Introduction to Sobolev Spaces and Weak Solutions of PDEs

Patrizia Donato
University of Rouen Normandie

Multi-Scale Analysis and Theory of Homogenization
Discussion-Meeting, ICTS Bangalore
26 August 2019 - 6 September 2019

What is a Partial Differential Equation ?

A **partial differential equation (PDE)** of order m is an equation of the form

$$F\left(x, u, D^1(u), D^2(u), \dots, D^m(u)\right) = f(x),$$

for $x = (x_1, x_2, \dots, x_N) \in \mathcal{O}$, where $\mathcal{O}$ is an open set in $\mathbb{R}^N$, $D^i(u)$ the set of all the derivatives of order i, and f is a given function.

A solution is a function $u = u(x)$ whose involved partial derivatives in are well-defined in $\mathcal{O}$ and which satisfies the equation at each point x of $\mathcal{O}$.

If $N = 1$, then it is simply an ordinary differential equation of order m.

The mapping

$$u \mapsto F\left(x, u, D^1(u), D^2(u), \dots, D^m(u)\right),$$

is called a **differential operator.**

Linear PDEs

The PDE is called **linear** if F is linear with respect to the unknown u and all its derivatives.

Then it can be written as

$$\sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha(u) = f(x),$$

where the functions $a_\alpha = a_\alpha(x)$ (for $|\alpha| \leq m$) are called the coefficients of the differential operator.

For homogeneous linear partial differential equations ($f \equiv 0$) a linear combination of its solutions is still a solution of the same equation. This is known as **the principle of superposition.**

We will discuss in particular some second order elliptic linear PDEs, which cover many interesting situations.

A classical example

The Laplace and Poisson equations.

The homogeneous Laplace equation in a bounded open set $\Omega \subset \mathbb{R}^N$ is:

$$\Delta u(x) = 0,$$

where Δ is the laplacian operator.

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_N^2}.$$

The corresponding nonhomogeneous equation

$$\Delta u(x) = f,$$

where f is a given function, is called Poisson equation.

It models for instance, the stationary thermal diffusion equation in an homogeneous material.
It also describes the displacements of a tight membrane or appear in electrical problems, where the potential of an electrostatic field satisfies the Poisson equation.

Let us describe the first model, i.e. steady heat conduction in an isotropic material.

Let γ be the thermal conductivity of a body occupying Ω, which describes the ability of the body to diffuse heat.
If the body is homogeneous and isotropic, then γ is a scalar.

Suppose that f represents the heat source and let $u = u(x)$ the temperature at the point $x \in \Omega$.

The flux of the temperature is defined by

$$q(x) = \gamma \nabla u(x)$$

Its divergence represents the flux density which is equal to the heat source, that is div $q = f$.

Since γ is constant, this can be rewritten in the form

$$-\gamma \Delta u = f \quad \text{in } \Omega,$$

which is exactly the Poisson equation.

If the body is **inhomogeneous and anisotropic**, the thermal conductivity is not described anymore by a constant γ as before, but by a matrix field

$$A(x) = (a_{ij}(x))_{1 \leq i,j \leq N},$$

so that the steady heat conduction in the material is described by the equation (the so-called in the divergence form)

$$-\text{div}\,(A(x)\nabla u) = f \quad \text{in } \Omega.$$

☛ This is an important equation here, since it is a model equation treated by the homogenization theory.

More classical examples

The heat equation.

If the heat source depends on time, then the energy conservation law leads to the following heat equation

$$\frac{\partial u}{\partial t} - \text{div}\,(A(x)\nabla u) = f.$$

Also, the Black-Scholes equation, well-known in financial mathematics can be reduced to a simiilar form.

Vibrations: the wave equation. Vibrations or waves in a inhomogeneous and anisotropic body are described by the equation

$$\frac{\partial^2 u}{\partial t^2} - \text{div}\,(A(x)\nabla u) = f.$$

Boundary and initial conditions

In general, a partial differential equation have more than one or even an infinite number of solutions.

