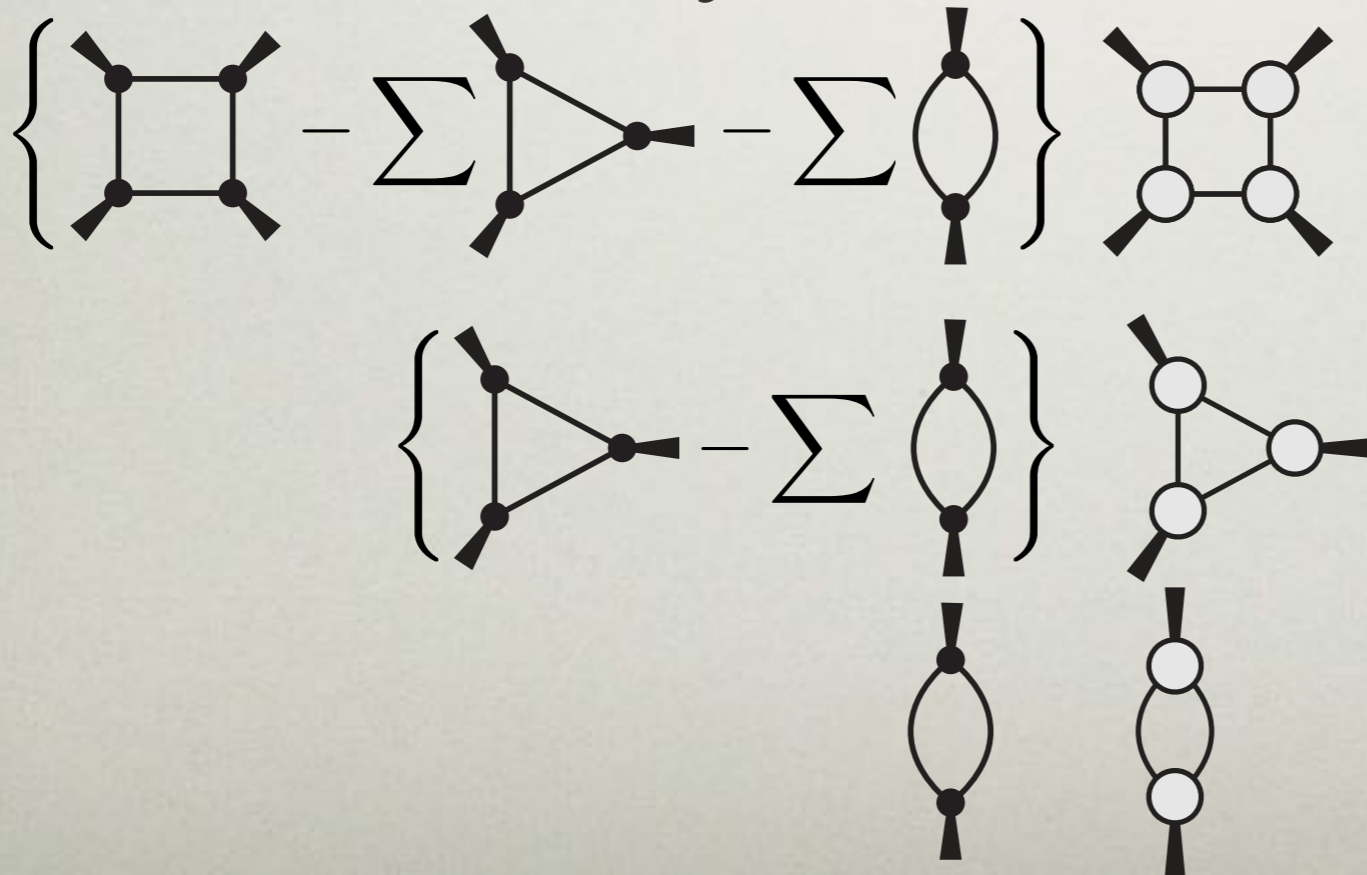
General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; ...]

- ◆ **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum a_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

- ◆ Once an independent basis is chosen, coefficients are determined by (*evaluations* / cuts on) **cuts**



General, Generalized Unitarity

[Bern, Dixon, Kosower; Dunbar; ...]

- ◆ **Integrands** are rational functions—so may be expanded into an *arbitrary* (but complete) **basis**:

$$\mathcal{A}^L = \sum a_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

- ◆ Once an independent basis is chosen, coefficients are determined by (*evaluations* / cuts on) **cuts**

¿What makes a basis a *good basis*?

*A Basis “Big Enough”
for Integrands in the
Standard Model*

Building a Basis 'Big Enough'

- ♦ WLOG: write loop-dependent numerators as **sums of products of (translates of) *inverse propagators***:

$$(\ell|Q)_m := (\ell + Q)^2 - m^2 + i\epsilon \quad (\ell|Q) := (\ell|Q)_{m=0}$$

$$[\ell] := \text{span}_Q \{(\ell|Q)\} \quad \text{rank}([\ell]) = (d+2)$$

$$= \text{span} \{ \ell^2, \ell \cdot k_i, 1 \} \quad \text{rank}([\ell]^k) = \binom{d+k}{d} + \binom{d+k-1}{d}$$

$$[\ell]^k := \text{span}_{Q_i} \left\{ \prod_{i=1}^k (\ell|Q_i) \right\} \quad [\ell]^0 \subset [\ell]^1 \subset [\ell]^2 \subset \dots \subset [\ell]^q$$

$$\widehat{[\ell]}^q := [\ell]^q \setminus [\ell]^{q-1} \quad [\ell]^q = 1 \oplus \widehat{[\ell]}^1 \oplus \widehat{[\ell]}^2 \oplus \dots \oplus \widehat{[\ell]}^q$$

Building a Basis 'Big Enough'

- ♦ WLOG: write loop-dependent numerators as **sums of products of (translates of) *inverse propagators***:

$$(\ell|Q)_m := (\ell + Q)^2 - m^2 + i\epsilon \quad (\ell|Q) := (\ell|Q)_{m=0}$$

$$[\ell] := \text{span}_Q \{(\ell|Q)\} \quad \ell^2 =: \widehat{\ell}^2 - \mu^2 \quad \ell^i \ell^j \notin [\ell]$$

$$= \text{span} \{ \ell^2, \ell \cdot k_i, 1 \} \quad \ell^2 (=: \widehat{\ell}^2 - \mu^2) \in [\ell]$$

$$[\ell]^k := \text{span}_{Q_i} \left\{ \prod_{i=1}^k (\ell|Q_i) \right\} \quad [\ell]^0 \subset [\ell]^1 \subset [\ell]^2 \subset \dots \subset [\ell]^q$$

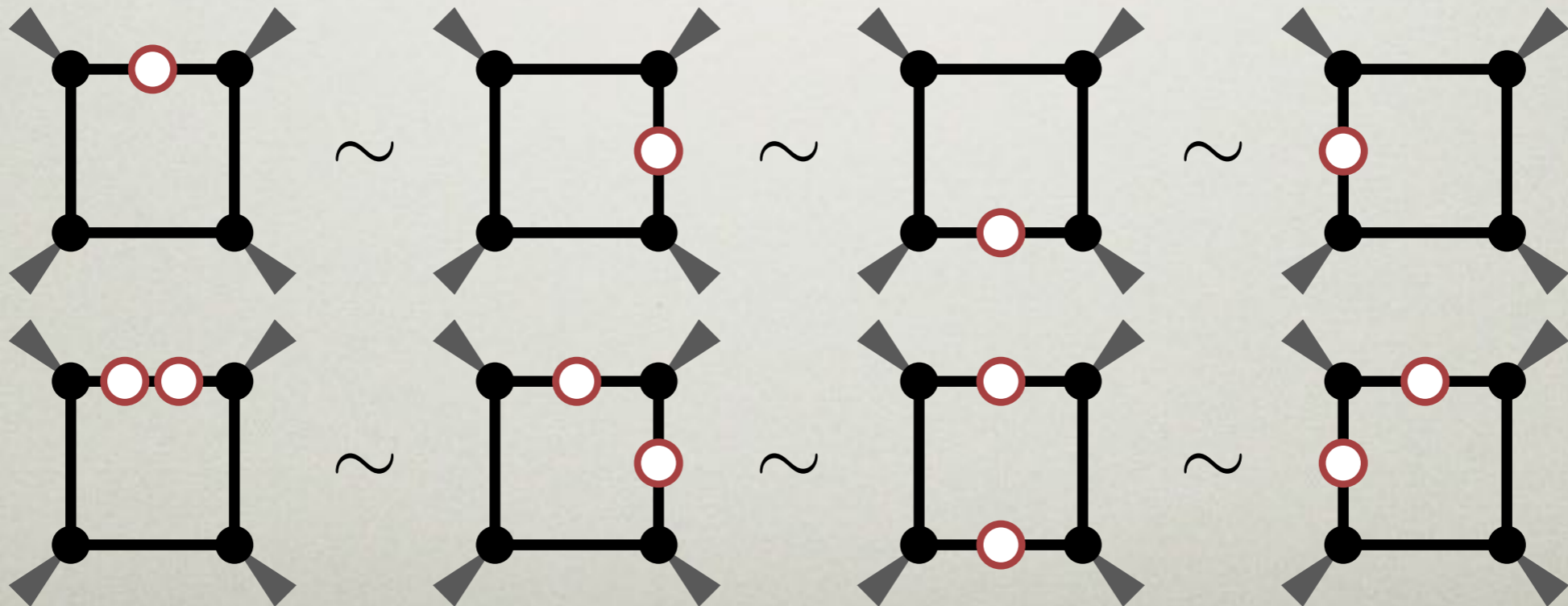
$$\widehat{[\ell]}^q := [\ell]^q \setminus [\ell]^{q-1} \quad [\ell]^q = 1 \oplus \widehat{[\ell]}^1 \oplus \widehat{[\ell]}^2 \oplus \dots \oplus \widehat{[\ell]}^q$$

Building a Basis 'Big Enough'

- ◆ In terms of these, define a generalized propagator:

$$\text{---} \circ \text{---} \xrightarrow{\vec{l}} := \frac{[l]}{l^2} \supset \left\{ \text{---} \sim \text{---} \xrightarrow{\vec{l}}, \text{---} \xrightarrow{\vec{l}}, \text{---} \text{---} \text{---} \xrightarrow{\vec{l}}, 1 \right\}$$

$$\text{---} \circ \circ \text{---} \xrightarrow{\vec{l}} := \frac{[l]^2}{l^2} \quad (\text{would include gravitons})$$

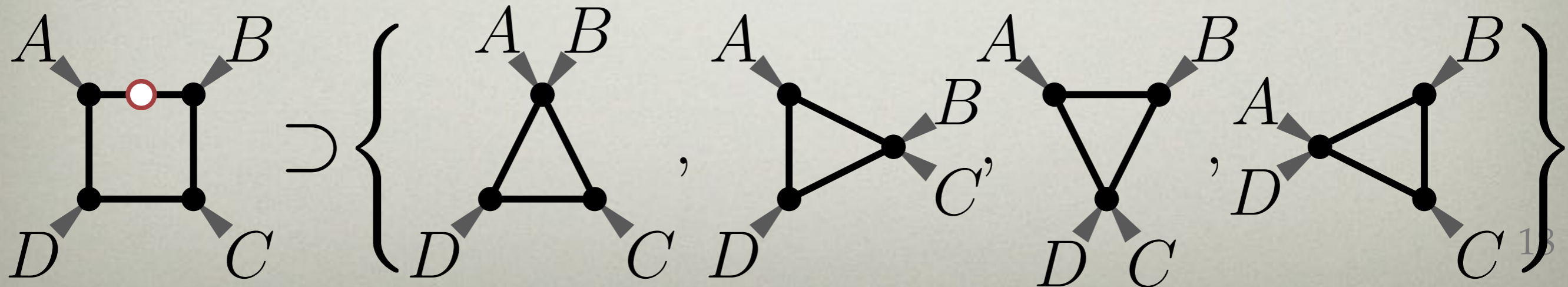
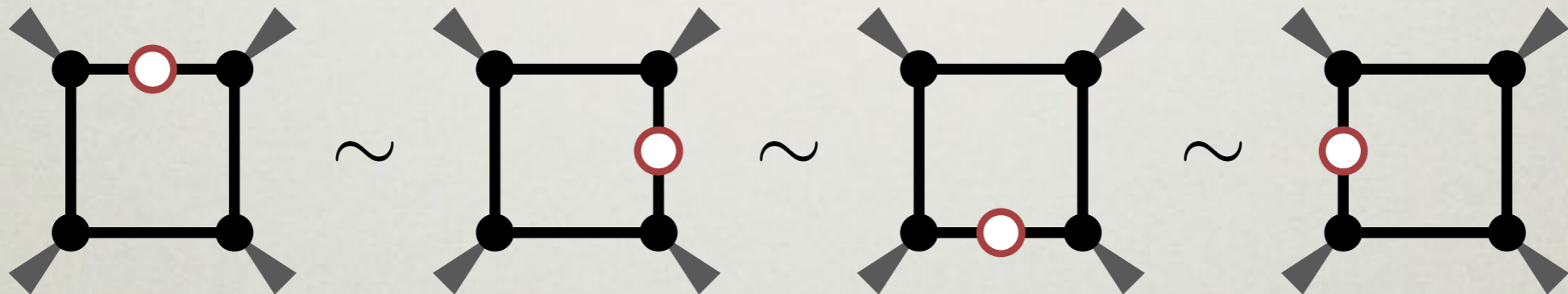


Building a Basis 'Big Enough'

◆ In terms of these, define a generalized propagator:

$$\text{---} \circ \text{---} \xrightarrow{\vec{l}} := \frac{[l]}{l^2} \supset \left\{ \text{---} \sim \text{---} \xrightarrow{\vec{l}}, \text{---} \xrightarrow{\vec{l}}, \text{---} \xrightarrow{\vec{l}}, 1 \right\}$$

$$\text{---} \circ \circ \text{---} \xrightarrow{\vec{l}} := \frac{[l]^2}{l^2} \quad (\text{would include gravitons})$$



Building a Basis 'Big Enough'

- ◆ In terms of these, define a generalized propagator:

$$\text{---} \circ \text{---} \xrightarrow{\vec{l}} := \frac{[l]}{l^2} \supset \left\{ \text{---} \text{---} \text{---} \xrightarrow{\vec{l}}, \text{---} \xrightarrow{\vec{l}}, \text{---} \text{---} \text{---} \xrightarrow{\vec{l}}, 1 \right\}$$

$$\text{---} \circ \text{---} \circ \text{---} \xrightarrow{\vec{l}} := \frac{[l]^2}{l^2} \quad (\text{would include gravitons})$$

- ◆ The loop-dependent part of any SM integrand will be spanned by the basis of "0-gons"—at L loops(!)

$$\mathcal{B}_0 := \left\{ 1, \text{---} \circ \text{---}, \text{---} \circ \text{---} \circ \text{---}, \text{---} \circ \text{---} \circ \text{---} \circ \text{---}, \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---}, \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---}, \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---}, \dots \right\}$$

$$\mathcal{B}_0 \supset \left\{ 1, \dots, \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---}, \dots, \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---}, \dots \right\}$$