The uniqueness of a solution of a physical problem in Ω can be obtained adding some conditions assigned on its boundary $\partial\Omega$, the so-called **boundary conditions.**

The most classical boundary conditions are the following:

1. The Dirichlet boundary condition: a given value is imposed for the solution u on $\partial\Omega$.
2. The Neumann boundary condition: the normal derivative of u, defined by $A\nabla u \cdot n$, where n denotes the outward unit normal to $\partial\omega$ has to take given values on $\partial\Omega$.
3. The Robin boundary condition: it involves both u and its normal derivative in an expression written on $\partial\Omega$.

These conditions have a physical meaning. For heat diffusion,

- Condition (1) signifies that the boundary is maintained at a given temperature.
- Condition (2) means that the flux of the heat through the boundary is prescribed.
- Condition (3) translates to the fact that through the boundary, there is an exchange of heat with the environment that has a given temperature.

☛ This makes an important difference with a ODE, even for N=1.

Additional conditions, **the initial conditions**, are imposed when the time t is a variable in the problem.
As for a ODE, one prescribes a value to u (and to some derivatives of u with respect to t), at the initial time $t = 0$ in Ω.

For the heat equation, one only prescribes the value of u.

Physically these conditions describe the initial state of the system.

The concept of well-posed problems

It is clear that the initial and boundary conditions to be prescribed depend on the physical situation and essentially on the order of the equation.

☛ If the number of conditions is insufficient, a solution may not have any relation with the physical phenomenon under study.

☛ If too many conditions, the equation may not have solution (this situation is called overdetermined).

A simple non existence example

☛ We cannot couple any equation with any boundary conditions.

Consider the 1D problem:

$$\begin{cases} u'' + u = 0 \quad \text{in } (0, 2\pi), \\ u(0) = a,\ u(2\pi) = b. \end{cases}$$

It is known that a solution of the equation has the form

$$u(x) = c_1 \sin x + c_2 \cos x, \quad c_1, c_2 \in \mathbb{R},$$

hence is a2π-periodic function. The choice of the constants is determined by the boundary conditions.

- Since $u(0) = u(2\pi)$, if $a \neq b$ problem above has no solution !!

A simple non uniqueness example

Consider the 1D problem:

$$\begin{cases} u'' = f \quad \text{in } (a,b), \\ u'(a) = u'(b) = 0. \end{cases}$$

Suppose that u is a solution of this problem.

Since the derivative of a constant is zero, then for every $c \in \mathbb{R}$ the function $u + c$ is also a solution.

- Hence, this problem admits an infinity of solution !!

☛ In general, a linear mathematical model is considered satisfactory if for a given data, if the equation with the given boundary and initial conditions admits one and only one solution.

☛ Nevertheless, even that is not sufficient. Indeed the data originate from measurements are never perfect, which is not important in the physical description. Then, a small change in the data should results in a small change in the solution.

This is called **the stability of a solution with respect to the data.**

☛ Not all the mathematical problems are stables.

The french mathematician **J. Hadamard** (1865-1963) gave an example of a sequence of problems where the data goes to zero and the solutions goes to infinity.

Well posed problems in the Hadamard sense

Consequently, Hadamard introduced the following concept:

Definition Let U and F two spaces of functions (for instance Banach spaces) and let us consider a partial differential equation with given boundary and initial conditions.

Let f a vector function representing the data of the problem.

The problem is called **well-posed** if one has

1. (Existence) For any element $f \in \mathcal{F}$ there exists a solution $u \in \mathcal{U}$ of the problem.
2. (Uniqueness) The solution is unique.
3. (Stability) The map which to $f \in \mathcal{F}$ associated the solution $u \in \mathcal{U}$ is continuous.

Existence of explicit solutions

This concern only special case (special types with radial simmetry, looking for power series...)

☛ Since in general it is not possible to have the explicit solutions, even for simple equations.

Then, it is important to prove that the problem is well posed, that is prove existence, uniqueness and stability theorems.

☞ This allows to validate the model (if no solutions, it cannot represent a physical situation....)

☛ Also, when no explicit solution, adapted numerical methods (finite difference method, finite element method,..) provide an approximates solution. Theorems allow to justify the use of the numerical methods.

A Classical classification

In analogy with the conics curves, most of equations can be classified in the following 3 types:

- **Elliptic equation**, the model case being the Poisson equation (no time).
- **Parabolic equation**, the model case being the heat equation (1 time-derivative).
- **Hyperbolic equation**, the model case being the wave equation (2 time-derivatives).

☛This depend somehow on the determinant of the second order coefficient matrix ....

☛ One can reduce a linear second order equations to a canonical form.

Proving theorems ...

As already mentioned, one look for theorems stating that a problem is well posed.

☛ These theorems are difficult to prove and require an important functional background and powerfull techniques.

For a given class of problem one need to prove suitable theorems, according on the assumption on:

- the type of equation,
- the type of boundary conditions,
- the kind of domain,
- the regularity of the coefficients.
- the regularity of the data.

For instance, one can prove that the problem is well posed for

- a linear elliptic equation,
- in a bounded domain Ω of $\mathbb{R}^N$,
- with Dirichlet conditions,

BUT

one has to require that

- the coefficients and the datum have a **strong regularity** (even $f \in C^0$ and Ω a ball are not sufficient, the Hölder continuous framework is needed).
- the domain has a smooth boundary.

☛ If not, there exist counter-examples of non existence.

☛ The solutions obtained in this context are usually called

classical solutions

The related results cover many interesting problems, but

the assumptions are not always satisfied

in many important real situations.

For instance if

- f a step function
- Ω a polygone or a polyedre

What we can do in these cases, where the experiences suggest that a solutions shoud exist?

A model problem

We consider, as model case, the stationary heat equation with, for instance, a Dirichlet condition on a bounded open set $\Omega \subset \mathbb{R}^N$, that is

$$\begin{cases} -\text{div } (A\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$-\text{div } (A\nabla) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big)$$

and (for instance) $f \in L^2(\Omega)$.

• We suppose that $A(x) = (a_{ij}(x))_{1\leq i,j\leq N}$ is a matrix field in $M(\alpha, \beta, \Omega)$, with $0 < \alpha < \beta$, i.e.

$$\begin{cases} (A(y)\lambda, \lambda) \geq \alpha|\lambda|^2, \\ |A(y)\lambda| \leq \beta|\lambda|, & \text{for any } \lambda \in \mathbb{R}^N. \end{cases}$$

Weak solutions

One introduces **weak solutions**, as follows.

- The derivatives are taken in a week sense, the so-called sense of distributions,
- The solutions belongs to a suitable functional space, a Sobolev space
- The solution satisfies a weak version of the above equation, called **the variational formulation**.

☛ Distributions and Sobolev spaces need a more sophisticated backgroud, and there existe a complete theory about.

A motivation

Consider the following one-dimensional problem in the interval $(0, 1)$:

$$(*) \begin{cases} -u''(x) + c(x)u(x) = f(x), & \forall x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $c \in C^0([0,1])$, $c \geq 0$, and $f \in C^0([0,1])$.

Let us introduce the space

$$V = \{ v \in C^1([0,1]) \mid v(0) = v(1) = 0 \}.$$

- It is known that V is dense in $L^2(0,1)$.

Define the differential operator

$$A : C^2([0,1]) \cap V \to C^0([0,1]),$$

given by

$$(Au)(x) = -u''(x) + c(x)u(x) \quad \text{for} \;\; x \in [0,1].$$

Theorem *Let $c \in C^0([0,1])$, $c \geq 0$ and $f \in C^0([0,1])$. Then a function $u \in C^2([0,1]) \cap V$ is a solution of (*) if and only if u is a solution of the following problem:*

$$(**) \quad \int_0^1 \big(u'\,v' + c\,u\,v\big)\,dx = \int_0^1 f\,v\,dx, \qquad \forall v \in V.$$

Proof. Let $u \in C^2([0,1]) \cap V$ be a solution of (*). Multiplying the equation in (*) by $v \in V$ and integrating on [0, 1], we obtain

$$\int_0^1 \big(- u'' + c\, u\big) v \; dx = \int_0^1 f \; v \; dx, \qquad \forall v \in V.$$

Integrating by parts and recalling that $v \in V$, we have

$$\int_0^1 -u'' \, v \; dx = \int_0^1 u' \, v' \; dx - u' \, v\Big|_0^1 = \int_0^1 u' \, v' \; dx.$$

These two equalities imply that u is a solution of (**).