[Feng, Huang (2012)]

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

◆ In any dimension, the 0-gons reduce to finite size:

$$\mathcal{B}_0 := \left\{ 1, \text{ (loop) }, \text{ (2-loop) }, \text{ (triangle) }, \text{ (square) }, \text{ (pentagon) }, \text{ (hexagon) }, \text{ (heptagon) }, \text{ (octagon) }, \dots \right\}$$

$$\mathcal{I}_p^q := \text{span} \left\{ \frac{[\ell]^q}{(\ell|P_1) \cdots (\ell|P_p)} \right\}$$

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

◆ In any dimension, the 0-gons reduce to finite size:

$$\mathcal{B}_0 := \left\{ 1, \text{ (loop) }, \text{ (2-loop) }, \text{ (triangle) }, \text{ (square) }, \text{ (pentagon) }, \text{ (hexagon) }, \text{ (heptagon) }, \text{ (octagon) }, \dots \right\}$$

$$\mathcal{I}_p^q := \text{span} \left\{ \frac{[\ell]^q}{(\ell|P_1) \cdots (\ell|P_p)} \right\}$$

$$\mathcal{B}_0 := \mathcal{I}_0^0 \cup \mathcal{I}_1^1 \cup \mathcal{I}_2^2 \cup \mathcal{I}_3^3 \cup \mathcal{I}_4^4 \cup \mathcal{I}_5^5 \cup \mathcal{I}_6^6 \cup \mathcal{I}_7^7 \cup \dots$$

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

◆ In any dimension, the 0-gons reduce to finite size:

$$\mathcal{B}_0 := \left\{ 1, \text{ (loop) }, \text{ (2-loop) }, \text{ (3-loop) }, \text{ (4-loop) }, \text{ (5-loop) }, \text{ (6-loop) }, \text{ (7-loop) }, \dots \right\}$$

$$\mathcal{I}_p^q := \text{span} \left\{ \frac{[\ell]^q}{(\ell|P_1) \cdots (\ell|P_p)} \right\}$$

$$\mathcal{B}_0 := \mathcal{I}_0^0 \cup \mathcal{I}_1^1 \cup \mathcal{I}_2^2 \cup \mathcal{I}_3^3 \cup \mathcal{I}_4^4 \cup \mathcal{I}_5^5 \cup \mathcal{I}_6^6 \cup \mathcal{I}_7^7 \cup \dots$$

$$\mathcal{I}_0^0 \subset \mathcal{I}_1^1 \subset \mathcal{I}_2^2 \subset \mathcal{I}_3^3 \subset \mathcal{I}_4^4 \subset \mathcal{I}_5^5 \subset \mathcal{I}_6^6 \subset \mathcal{I}_7^7 \subset \dots$$

$$\widehat{\mathcal{I}}_p^p := \mathcal{I}_p^p \setminus \mathcal{I}_{p-1}^{p-1}$$

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

◆ In any dimension, the 0-gons reduce to finite size:

$$\mathcal{B}_0 := \left\{ 1, \text{ (loop) }, \text{ (2-loop) }, \text{ (triangle) }, \text{ (square) }, \text{ (pentagon) }, \text{ (hexagon) }, \text{ (heptagon) }, \text{ (octagon) }, \dots \right\}$$

$$\mathcal{I}_p^q := \text{span} \left\{ \frac{[\ell]^q}{(\ell|P_1) \cdots (\ell|P_p)} \right\}$$

$$\mathcal{B}_0 := \widehat{\mathcal{I}}_0^0 \oplus \widehat{\mathcal{I}}_1^1 \oplus \widehat{\mathcal{I}}_2^2 \oplus \widehat{\mathcal{I}}_3^3 \oplus \widehat{\mathcal{I}}_4^4 \oplus \widehat{\mathcal{I}}_5^5 \oplus \widehat{\mathcal{I}}_6^6 \oplus \widehat{\mathcal{I}}_7^7 \oplus \dots$$

$$\mathcal{I}_0^0 \subset \mathcal{I}_1^1 \subset \mathcal{I}_2^2 \subset \mathcal{I}_3^3 \subset \mathcal{I}_4^4 \subset \mathcal{I}_5^5 \subset \mathcal{I}_6^6 \subset \mathcal{I}_7^7 \subset \dots$$

$$\widehat{\mathcal{I}}_p^p := \mathcal{I}_p^p \setminus \mathcal{I}_{p-1}^{p-1}$$

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

◆ In any dimension, the 0-gons reduce to finite size:

$$\mathcal{B}_0 := \left\{ 1, \text{ (loop) }, \text{ (2-loop) }, \text{ (triangle) }, \text{ (square) }, \text{ (pentagon) }, \text{ (hexagon) }, \text{ (heptagon) }, \text{ (octagon) }, \dots \right\}$$

$$\mathcal{I}_p^q := \text{span} \left\{ \frac{[\ell]^q}{(\ell|P_1) \cdots (\ell|P_p)} \right\}$$



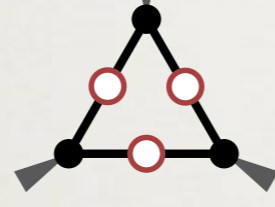
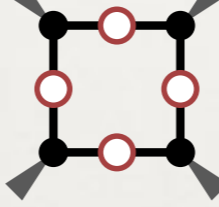
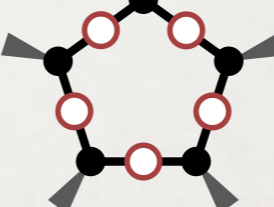
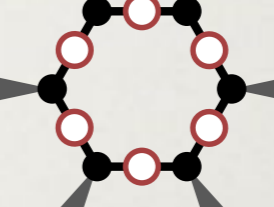
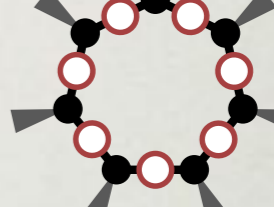
$$\mathcal{B}_0 := \widehat{\mathcal{I}}_0^0 \oplus \widehat{\mathcal{I}}_1^1 \oplus \widehat{\mathcal{I}}_2^2 \oplus \widehat{\mathcal{I}}_3^3 \oplus \widehat{\mathcal{I}}_4^4 \oplus \widehat{\mathcal{I}}_5^5 \oplus \widehat{\mathcal{I}}_6^6 \oplus \widehat{\mathcal{I}}_7^7 \oplus \dots$$

$$\mathfrak{d}_d[p] := \text{rank}([\ell]_d^p) = \binom{d+p}{d} + \binom{d+p-1}{d}$$

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

◆ In any dimension, the 0-gons reduce to finite size:

$\mathcal{B}_0 :=$	$\left\{ 1, \right.$		$,$		$,$		$,$		$,$		$,$		$,$		$, \dots \left. \right\}$
$d=2$	1	4		9		16		25		36		49		64	
$d=3$	1	5	14	30	55	91	140	204							
$d=4$	1	6	20	50	105	196	336	540							

$$\begin{aligned} \mathfrak{d}_d[p] &:= \text{rank}([\ell]_d^p) = \binom{d+p}{d} + \binom{d+p-1}{d} \\ &=: \widehat{\mathfrak{d}}_d[p] + \sum_{j=1}^p \binom{p}{j} \widehat{\mathfrak{d}}_d[p-j] \end{aligned}$$

total rank = top rank + contact terms

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

◆ In any dimension, the 0-gons reduce to finite size:

	$\mathcal{B}_0 := \left\{ 1, \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \\ \text{diagram 4} \\ \text{diagram 5} \\ \text{diagram 6} \\ \text{diagram 7} \\ \text{diagram 8} \end{array}, \dots \right\}$							
$d=2$	1	4	9	16	25	36	49	64
	$1+0$							
$d=3$	1	5	14	30	55	91	140	204
	$1+0$							
$d=4$	1	6	20	50	105	196	336	540
	$1+0$							

$$\begin{aligned} \mathfrak{d}_d[p] &:= \text{rank}([\ell]_d^p) = \binom{d+p}{d} + \binom{d+p-1}{d} \\ &=: \widehat{\mathfrak{d}}_d[p] + \sum_{j=1}^p \binom{p}{j} \widehat{\mathfrak{d}}_d[p-j] \end{aligned}$$

total rank = top rank + contact terms

Reducibility and Completeness

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

◆ In any dimension, the 0-gons reduce to finite size:

	$\mathcal{B}_0 := \left\{ 1, \text{diagram 1}, \text{diagram 2}, \text{diagram 3}, \text{diagram 4}, \text{diagram 5}, \text{diagram 6}, \text{diagram 7}, \text{diagram 8}, \dots \right\}$								
$d=2$	1	4	9	16	25	36	49	64	
	1+0	3+1	2+7	0+16	0+25	0+36	0+49	0+64	
$d=3$	1	5	14	30	55	91	140	204	
	1+0	4+1	5+9	2+28	0+55	0+91	0+140	0+204	
$d=4$	1	6	20	50	105	196	336	540	
	1+0	5+1	9+11	7+41	2+103	0+196	0+336	0+540	



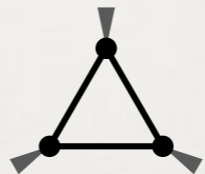
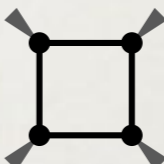
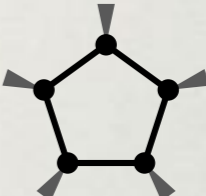
$$\begin{aligned} \mathfrak{d}_d[p] &:= \text{rank}([\ell]_d^p) = \binom{d+p}{d} + \binom{d+p-1}{d} \\ &=: \widehat{\mathfrak{d}}_d[p] + \sum_{j=1}^p \binom{p}{j} \widehat{\mathfrak{d}}_d[p-j] \end{aligned}$$

total rank = top rank + contact terms

Integrand Reduction at 1 Loop

◆ Re-considering one-loop bases in four dimensions

[Ossola, Papadopoulos, Pittau; Vermaseren, van Nerveen; Forde, Kosower]

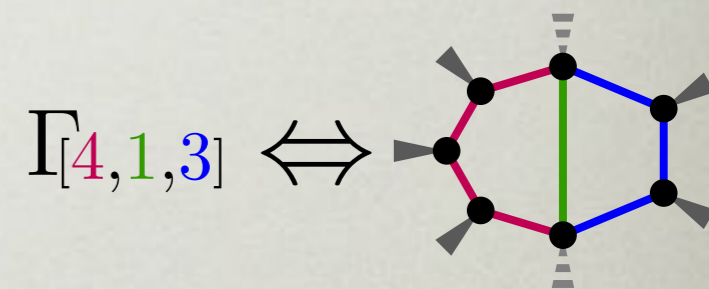
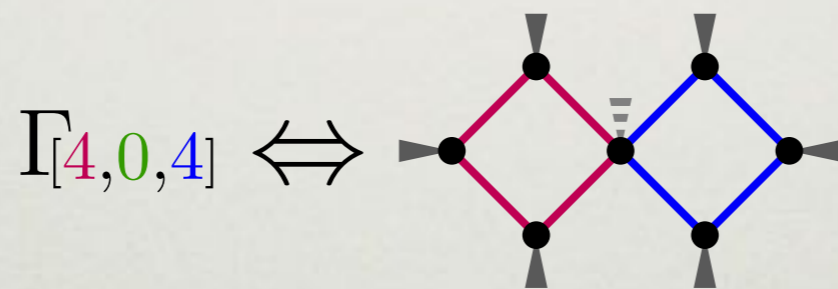
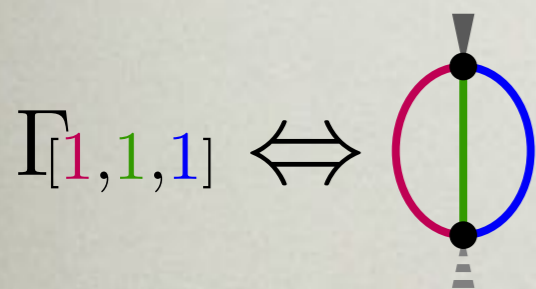
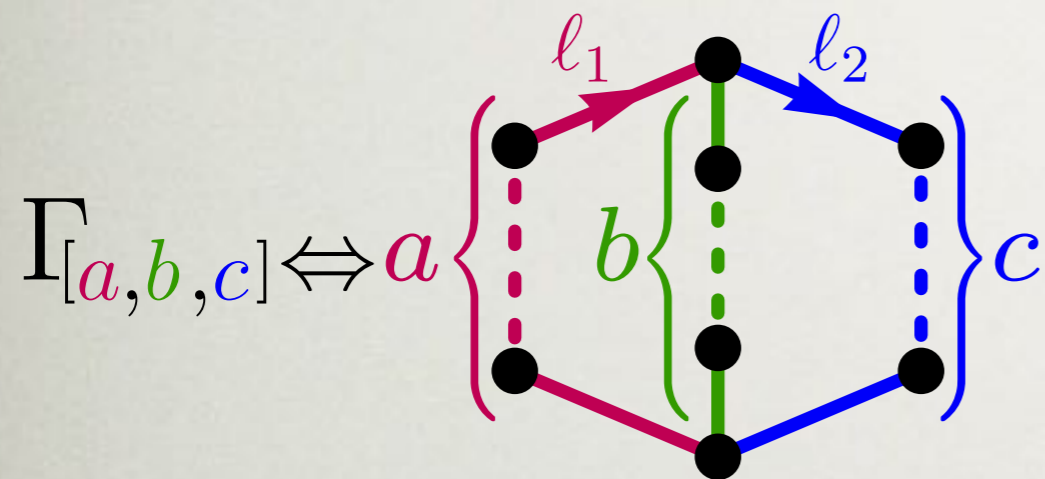
	1						...
$\mathcal{B}_0^{(4)}$	$[\ell]^0 = \widehat{b}_0^0$ $1 = \mathbf{1}$	$[\ell]^1 = \widehat{b}_1^0 \oplus \dots$ $6 = \mathbf{5} + 1$	$[\ell]^2 = \widehat{b}_2^0 \oplus \dots$ $20 = \mathbf{9} + 11$	$[\ell]^3 = \widehat{b}_3^0 \oplus \dots$ $50 = \mathbf{7} + 41$	$[\ell]^4 = \widehat{b}_4^0 \oplus \dots$ $105 = \mathbf{2} + 103$	$[\ell]^5 = \widehat{b}_5^0 \oplus \dots$ $196 = \mathbf{0} + 196$	
$\mathcal{B}_1^{(4)}$		$[\ell]^0 = \widehat{b}_1^1$ $1 = \mathbf{1}$	$[\ell]^1 = \widehat{b}_2^1 \oplus \dots$ $6 = \mathbf{4} + 2$	$[\ell]^2 = \widehat{b}_3^1 \oplus \dots$ $20 = \mathbf{5} + 15$	$[\ell]^3 = \widehat{b}_4^1 \oplus \dots$ $50 = \mathbf{2} + 48$	$[\ell]^4 = \widehat{b}_5^1 \oplus \dots$ $105 = \mathbf{0} + 105$	
$\mathcal{B}_2^{(4)}$			$[\ell]^0 = \widehat{b}_2^2$ $1 = \mathbf{1}$	$[\ell]^1 = \widehat{b}_3^2 \oplus \dots$ $6 = \mathbf{3} + 3$	$[\ell]^2 = \widehat{b}_4^2 \oplus \dots$ $20 = \mathbf{2} + 18$	$[\ell]^3 = \widehat{b}_5^2 \oplus \dots$ $50 = \mathbf{0} + 50$	
$\mathcal{B}_3^{(4)}$				$[\ell]^0 = \widehat{b}_3^3$ $1 = \mathbf{1}$	$[\ell]^1 = \widehat{b}_4^3 \oplus \dots$ $6 = \mathbf{2} + 4$	$[\ell]^2 = \widehat{b}_5^3 \oplus \dots$ $20 = \mathbf{0} + 20$	
$\mathcal{B}_4^{(4)}$					$[\ell]^0 = \widehat{b}_4^4$ $1 = \mathbf{1}$	$[\ell]^1 = \widehat{b}_5^4 \oplus \dots$ $6 = \mathbf{1} + 5$	