Conversely, let $u \in C^2([0,1]) \cap V$ be a solution of (**). Hence,

$$\int_0^1 \big(- u'' + c\, u - f\big) \; v \; dx = 0, \qquad \forall v \in V.$$

Since V is dense in $L^2(0,1)$, we have

$$\int_0^1 \big(-u'' + c\,u - f\big)\, v\, dx = 0, \qquad \forall v \in L^2(0,1).$$

This means that the function $\phi = -u'' + cu - f$ is an element of $L^2(0,1)$ satisfying

$$\big(\phi, v\big) = 0, \qquad \forall v \in L^2(0,1).$$

Since the zero function is the only function orthogonal to all elements of $L^2(0,1)$,

$$0 = \phi = -u'' + cu - f \qquad \text{a.e. in } (0,1).$$

Since $\phi \in C^0([0,1])$, it follows that $\phi(x) = 0$ for every x in $(0,1)$. This implies that u satisfies the equation in (*). Moreover, $u \in V$, so u also satisfies the boundary conditions.
This completes the proof. □

The idea of weak solutions.

Problem (**) is called a variational formulation of problem (*). The fact that u is a solution of (**) means that the equality is satisfied by every v belonging to the given space V. The function v is called a test function.

Observe now that (**) makes sense even if f, u, v, u', and v' are only in $L^2(0,1)$ and not in spaces of continuous functions (in a sense to be precised).

Therefore one can try to find a function in H, a bigger space than $C^2([0,1]) \cap V$, satisfying (**) for any test in H. In this case, u is called the weak solution of (*) and (**) is called the variational formulation of (*).

Then, one has to prove that if the solutions and the data are smooth, then u is a solution in the usual (classical) sense.

☛ Sobolev spaces are the suitable spaces for that. To define them, one need to introduce distributions.

Distributions

☛ A fundamental set of functions:

Let $\mathcal{O}$ an open set of $\mathbb{R}^N$. We define $\mathcal{D}(\mathcal{O})$ as the set of the infinitely differentiable functions v on $\mathcal{O}$ such taht supp v is a compact set in $\mathbb{R}^N$ contained in $\mathcal{O}$.

Definition A sequence $\{\varphi_n\}$ in $\mathcal{D}(\mathcal{O})$ is said to converge to an element φ in $\mathcal{D}(\mathcal{O})$ if the following are satisfied:

(1) there exists a compact set $K \subset \mathcal{O}$ such that supp $\varphi \subset K$ and for any $n \in \mathbb{N}$, supp $\varphi_n \subset K$,

(2) for any $\alpha \in \mathbb{N}^N$, $\partial^\alpha \varphi_n$ converges uniformly to $\partial^\alpha \varphi$ on K.

☛ The space $\mathcal{D}(\mathcal{O})$ is not a metric space and the above definition does not induce a topology, but a suitable (complicated and not neede here) topology may be defined such that the convergence of sequences is exactly that one.

Definition

A **distribution** on $\mathcal{O}$ is a map $T : \mathcal{D}(\mathcal{O}) \to \mathbb{R}$ such that

(1) T is linear,

(2) if $\varphi_n \to \varphi$ in $\mathcal{D}(\mathcal{O})$, then $T(\varphi_n) \to T(\varphi)$ in $\mathbb{R}$.

We denote by $\mathcal{D}'(\mathcal{O})$ the set of distributions on $\mathcal{O}$.

Exercise

1. Let $x_0 \in \mathbb{R}^N$. The mapping defined by

$$\delta_{x_0}(\varphi) = \varphi(x_0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N),$$

is a distribution (the so-called Dirac mass).