Integrand Reduction at 2 Loops

◆ At two loops, all loop integrands can be labeled by:

[Gluza, Kajda, Kosower]

[JB, Herrmann, Langer, Trnka (2020)]

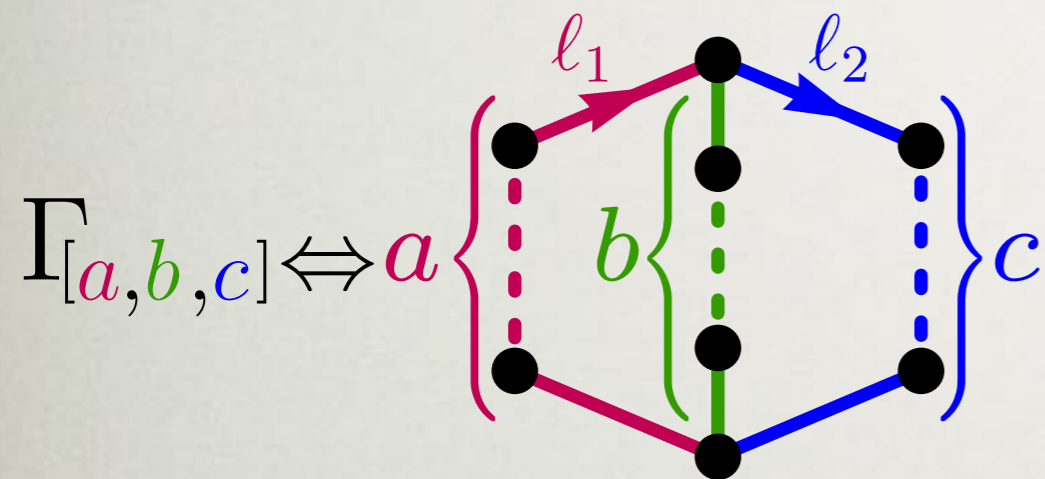


Integrand Reduction at 2 Loops

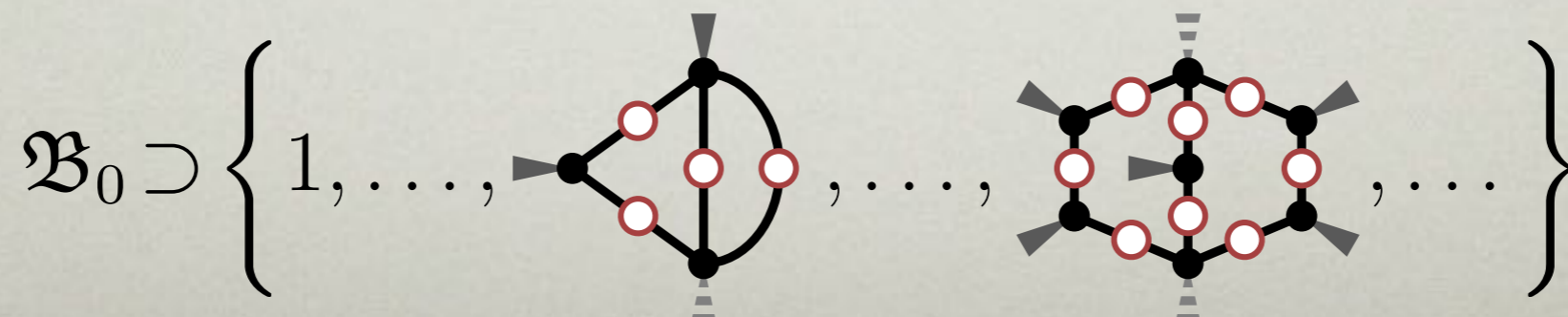
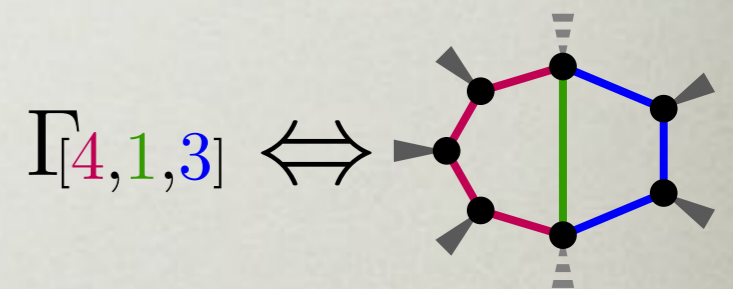
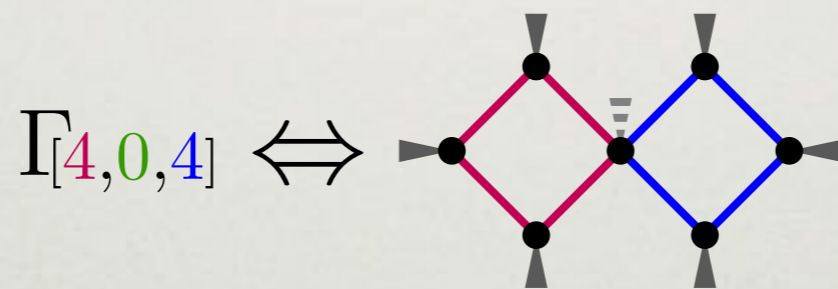
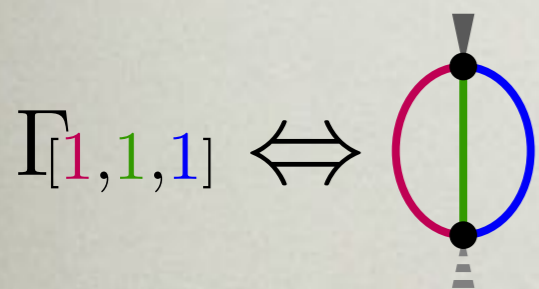
◆ At two loops, all loop integrands can be labeled by:

[Gluza, Kajda, Kosower]

[JB, Herrmann, Langer, Trnka (2020)]



$$\mathfrak{N}[a, b, c] := [\ell_1]^a [\ell_1 - \ell_2]^b [\ell_2]^c$$

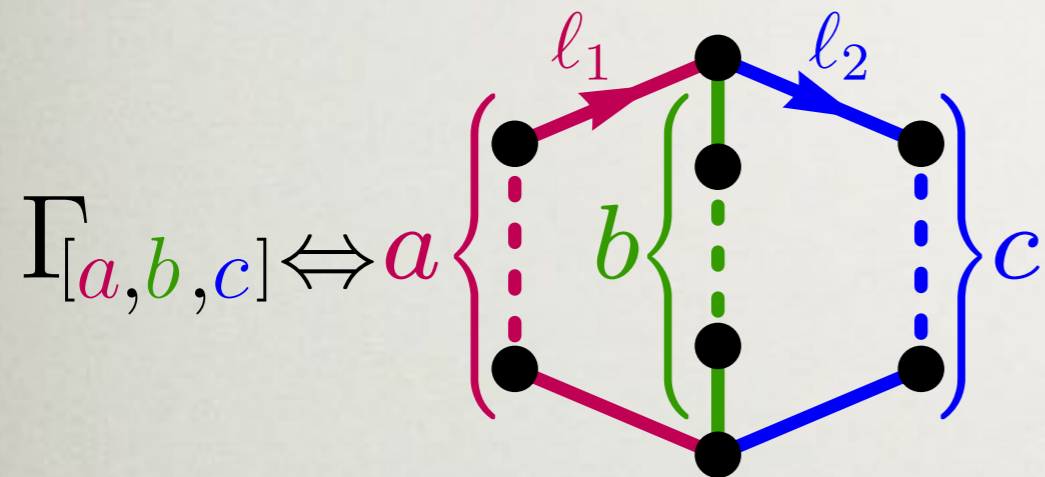


Integrand Reduction at 2 Loops

◆ At two loops, all loop integrands can be labeled by:

[Gluza, Kajda, Kosower]

[JB, Herrmann, Langer, Trnka (2020)]



$$\mathfrak{N}[a, b, c] := [\ell_1]^a [\ell_1 - \ell_2]^b [\ell_2]^c$$

$$\mathfrak{d}_d[a, b, c] := \text{rank}(\mathfrak{N}[a, b, c]) = \text{rank}\left([\ell_1]_d^a [\ell_1 - \ell_2]_d^b [\ell_2]_d^c\right)$$

$$=: \widehat{\mathfrak{d}}_d[a, b, c] + \sum_{(i,j,k) > (0,0,0)}^{(a,b,c)} \binom{a}{i} \binom{b}{j} \binom{c}{k} \widehat{\mathfrak{d}}_d[a-i, b-j, c-k]$$

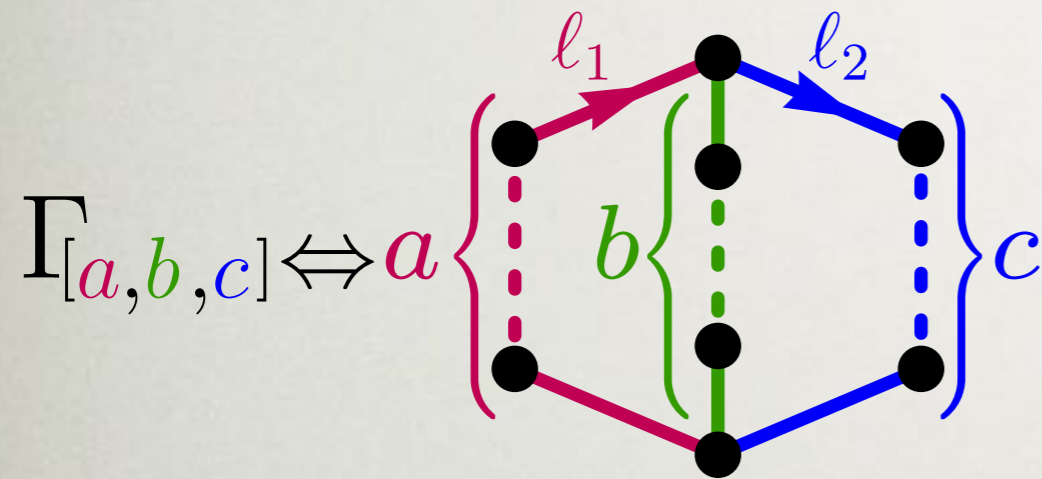
total rank = top rank + contact terms

Integrand Reduction at 2 Loops

◆ At two loops, all loop integrands can be labeled by:

[Gluza, Kajda, Kosower]

[JB, Herrmann, Langer, Trnka (2020)]



$$\mathfrak{N}[a, b, c] := [\ell_1]^a [\ell_1 - \ell_2]^b [\ell_2]^c$$

$$\mathfrak{N}_p(\Gamma_{[a,b,c]}) = \underbrace{\hat{\mathfrak{N}}_p(\Gamma_{[a,b,c]})}_{\text{top-level numerators}}$$

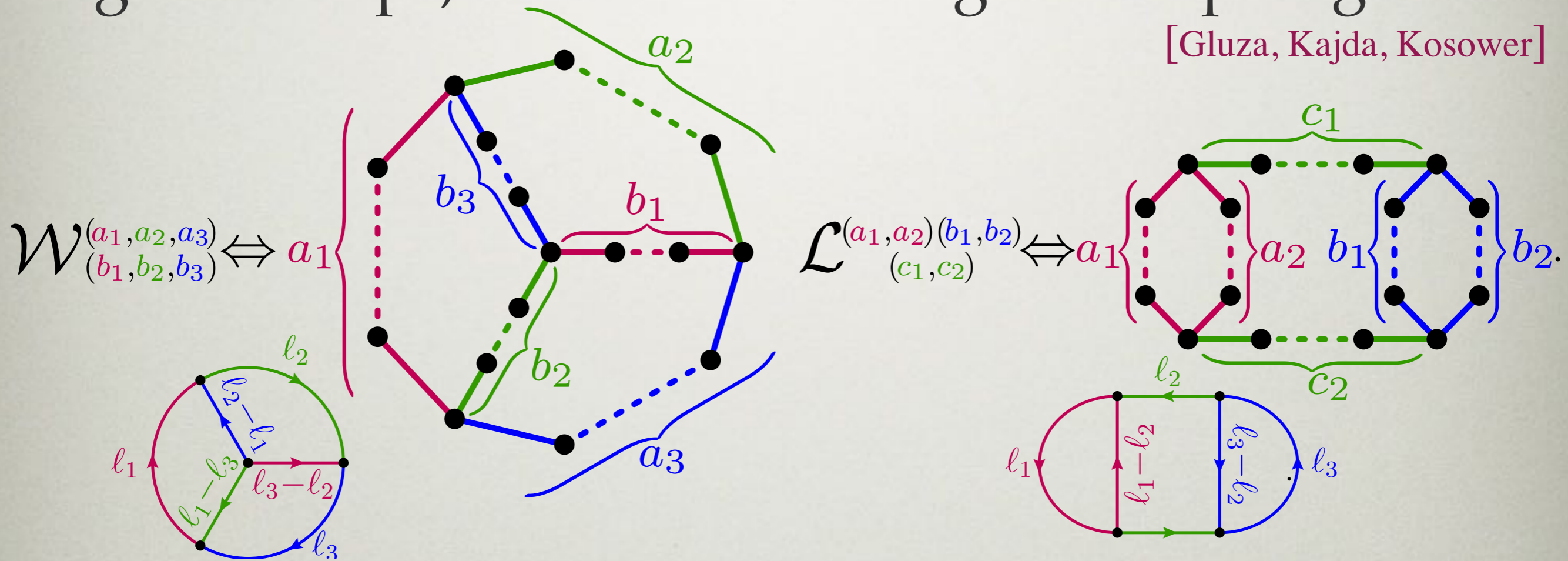
$$\underbrace{\bigoplus_{(i,j,k) > \vec{0}} [(\ell_A | Q_{a_1}) \cdots (\ell_A | Q_{a_i})][(\ell_B | Q_{b_1}) \cdots (\ell_B | Q_{b_j})][(\ell_C | S_{c_1}) \cdots (\ell_C | S_{c_k})] \hat{\mathfrak{N}}_p(\Gamma_{[a-i, b-j, c-k]})}_{\text{contact-term numerators}}$$

Integrand Reduction at 3 Loops

[JB, Herrmann, Langer, Trnka (2020)]

- ◆ We can obviously continue this to higher loops— e.g. at 3 loops, we have the integrand topologies:

[Gluza, Kajda, Kosower]



$$\mathfrak{w}_d^0(a_1, \dots, b_3) = \text{rank} \left([l_1]^{a_1} [l_2]^{a_2} [l_3]^{a_3} [l_3 - l_2]^{b_1} [l_1 - l_3]^{b_2} [l_2 - l_1]^{b_3} \right),$$

$$\mathfrak{l}_d^0(a_1, \dots, c_2) = \text{rank} \left([l_1]^{a_1} [l_1 - l_2]^{a_2} [l_2]^{c_1 + c_2} [l_3]^{b_1} [l_3 - l_2]^{b_2} \right).$$

¿Can someone derive these formulae?