2. Let $f \in L^1_{loc}(\mathcal{O})$ and define the mapping T_f on $\mathcal{D}(\mathcal{O})$ by

$$\langle T_f, \varphi \rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})} = \int_{\mathcal{O}} f\varphi \, dx.$$

Then T_f is a distribution on $\mathcal{O}$. Moreover, the linear map

$$L : f \in L^1_{loc}(\mathcal{O}) \longmapsto T_f \in \mathcal{D}'(\mathcal{O}),$$

is one-to-one.

As a consequence $L^1_{loc}(\mathcal{O})$ (in particular $L^p(\mathcal{O})$) can be regarded as a subspace of $\mathcal{D}'(\mathcal{O})$ by identifying f with T_f. With this identification done in this remark, we have the following definition:

Definition A distribution T is in $L^1_{loc}(\mathcal{O})$ (respectively in $L^p(\mathcal{O})$), if there exists $f \in L^1_{loc}(\mathcal{O})$ (respectively in $L^p(\mathcal{O})$), such that $T = T_f$). In this case, T is called a regular distribution.

☛ There exist distributions which are not regular, for instance the Dirac function at a point $x_0 \in \mathbb{R}$ (Exercice !).

Definition Let $T \in \mathcal{D}'(\mathcal{O})$. The derivative of T with respect to x_i for any $i = 1, \dots, N$, denoted $\dfrac{\partial T}{\partial x_i}$, is defined as

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})} = -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})},$$

for all $\varphi \in \mathcal{D}(\mathcal{O})$.

Exercices

- Show that if $T \in \mathcal{D}'(\mathcal{O})$, then

$$\frac{\partial T}{\partial x_i} \in \mathcal{D}'(\mathcal{O}) \quad \text{for} \quad i = 1, \dots, N.$$

  That is, the derivative of a distribution is also a distribution.
- The derivative of the Dirac function δ_{x_0} is given by

$$\left\langle \frac{\partial \delta_{x_0}}{\partial x_i}, \varphi \right\rangle_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = -\frac{\partial \varphi}{\partial x_i}(x_0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).$$

Remark

Let us suppose that f and its derivatives in the sense of distributions are in $L^1_{loc}(\mathcal{O})$ for $i = 1, \ldots, N$, then

$$\int_{\mathcal{O}} \frac{\partial f}{\partial x_i}\, \varphi \, dx = - \int_{\mathcal{O}} f \frac{\partial \varphi}{\partial x_i}\, dx, \quad \forall \varphi \in \mathcal{D}(\mathcal{O}).$$

If $f \in C^1(\mathcal{O})$, its derivatives in the sense of distributions coincide with the usual partial derivatives since by Green's formula,

$$\int_{\mathcal{O}} f \frac{\partial \varphi}{\partial x_i}\, dx = - \int_{\mathcal{O}} f_{x_i}\, \varphi \, dx,$$

where f_{x_i} denotes the usual derivative. Hence,

$$f_{x_i} = \frac{\partial f}{\partial x_i} \quad \text{for every} \;\; i = 1, \ldots, N.$$

☛ This is not true if f is differentiable only a.e. (Exercice !).

Definition Let $1 \leq p \leq +\infty$. We define the Sobolev space $W^{1,p}(\mathcal{O})$ as

$$W^{1,p}(\mathcal{O}) = \left\{ u \in L^p(\mathcal{O}) \middle| \frac{\partial u}{\partial x_i} \in L^p(\mathcal{O}),\ i = 1, \ldots, N \right\},$$

where the derivatives are taken in the sense of distributions, together with the associated norm

$$\|u\|_{W^{1,p}(\mathcal{O})} = \|u\|_{L^p(\mathcal{O})} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathcal{O})}.$$

In particular, for $p = 2$, we denote $W^{1,2}(\mathcal{O})$ by $H^1(\mathcal{O})$.

Proposition

The norm in $W^{1,p}(\mathcal{O})$ is equivalent to

$$\|u\| = \left(\|u\|^p_{L^p(\mathcal{O})} + \|\nabla u\|^p_{L^p(\mathcal{O})} \right)^{\frac{1}{p}},$$

where, we set

$$\|\nabla u\|_{L^p(\mathcal{O})} = \Big(\sum_{i=1}^{N} \Big\| \frac{\partial u}{\partial x_i} \Big\|^p_{L^p(\mathcal{O})} \Big)^{\frac{1}{p}}.$$

☛ This is very useful for proving estimates by taking the square of the norm.