*A (modest) Proposal for
non-Planar Power-Counting*

Stratifying Theories by Unitarity

[JB, Herrmann, Langer, Trnka (2020)]

- ◆ **QFTs** can be (partially) **ordered** by the scope of the integrands needed to represent amplitudes

$$\mathcal{A}^L = \sum a_i \mathcal{I}_i^L \quad \mathcal{I}_i \in \mathfrak{B}$$

(Standard Model) \succ (SM \ Higgs) \succ (QCD) \succ (Yang-Mills)
 \succ ($\mathcal{N} = 2$ sYM) \succ ($\mathcal{N} = 4$ sYM) \succ (planar $\mathcal{N} = 4$ sYM)
 \succ (fishnet theory) \succ \dots

$$\mathfrak{B}^{\text{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$$

This reflects **UV behavior** (“*power-counting*”) of theories; can be used to **stratify integrand bases**

¿Can we define $\mathfrak{B}^{\mathcal{N}=4}$?

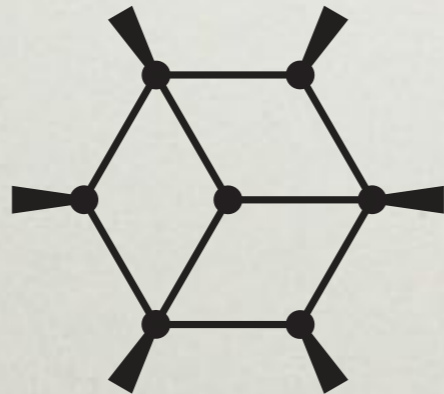
—a *basis of just the best UV-behaved amplitudes?*

Power-Counting when Planar

- ◆ For a *planar* graph, there is a natural *routing* of the loop momenta associated with its dual graph.
- ◆ A *planar integrand* \mathcal{I} has “*p-gon power-counting*” if

$$\lim_{\ell_i \rightarrow \infty} (\mathcal{I}) = \frac{1}{(\ell_i^2)^{q \geq p}} (1 + \mathcal{O}(1/\ell_i^2)) \quad \text{for all } \ell_i$$

- ◆ Let \mathcal{B}_p denote the complete basis of integrands with *p-gon power-counting*.



Power-Counting when Planar

◆ For a *planar* graph, there is a natural *routing* of the loop momenta associated with its dual graph.

◆ A *planar integrand* \mathcal{I} has “*p-gon power-counting*” if

$$\lim_{\ell_i \rightarrow \infty} (\mathcal{I}) = \frac{1}{(\ell_i^2)^{q \geq p}} (1 + \mathcal{O}(1/\ell_i^2)) \quad \text{for all } \ell_i$$

◆ Let \mathcal{B}_p denote the complete basis of integrands with *p-gon power-counting*.

$$\mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \mathcal{B}_3 \supset \mathcal{B}_4 \supset \mathcal{B}_5 \supset \dots$$

$$\widehat{\mathcal{B}}_p := \mathcal{B}_p \setminus \mathcal{B}_{p+1} \quad \mathcal{B}_p = \widehat{\mathcal{B}}_p \oplus \widehat{\mathcal{B}}_{p+1} \oplus \dots$$

◆ An *amplitude* is “*p-gon constructible*” if $A \subset \mathcal{B}_p$

$$A_p := A \cap \widehat{\mathcal{B}}_p \quad A = A_d \oplus A_{d-1} \oplus \dots$$

(Optimality of Dual-Conformality?)

- ◆ For *planar* $\mathcal{N}=4$ sYM, we know that amplitude integrands are *dual-conformally invariant*

[Drummond, Henn, Smirnov, Sokatchev;
Drummond, Korchemsky, Henn;
Alday, Maldacena;...]

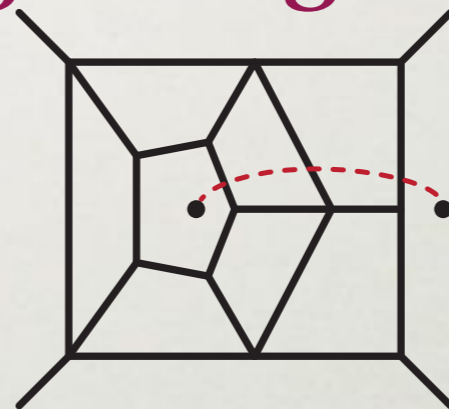
(Optimality of Dual-Conformality?)

- ◆ For *planar* $\mathcal{N}=4$ sYM, we know that amplitude integrands are *dual-conformally invariant*

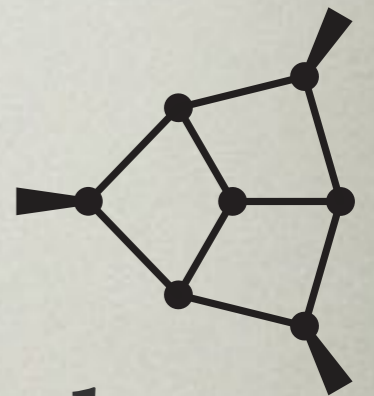
$$\mathcal{B}_4 \supset \mathcal{B}^{\text{DCI}} \supset \mathcal{B}^{\mathcal{N}=4} \quad [\text{Drummond, Henn, Smirnov, Sokatchev}; \\ \text{Drummond, Korchemsky, Henn};$$

But even DCI is far from strong enough! [Alday, Maldacena;...]

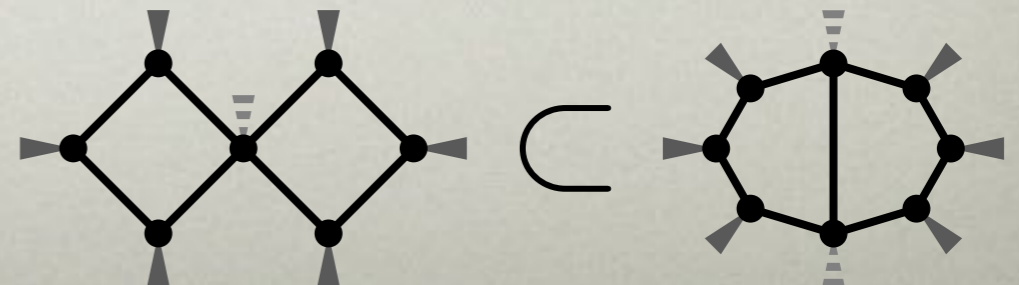
- ◆ doesn't ensure UV finiteness



- ◆ doesn't ensure maximal transcendentality



- ◆ it *forces* a topological over-completeness and non-triangularity of bases



Power-Counting Strata at 1-Loop

[JB, Herrmann, Langer, Trnka (2020)]

◆ Recall how “0-gon” integrands could be defined:

$$\mathfrak{B}_0 := \left\{ 1, \text{loop}, \text{2-loop}, \text{triangle}, \text{square}, \text{pentagon}, \text{hexagon}, \text{heptagon}, \text{octagon}, \dots \right\}$$

$$\mathcal{I}_p^q := \text{span} \left\{ \frac{[\ell]^q}{(\ell|P_1) \cdots (\ell|P_p)} \right\} \in \mathfrak{B}_{p-q}$$

Power-Counting Strata at 1-Loop

[JB, Herrmann, Langer, Trnka (2020)]

◆ Recall how “0-gon” integrands could be defined:

$$\mathcal{B}_0 := \left\{ 1, \text{loop with 1 red vertex}, \text{loop with 2 red vertices}, \text{triangle with 3 red vertices}, \text{square with 4 red vertices}, \text{pentagon with 5 red vertices}, \text{hexagon with 6 red vertices}, \text{heptagon with 7 red vertices}, \dots \right\}$$

$$\mathcal{B}_1 := \left\{ \text{loop with 0 red vertices}, \text{loop with 1 red vertex}, \text{triangle with 2 red vertices}, \text{square with 3 red vertices}, \text{pentagon with 4 red vertices}, \text{hexagon with 5 red vertices}, \text{heptagon with 6 red vertices}, \dots \right\}$$

$$\mathcal{B}_2 := \left\{ \text{loop with 2 red vertices}, \text{triangle with 1 red vertex}, \text{square with 2 red vertices}, \text{pentagon with 3 red vertices}, \text{hexagon with 4 red vertices}, \text{heptagon with 5 red vertices}, \dots \right\}$$

$$\mathcal{B}_3 := \left\{ \text{triangle with 0 red vertices}, \text{square with 1 red vertex}, \text{pentagon with 2 red vertices}, \text{hexagon with 3 red vertices}, \text{heptagon with 4 red vertices}, \dots \right\}$$

$$\mathcal{B}_4 := \left\{ \text{square with 0 red vertices}, \text{pentagon with 1 red vertex}, \text{hexagon with 2 red vertices}, \text{heptagon with 3 red vertices}, \dots \right\}$$

Power-Counting Strata at 1-Loop

[JB, Herrmann, Langer, Trnka (2020)]

◆ Recall how “0-gon” integrands could be defined:

$$\mathcal{B}_0 := \left\{ 1, \text{diagram 1}, \text{diagram 2}, \text{diagram 3}, \text{diagram 4}, \text{diagram 5}, \text{diagram 6}, \text{diagram 7}, \dots \right\}$$

$$\mathcal{B}_1 := \left\{ \text{diagram 1}, \text{diagram 2}, \text{diagram 3}, \text{diagram 4}, \text{diagram 5}, \text{diagram 6}, \text{diagram 7}, \dots \right\}$$

$$\mathcal{B}_2 := \left\{ \text{diagram 1}, \text{diagram 2}, \text{diagram 3}, \text{diagram 4}, \text{diagram 5}, \text{diagram 6}, \dots \right\}$$

$$\mathfrak{d}_d^q[p] := \text{rank} \left([\ell]_d^{p-q} \right)$$

$$\mathcal{B}_3 := \left\{ \text{diagram 1}, \text{diagram 2}, \text{diagram 3}, \text{diagram 4}, \text{diagram 5}, \text{diagram 6}, \dots \right\}$$

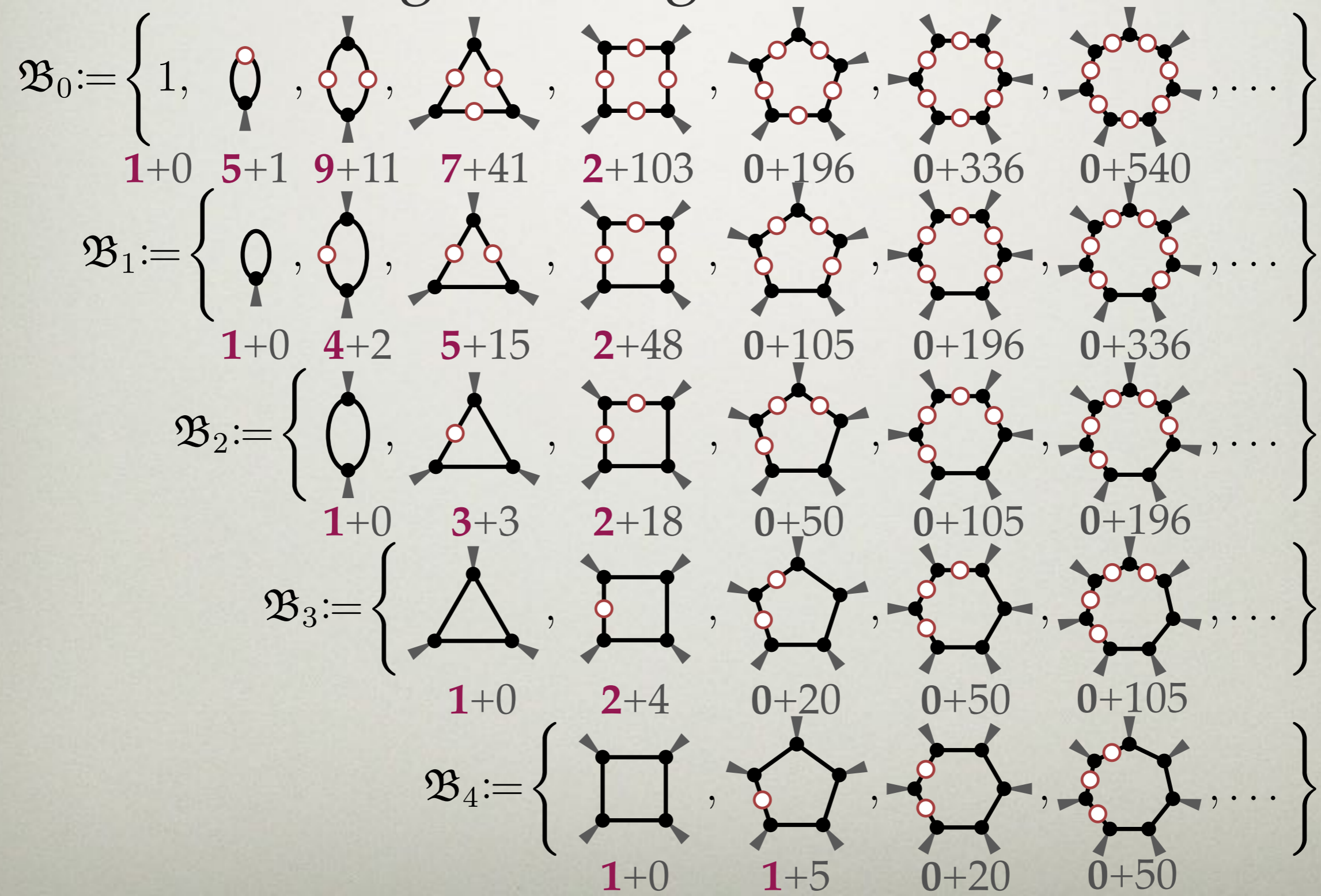
$$=: \widehat{\mathfrak{d}}_d^q[p] + \sum_{j=1}^q \binom{p}{j} \widehat{\mathfrak{d}}_d^q[p-j]$$

total rank = top rank + contact terms

Power-Counting Strata at 1-Loop

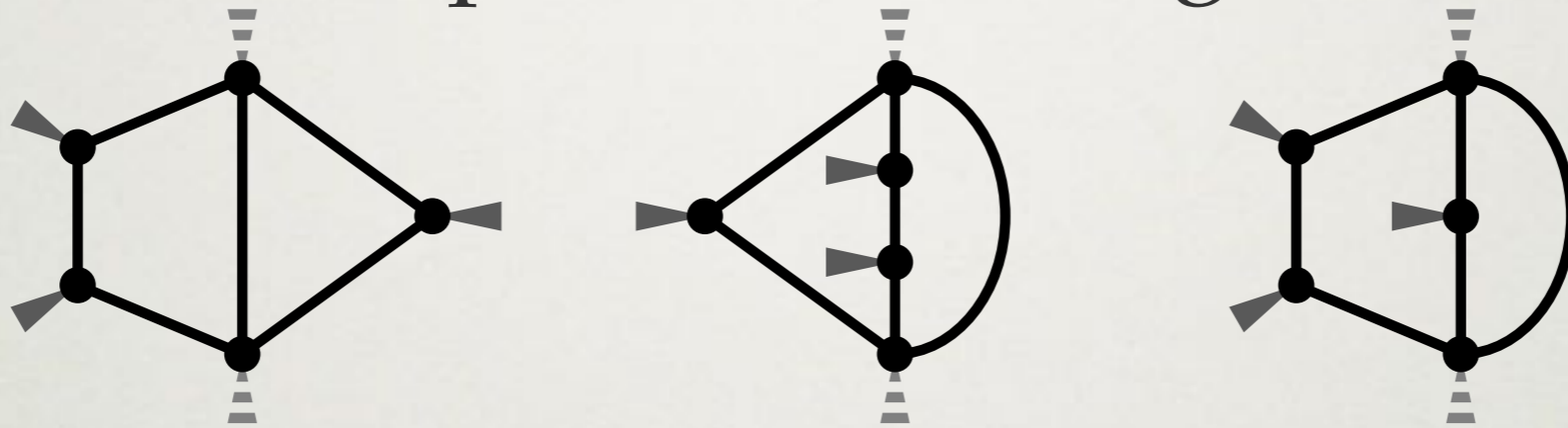
[JB, Herrmann, Langer, Trnka (2020)]

◆ Recall how “0-gon” integrands could be defined:



Power-Counting Beyond Planar?

- ◆ With no preferred *routing* of loop momenta, the earlier notion of “power-counting” is ill-defined

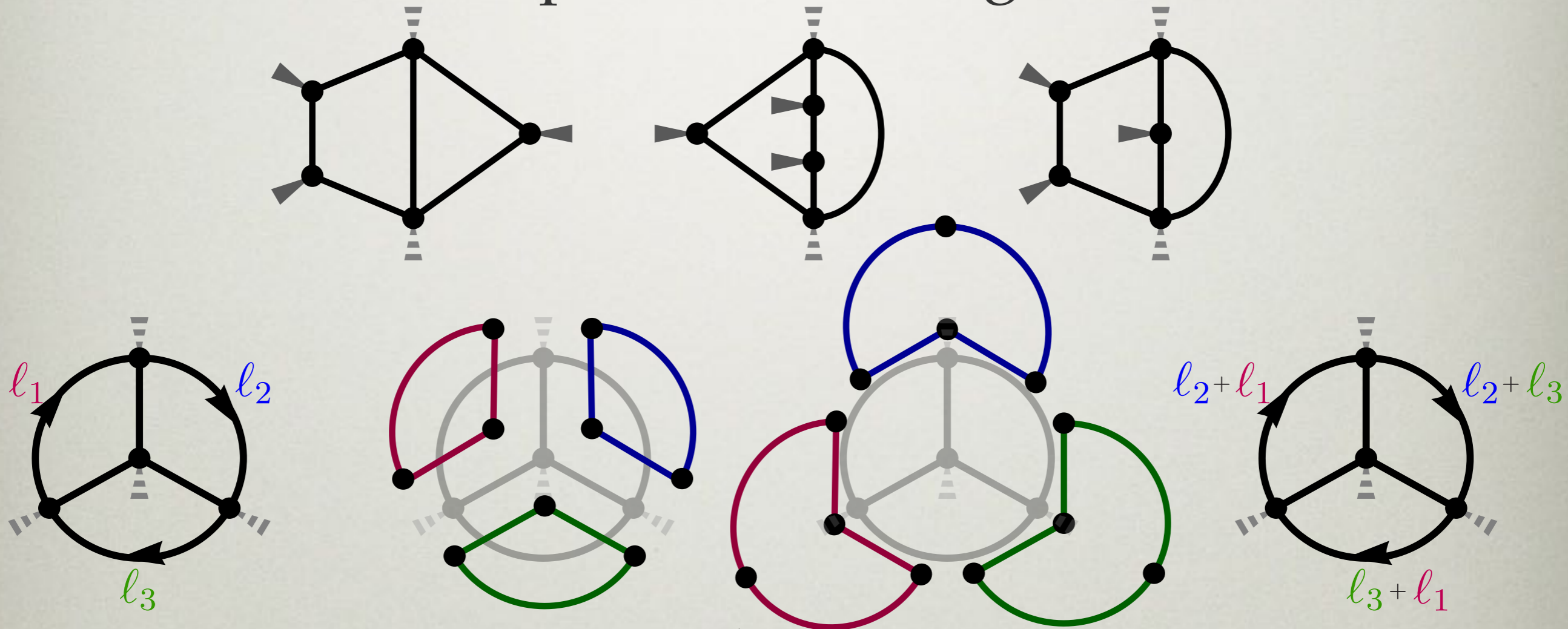


Recall: *planar integrand* \mathcal{I} has *p-gon power-counting* if

$$\lim_{\ell_i \rightarrow \infty} (\mathcal{I}) = \frac{1}{(\ell_i^2)^{q \geq p}} \left(1 + \mathcal{O}(1/\ell_i^2)\right) \quad \text{for all } \ell_i$$

Power-Counting Beyond Planar?

- ◆ With no preferred *routing* of loop momenta, the earlier notion of “power-counting” is ill-defined



Recall: *planar integrand* \mathcal{I} has *p-gon power-counting* if

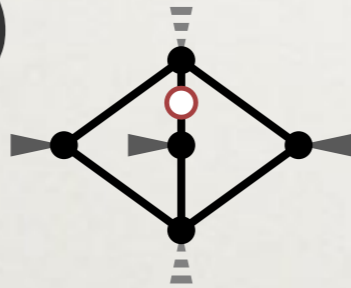
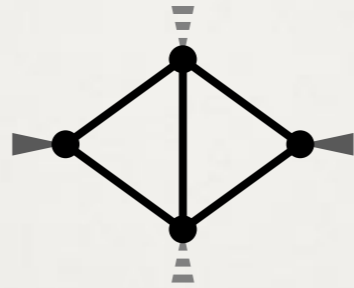
$$\lim_{\ell_i \rightarrow \infty} (\mathcal{I}) = \frac{1}{(\ell_i^2)^{q \geq p}} (1 + \mathcal{O}(1/\ell_i^2)) \quad \text{for all } \ell_i$$

Proposal: Graph Power-Counting

- ◆ What *would* make sense *independent of routing*

would be: to define the **power-counting** *relative to some graph (or graphs)*

[JB, Herrmann, Langer, Trnka (2020)]

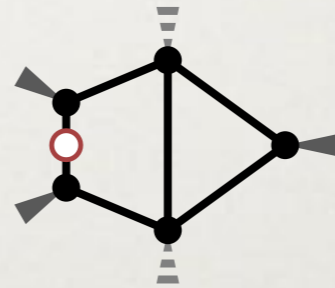
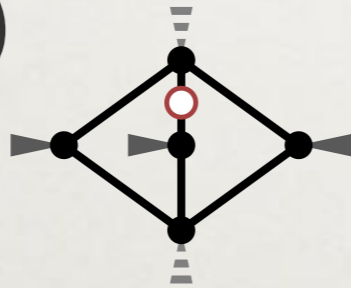
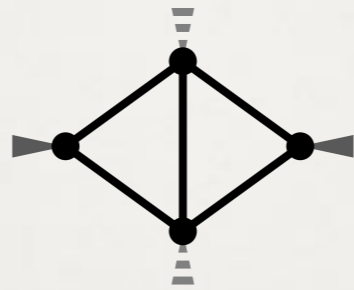


Proposal: *Graph Power-Counting*

- ◆ What *would* make sense *independent of routing*

would be: to define
the **power-counting**
relative to some
graph (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]

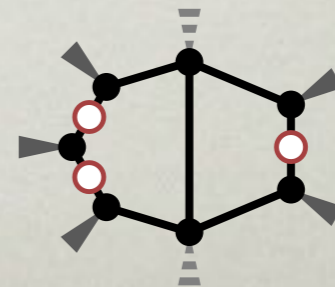
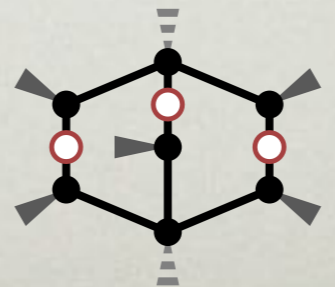
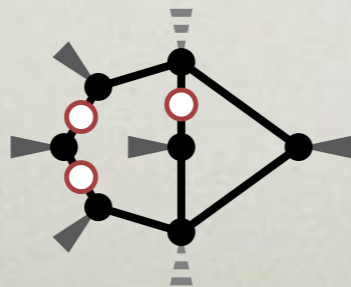
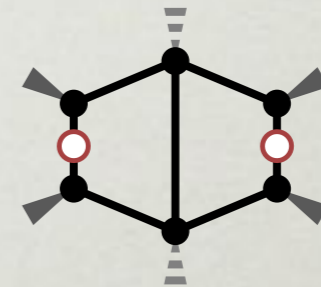
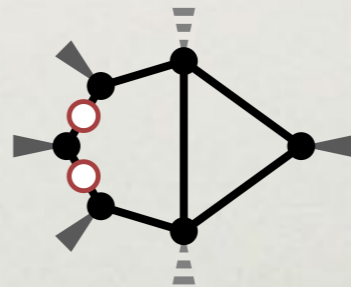
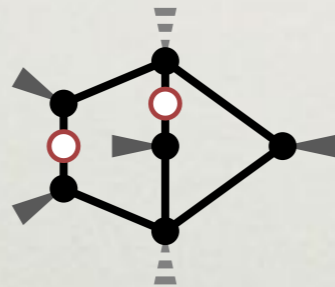
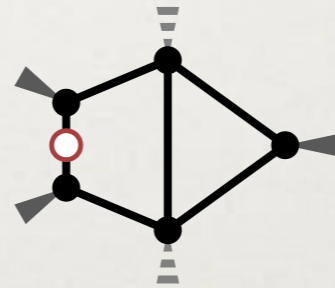
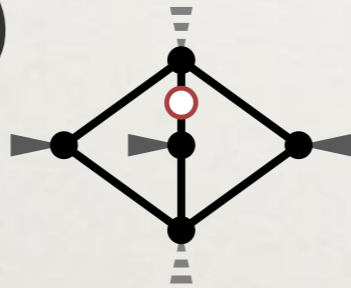
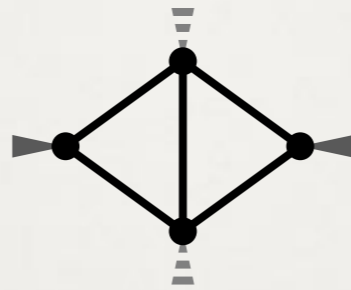


Proposal: Graph Power-Counting

◆ What *would* make sense *independent of routing*

would be: to define
the **power-counting**
relative to some
graph (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]

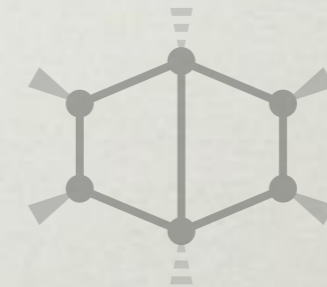
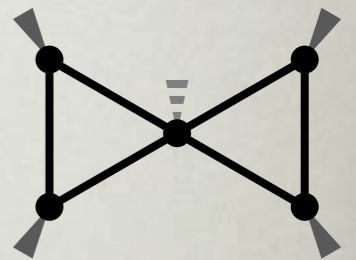
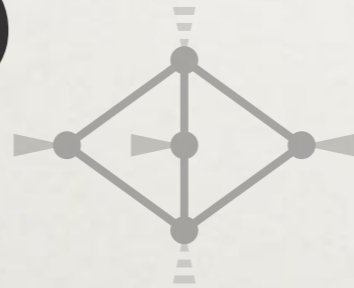
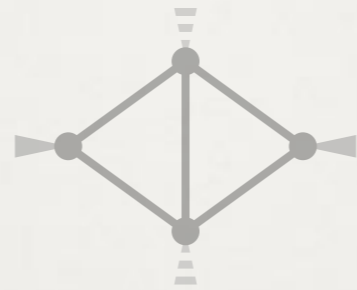


Proposal: Graph Power-Counting

◆ What *would* make sense *independent of routing*

would be: to define
the **power-counting**
relative to some
graph (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]

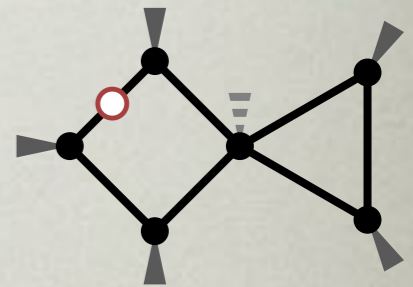
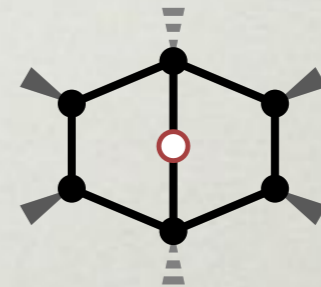
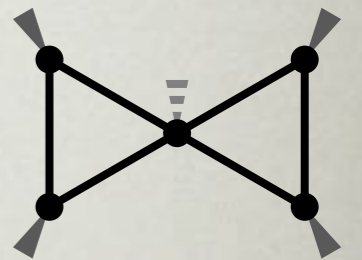
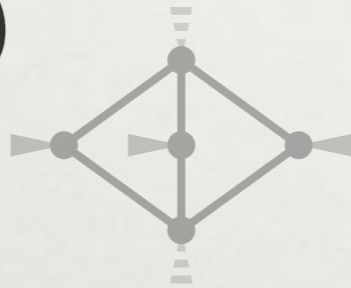
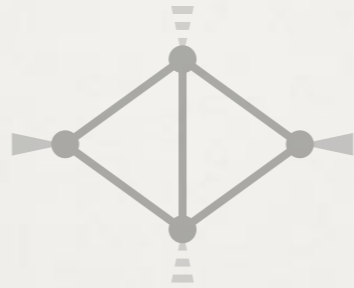


Proposal: *Graph Power-Counting*

◆ What *would* make sense *independent of routing*

would be: to define
the **power-counting**
relative to some
graph (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]