Main properties

- For $1 \le p \le +\infty$, the space $W^{1,p}(\mathcal{O})$ is a Banach space and $H^1(\mathcal{O})$ is a Hilbert space for the scalar product defined by

$$(v, w)_{H^1(\mathcal{O})} = (v, w)_{L^2(\mathcal{O})} + \sum_{i=1}^{N} \Big(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \Big)_{L^2(\mathcal{O})},$$

  for all $v, w \in H^1(\mathcal{O})$.

- $W^{1,p}(\mathcal{O})$ is separable if $1 \le p < +\infty$, reflexive if $1 < p < +\infty$.
- Let $1 \le p < +\infty$. Then $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$.
- Let Ω a bounded open set with a Lipschitz continuous boundary. Then the set $\mathcal{D}\big(\overline{\Omega}\big)$ of the restrictions to $\overline{\Omega}$ of functions in $\mathcal{D}(\mathbb{R}^N)$is dense in $W^{1,p}(\Omega)$.

The space H_0^1 and its properties

The space $H_0^1(\mathcal{O})$ is well adapted to treat variational elliptic problems with homogeneous boundary Dirichlet conditions.

Definition *Let* $1 \leq p \leq +\infty$. *The Sobolev space* $W_0^{1,p}(\mathcal{O})$ *is the closure of* $\mathcal{D}(\mathcal{O})$ *with respect to the norm of* $W^{1,p}(\mathcal{O})$. *In particular, we set*

$$H_0^1(\mathcal{O}) = W_0^{1,2}(\mathcal{O}).$$

We give now two important properties of this space.

We recall that if u is a function defined on a open set $\mathcal{O} \subset \mathbb{R}^N$, then its **zero extension** to $\mathbb{R}^N$, denoted $\tilde{u}$, is defined by

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition. *Let $1 \leq p \leq +\infty$. If u is a function in $W_0^{1,p}(\mathcal{O})$, then its zero extension ũ defined by*

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \mathcal{O}, \\ 0 & \text{otherwise}, \end{cases}$$

belongs to $W_0^{1,p}(\mathcal{O}_1)$ for every open subset $\mathcal{O}_1$ containing $\mathcal{O}$. Moreover,

$$\|u\|_{W_0^{1,p}(\mathcal{O})} = \|\widetilde{u}\|_{W_0^{1,p}(\mathcal{O}_1)}.$$

Proof. The delicate point is to prove that:

$$\frac{\partial \widetilde{u}}{\partial x_i} = \widetilde{\frac{\partial u}{\partial x_i}} \qquad \text{for } i = 1, \ldots, N,$$

in the distribution sense. Then, the equality is immediate. □

The Poincaré Inequality

Theorem. *Let* Ω *be a bounded open set in* $\mathbb{R}^N$. *Then there exists a constant* C_Ω, *such that*

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \, \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega),$$

and C_Ω *depends only on the diameter of* Ω.

☛ As a consequence,

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)},$$

defines a norm on $H_0^1(\Omega)$ equivalent to the norm of $H^1(\Omega)$.

Weak solutions for Dirichlet boundary conditions

Let us come back to our model problem

$$\begin{cases} -\text{div}\,(A\nabla u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

- We look for a solution that satisfies a weak version of the above equation, called the variational formulation. This is formally obtained multiplying the equation by a smooth function φ which satisfies the Dirichlet boundary condition and then integrating by parts.
- It has to be satisfied by any such function φ and by density by any function $v \in H_0^1(\Omega)$ (called a test function).

- For our model problem, the variational formulation is :

$$\text{Find } u \in H_0^1(\Omega) \text{ such that}$$

$$\int_\Omega A\nabla u \, \nabla v \; dx = \int_\Omega f \; v \; dx \quad \forall v \in H_0^1(\Omega).$$

- The (weak) sense of the Dirichlet boundary condition is contained in the condition $u \in H_0^1(\Omega)$. When u is continuous in $\bar{\Omega}$, this implies the Dirichlet boundary condition in every point.