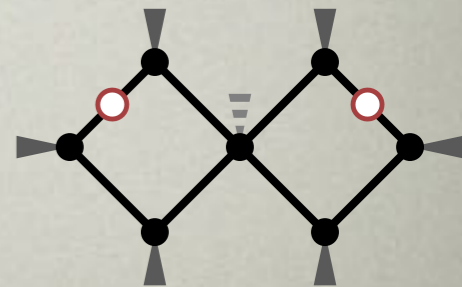
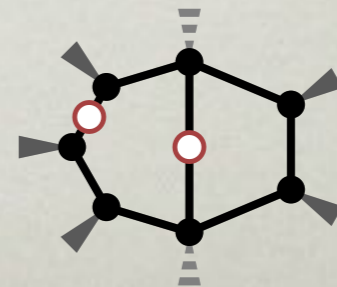
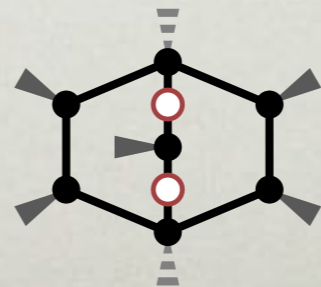
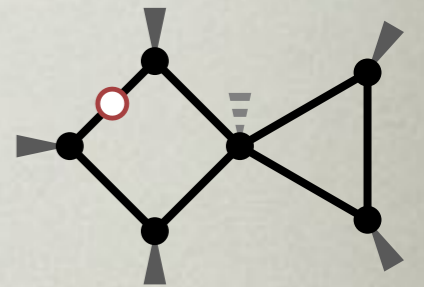
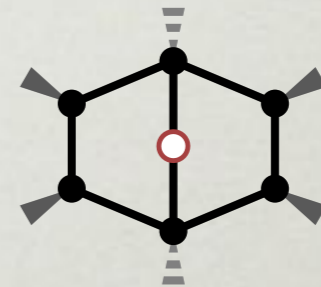
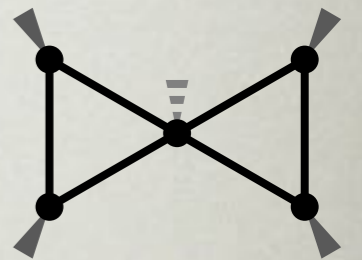
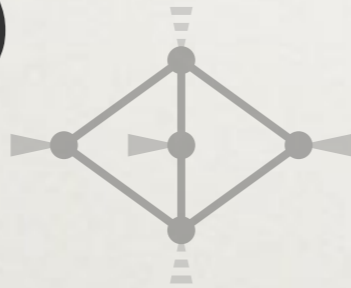
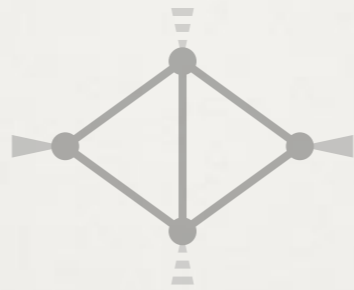


Proposal: Graph Power-Counting

◆ What *would* make sense *independent of routing*

would be: to define the **power-counting** *relative to some graph (or graphs)*

[JB, Herrmann, Langer, Trnka (2020)]

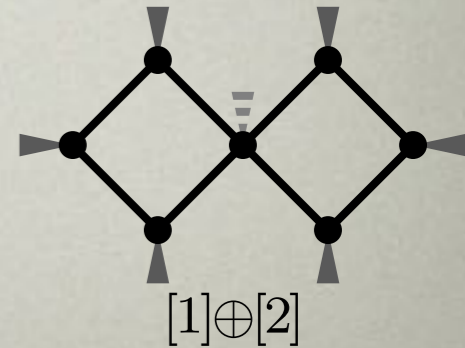
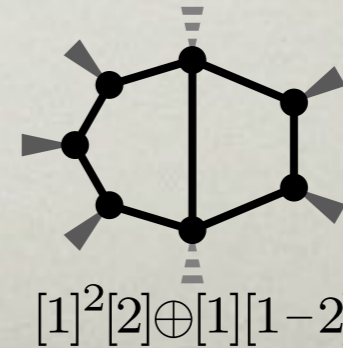
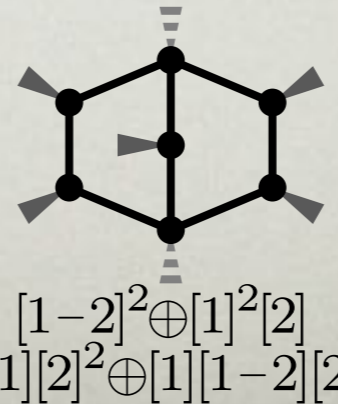
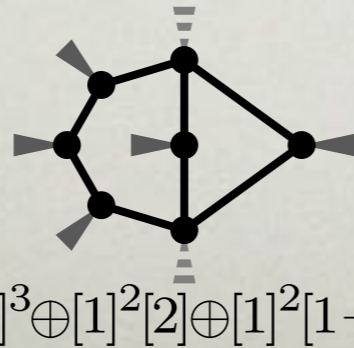
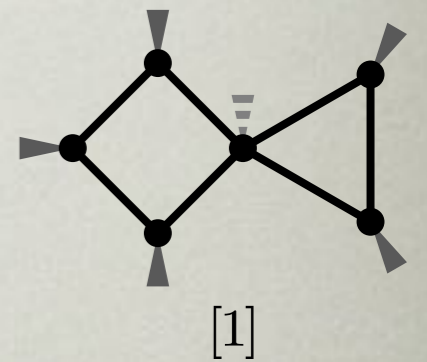
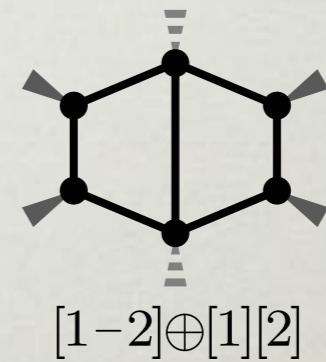
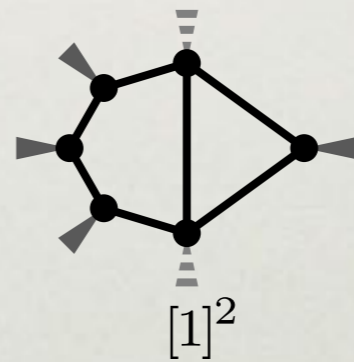
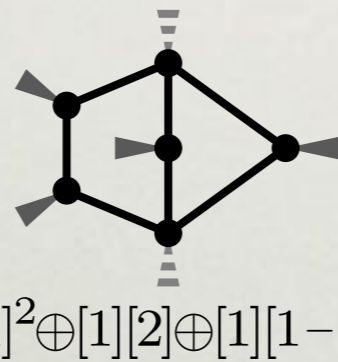
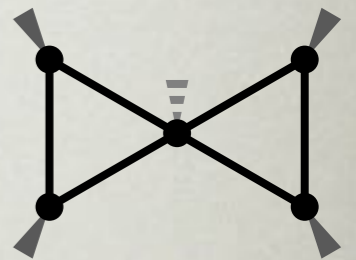
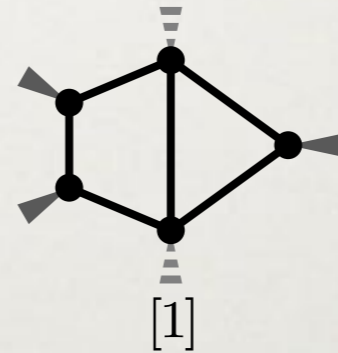
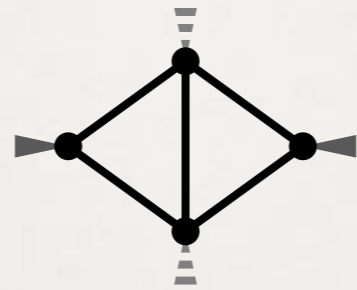


Proposal: Graph Power-Counting

♦ What *would* make sense *independent of routing*

would be: to define the **power-counting** relative to some graph (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]

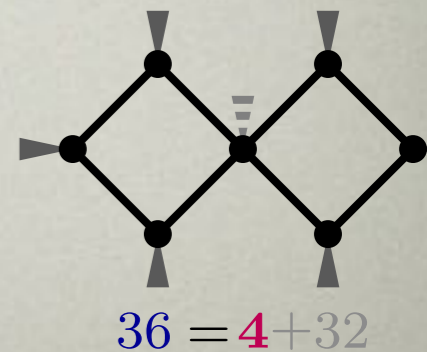
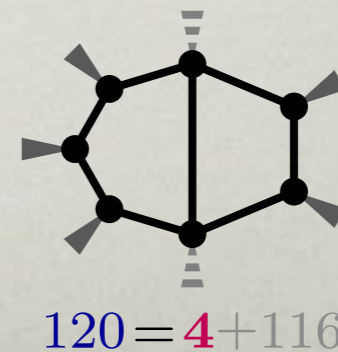
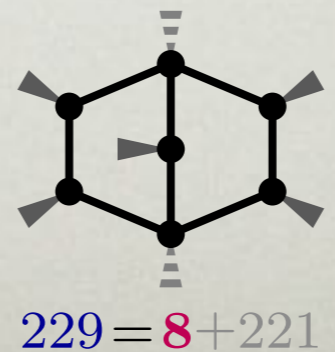
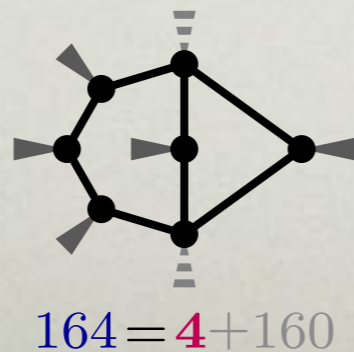
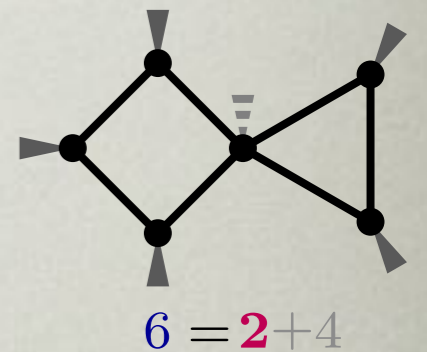
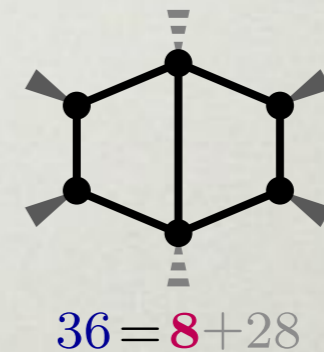
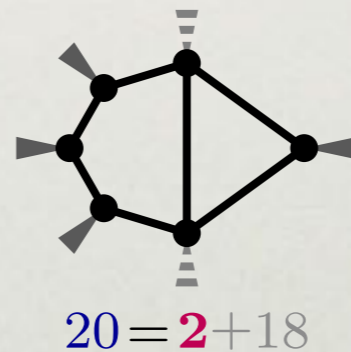
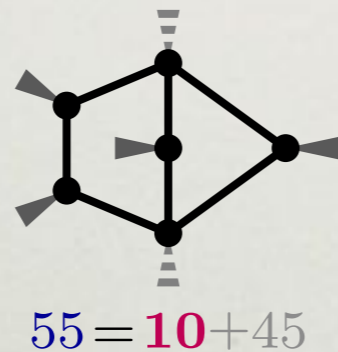
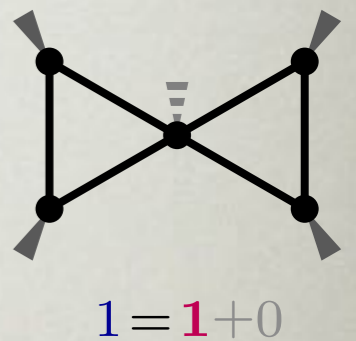
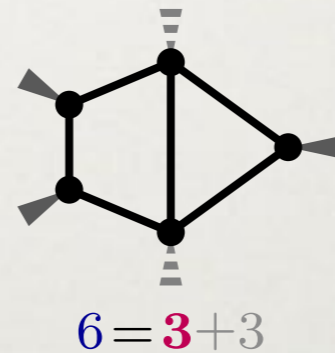
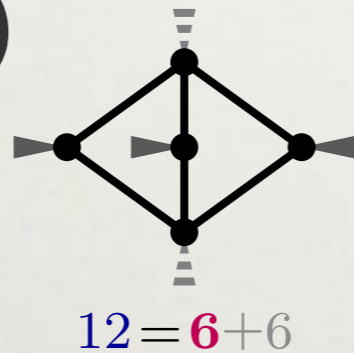
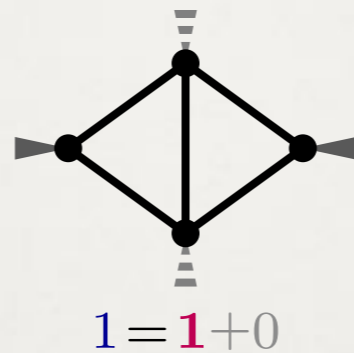


Proposal: Graph Power-Counting

♦ What *would* make sense *independent of routing*

would be: to define the **power-counting** *relative to some graph* (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]



Proposal: Graph Power-Counting

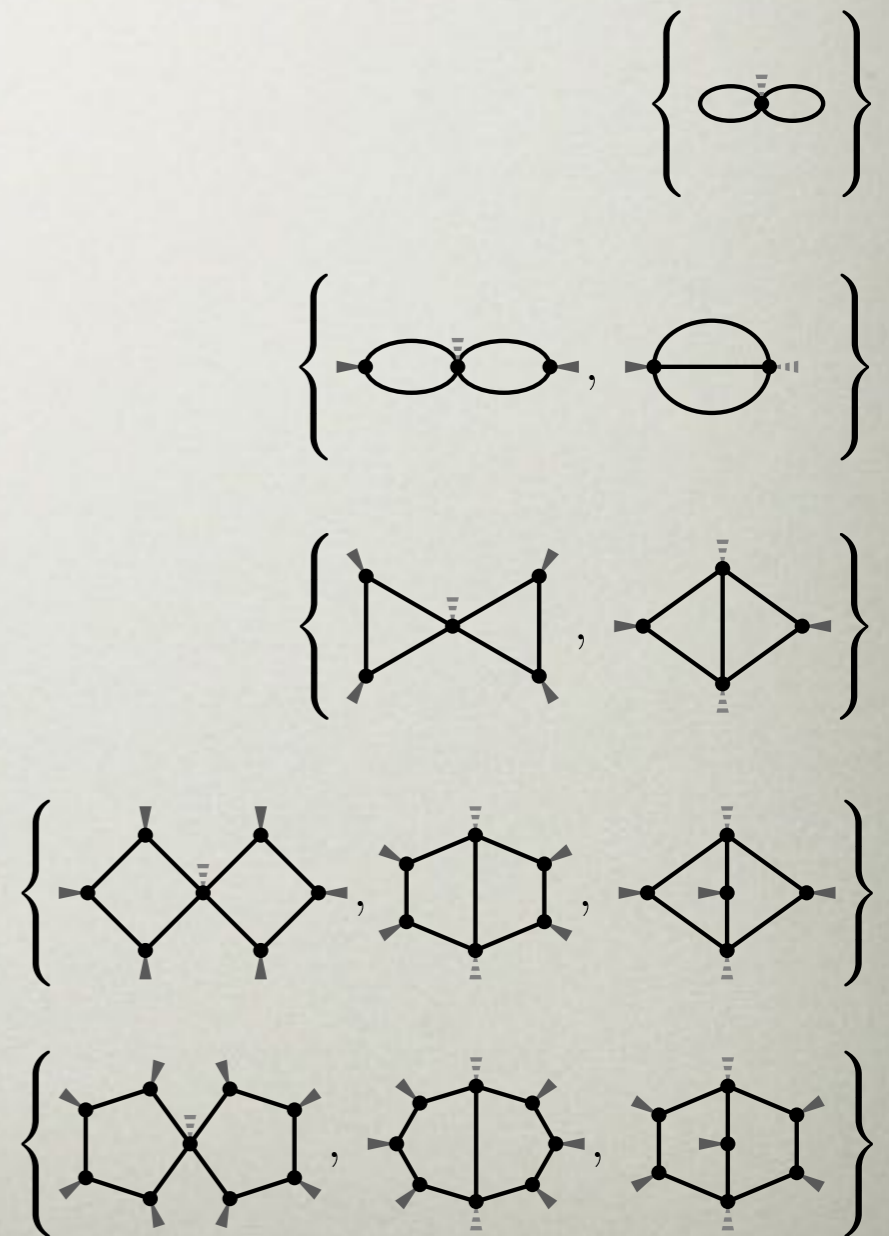
♦ What *would* make sense *independent of routing*

would be: to define
the **power-counting**
relative to some
graph (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]

Definition

a **scalar p -gon** is
a graph of *girth*
 p such that all its
edge contractions
have $\text{girth} < p$

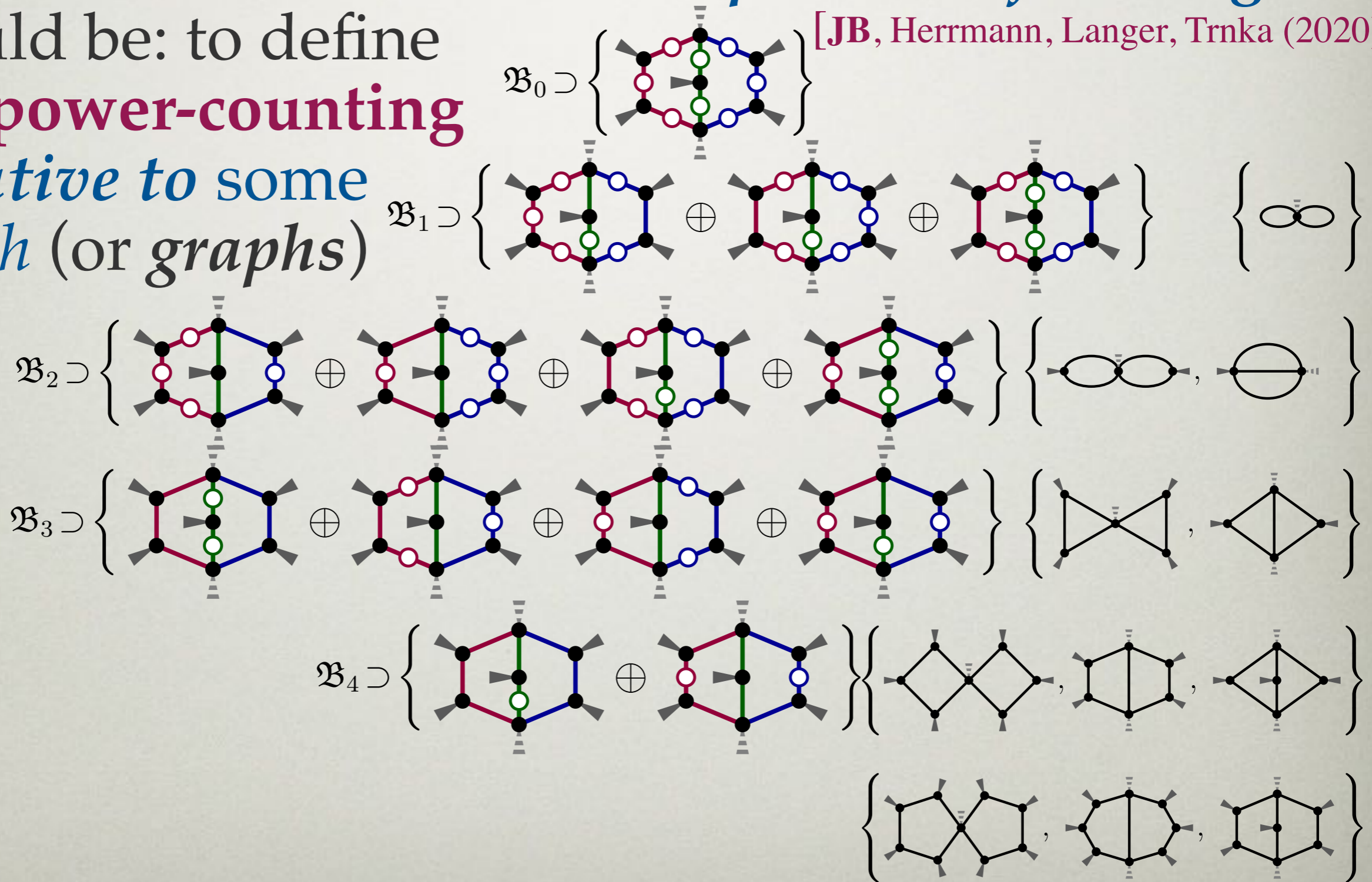


Proposal: Graph Power-Counting

♦ What *would* make sense *independent of routing*

would be: to define the **power-counting** relative to some graph (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]

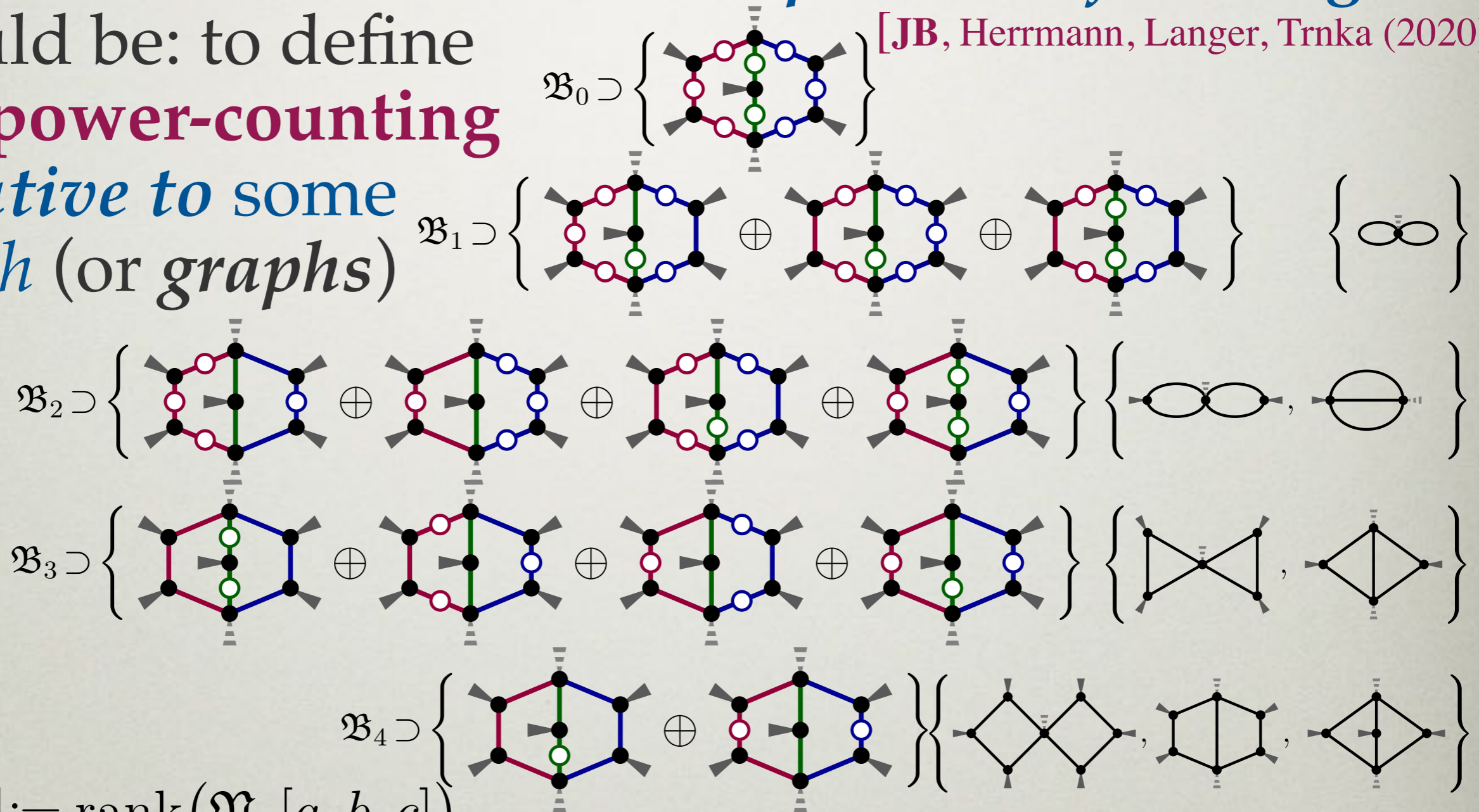


Proposal: Graph Power-Counting

◆ What *would* make sense *independent of routing*

would be: to define the **power-counting** relative to some graph (or graphs)

[JB, Herrmann, Langer, Trnka (2020)]



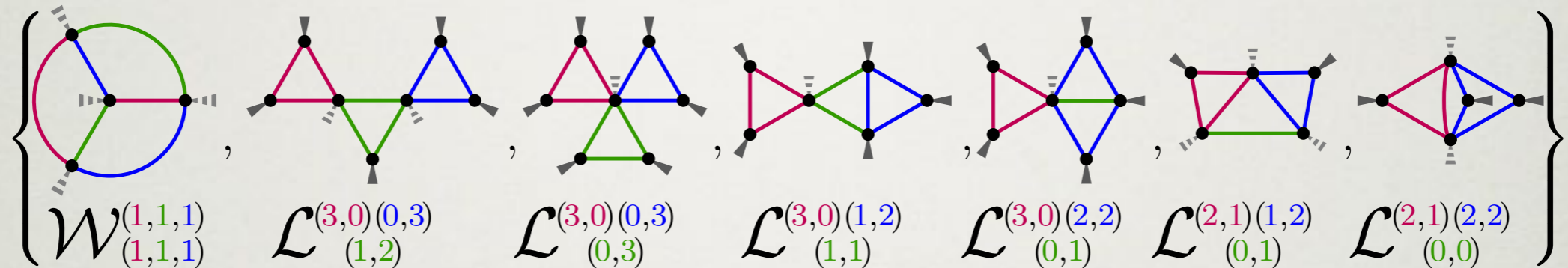
$$\mathfrak{d}_d^p[a, b, c] := \text{rank}(\mathfrak{N}_p[a, b, c])$$

$$=: \widehat{\mathfrak{d}}_d^p[a, b, c] + \sum_{(i,j,k) > (0,0,0)} \binom{a}{i} \binom{b}{j} \binom{c}{k} \widehat{\mathfrak{d}}_d^p[a-i, b-j, c-k]$$

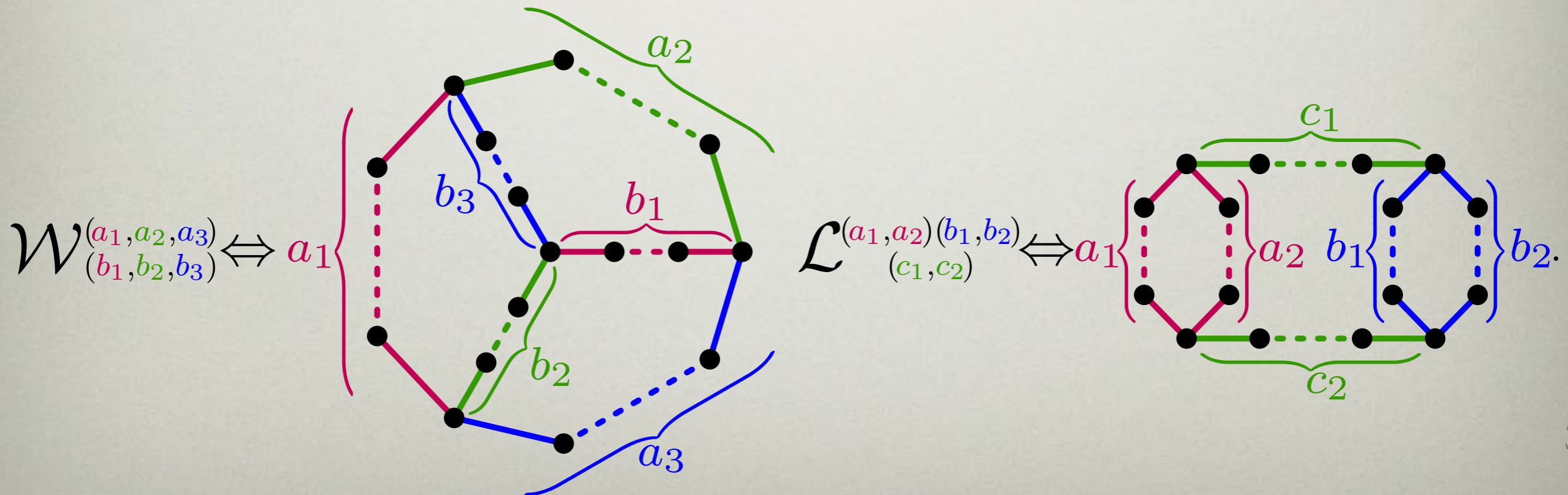
Graph Power-Counting @ $L=3$

[JB, Herrmann, Langer, Trnka (2020)]

◆ At 3 loops, the 3-gon power-counting scalars are:



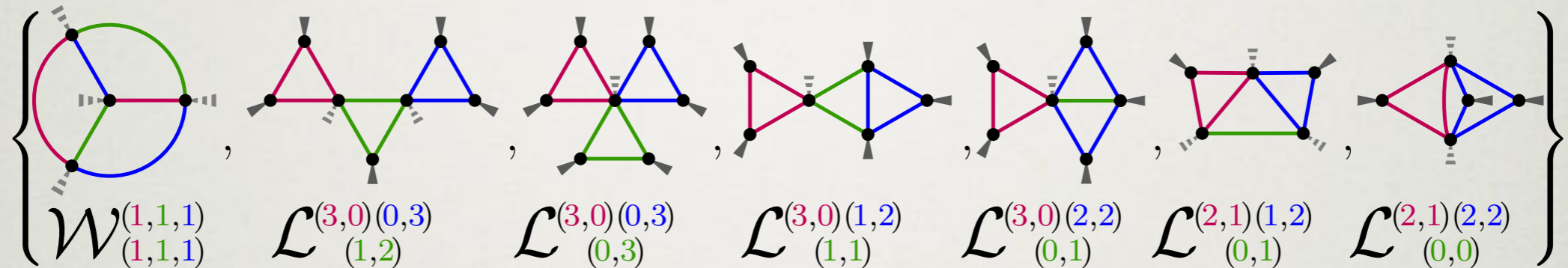
◆ What would be the numerator for $\mathcal{L}^{(3,1)(2,2)}_{(1,2)} =$?



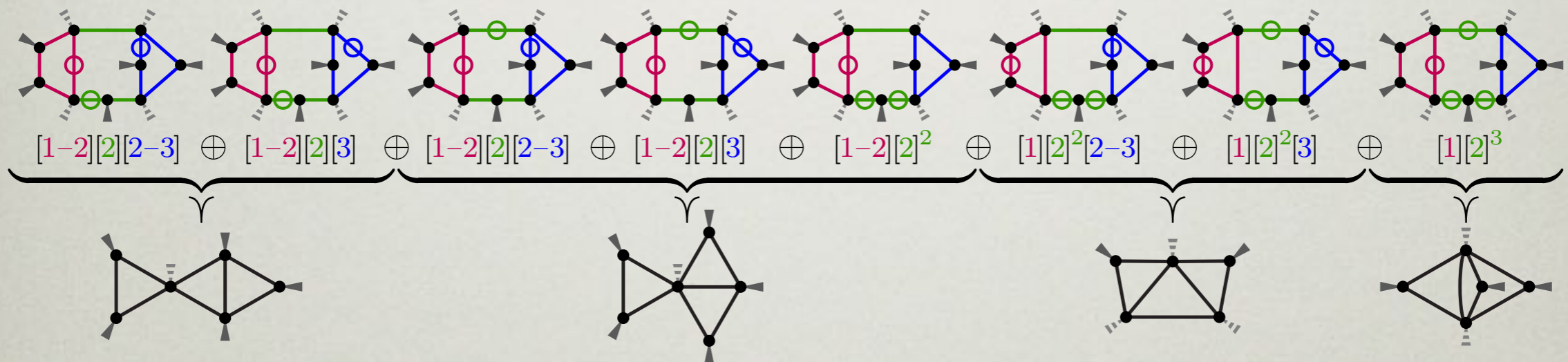
Graph Power-Counting @ $L=3$

[JB, Herrmann, Langer, Trnka (2020)]

◆ At 3 loops, the 3-gon power-counting scalars are:



◆ What would be the numerator for $\mathcal{L}_{(1,2)}^{(3,1)(2,2)} =$  ?



$$\mathfrak{N}_3 \left(\mathcal{L}_{(1,2)}^{(3,1)(2,2)} \right) = [1-2][2][2-3] \oplus [1-2][2][3] \oplus [1-2][2]^2 \oplus [1][2]^2[2-3] \oplus [1][2]^2[3] \oplus [1][2]^3$$

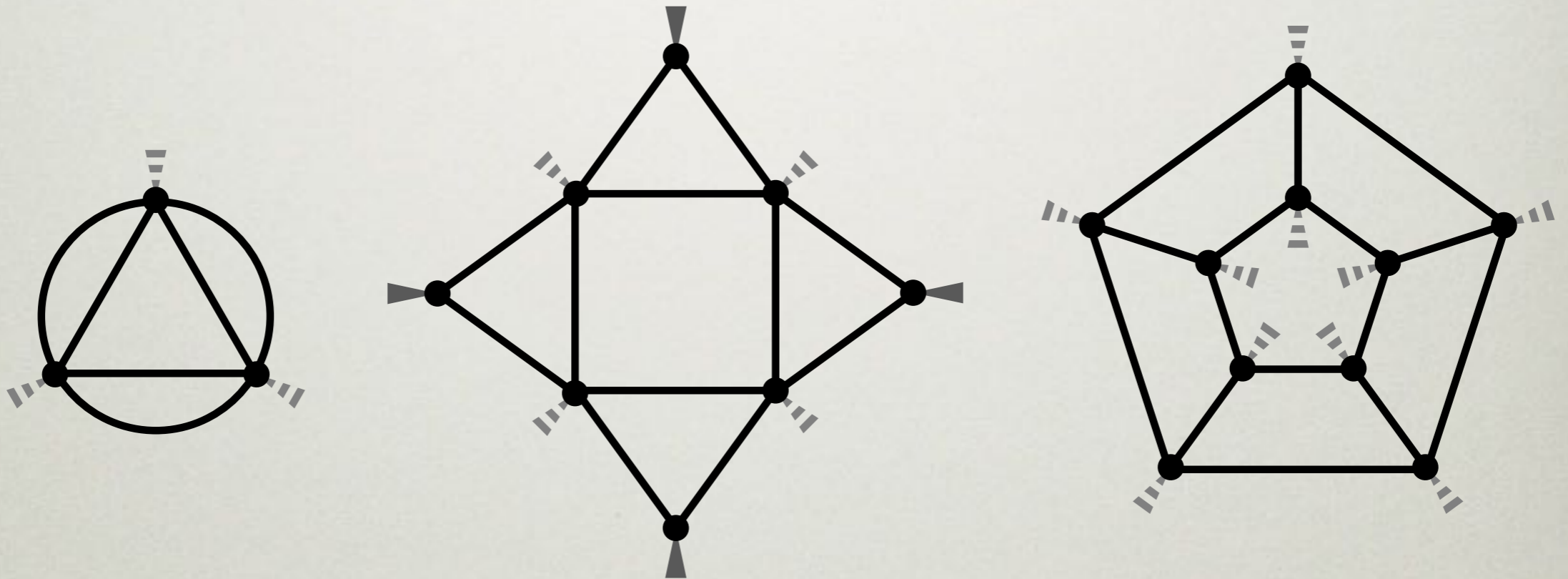
$$\left| \mathfrak{N}_3 \left(\mathcal{L}_{(1,2)}^{(3,1)(2,2)} \right) \right| = \text{rank}_{d=4} \left([1-2][2][2-3] \oplus [1-2][2][3] \oplus [1-2][2]^2 \oplus [1][2]^2[2-3] \oplus [1][2]^2[3] \oplus [1][2]^3 \right)$$

$$= 984 = 32 + 952. \quad (3.7)$$

What Goes Wrong at Five Loops?

[JB, Herrmann, Langer, Trnka (2020)]

- ◆ Unfortunately, the “ p -gon power-counting” basis proposed for non-planar is *not compatible with planar power-counting* (at high loops): $\mathcal{B}_p^{\text{Pl}} \not\subset \mathcal{B}_p^{\text{NP}}$



$$\mathcal{B}^{\mathcal{N}=4} \subset \mathcal{B}^{\text{DCI}} \subset \mathcal{B}_4^{\text{Pl}} \not\subset \mathcal{B}_4^{\text{NP}}$$

¿Can someone propose a better definition?

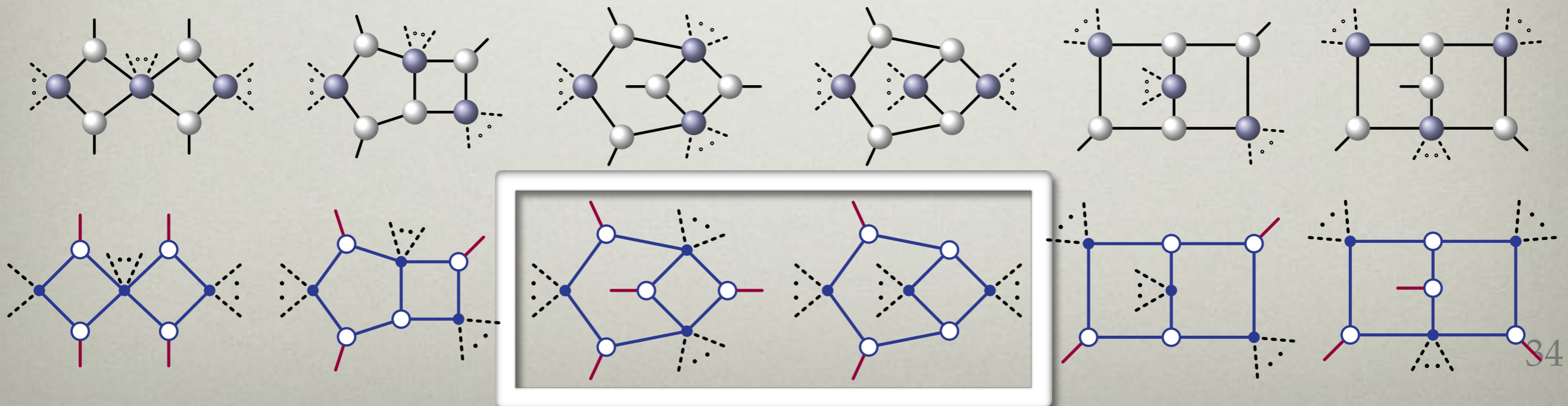
*Room for Improvement:
Building **Better** (Wiser) **Bases***

Normalizing Integrands Wisely

- ◆ It is often a good idea to normalize as much of the basis as possible on places in loop-momentum space where many amplitudes vanish
- ◆ This works well for *nice* amplitudes: those with **low multiplicity**, **low loops**, or **low N^k MHV-degree** (i.e. *where polylogs abound*)

For example, two-loop MHV amplitudes in sYM:

[JB, Herrmann, Langer, McLeod, Trnka (2019)]



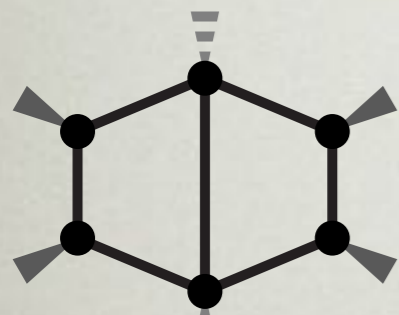
What is 'Purity' Beyond Polylogs?

- ◆ When the basis is $d\log$ and all 'evaluations' are (residues) on poles, then diagonalization ensures each basis integrand is in *canonical form* (UT / etc.)
 - ¿ *What about when no $d\log$ form exists?*

What is 'Purity' Beyond Polylogs?

- ◆ When the basis is $d\log$ and all 'evaluations' are (residues) on poles, then diagonalization ensures each basis integrand is in *canonical form* (UT / etc.)

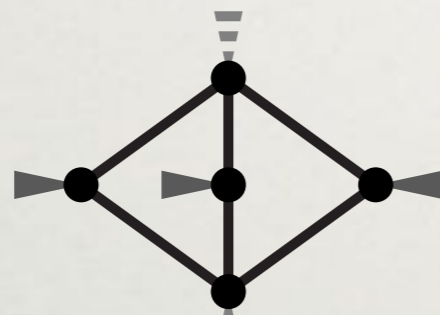
¿What about when no $d\log$ form exists?



CY₁

1

1 1
1



CY₂

1

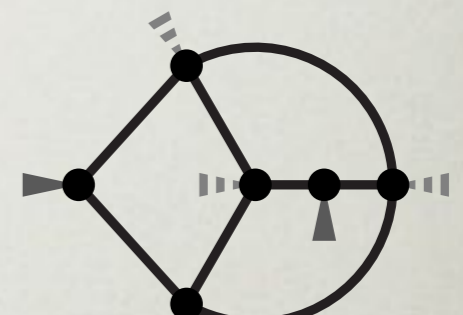
0 0
1 20 1
0 0
1



CY₃

1

0 0
0 1 0
1 149 149 1
0 1 0
0 0
1



CY₄

1

0 0
0 1 0
0 0 0 0
1 976 3952 976 1
0 0 0 0
0 1 0
0 0
1

[JB, McLeod, von Hippel, Wilhelm (2018)]

[JB, He, McLeod, von Hippel, Wilhelm (2018)]

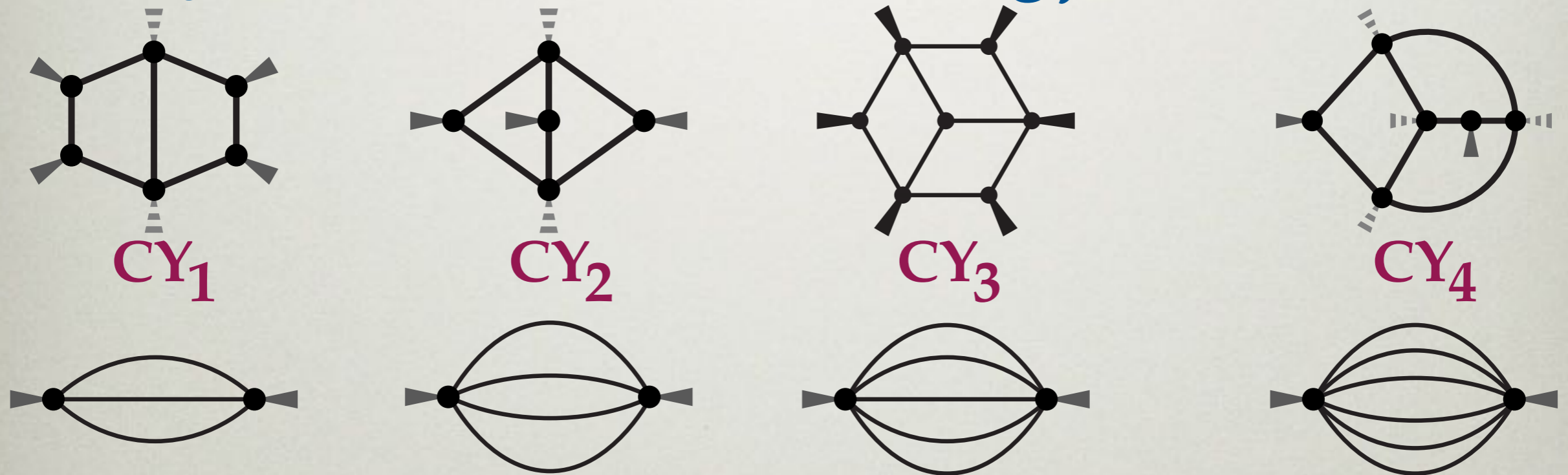
[JB, McLeod, Spradlin, von Hippel, Wilhelm (2018)]

[JB, McLeod, von Hippel, Vergu, Volk, Wilhelm (2019)]

What is 'Purity' Beyond Polylogs?

- ◆ When the basis is $d\log$ and all 'evaluations' are (residues) on poles, then diagonalization ensures each basis integrand is in *canonical form* (UT / etc.)

¿What about when no $d\log$ form exists?



[Bloch, Kerr, Vanhove; Broadhurst;...]

[JB, McLeod, von Hippel, Wilhelm (2018)]

[JB, He, McLeod, von Hippel, Wilhelm (2018)]

[JB, McLeod, Spradlin, von Hippel, Wilhelm (2018)]

[JB, McLeod, von Hippel, Vergu, Volk, Wilhelm (2019)]

Great Room for Improvement

- ◆ Prescriptive unitarity has made great progress, but the results raise (or sharpen) bigger questions

$$\mathcal{A} = \mathcal{A}_d \oplus \mathcal{A}_{d-1} \oplus \cdots \quad \mathfrak{B}_p = \widehat{\mathfrak{B}}_p \oplus \widehat{\mathfrak{B}}_{p+1} \oplus \cdots$$

Better integrand bases would:

- ◆ trade *evaluations* for *periods* on all topologies (does this ensure “purity”?)
- ◆ *stratify* integrands by more refined criteria—e.g.
 - ▶ *actual* UV behavior (do finite bases exist?)
 - ▶ *transcendental weight* (what does this mean?)
 - ▶ *dim-reg* partitioning of numerator monomials
 - ▶ ...