☛ Justification of this framework: One has to prove that if all the data are smooth, a weak solution is a classical one, i.e. a function of class $C^2(\bar{\Omega})$ verifying pointwise the equation.

Existence and uniqueness of a weak solution

We make use of an important abstract result, the Lax-Milgram Theorem.

Theorem Let a be a continuous bilinear form on a Hilbert space H and $F \in H'$. If a is H-elliptic with constant α_0, then the variational equation

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u,v) = \langle F, v \rangle_{H',H}, \quad \forall v \in H, \end{cases}$$

has a unique solution $u \in H$.
Moreover, we have the following *a priori* estimate:

$$\|u\|_H \leq \frac{1}{\alpha_0} \|F\|_{H'}.$$

☛ It applies to our model problem as follows.

Theorem *Consider the model problem above with* $A \in M(\alpha, \beta, \Omega)$ *and* $f \in H^{-1}(\Omega) = (H_0^1(\Omega))'$ *(in particular in* $L^2(\Omega)$*). There exists a unique solution* $u \in H_0^1(\Omega)$ *such that*

$$\int_\Omega A\nabla u \nabla v \, dx = \int_\Omega f \, v \, dx \quad \forall v \in H_0^1(\Omega).$$

Moreover, one has the following a priori estimate

$$\|u\|_{H_0^1(\Omega)} \leq \frac{C_\Omega}{\alpha} \|f\|_{L^2(\Omega)},$$

where C_Ω is the Poincaré constant.

☛ The problem is well posed, in the sense of Hadamard.

The notion of trace

The notion of trace gives a meaning to the restriction of functions in $W^{1,p}(\Omega)$ to $\partial\Omega$ ($\partial\Omega$ Lipschitz continuous), where one can still define a normal a.e. and integrals on $\partial\Omega$ and consequently $L^p(\partial\Omega)$.

The trace theorem. *If Ω is a bounded open subset of $\mathbb{R}^N$ such that $\partial\Omega$ is Lipschitz continuous, then there exists a unique linear continuous map*

$$\gamma \,:\, H^1(\Omega) \longmapsto L^2(\partial\Omega),$$

such that $\gamma(u) = u|_{\partial\Omega}$ for any $u \in H^1(\Omega) \cap C^0\big(\overline{\Omega}\big)$. The function $\gamma(u)$ is called the trace of u on $\partial\Omega$.

☛ This give a meaning to boundary conditions for functions in H^1.

Theorem. *For $N \geq 2$, let Ω be a bounded open subset of $\mathbb{R}^N$ such that $\partial\Omega$ is Lipschitz continuous. Then*

$$H^1_0(\Omega) = \big\{\, u \in H^1(\Omega) \,|\, \gamma(u) = 0 \;\; on \;\; \partial\Omega \big\}.$$

☛ This give a meaning to the homogeneous Dirichlet boundary condition for functions in H^1_0.

The space $H^1{}_{\text{per}}(Y)$

We introduce now a notion of periodicity for functions in the Sobolev space H^1. Let $Y =]0, \ell_1[\times \ldots \times]0, \ell_N[$, be the reference cell where $\ell_1, \ldots, \ell_N$ are given positive numbers.

Definition. Let $C^\infty_{\text{per}}(Y)$ be the subset of $C^\infty(\mathbb{R}^N)$ of Y-periodic functions. We denote by $H^1{}_{\text{per}}(Y)$ the closure of $C^\infty_{\text{per}}(Y)$ for the H^1-norm.

Proposition. *Let $u \in H^1{}_{per}(Y)$. Then, u has the same trace on the opposite faces of Y.*
Moreover, let g be a function defined a.e. on Y and denote by $g^\#$ its extension by periodicity to the whole of $\mathbb{R}^N$, defined by

$$g^\#(x + k\,\ell_i\, e_i) = g(x) \quad \text{a.e. on } Y, \quad \forall\, k \in \mathbb{Z}, \quad \forall\, i \in \{1, \ldots, N\},$$

where $\{e_1, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^N$. Then $u^\#$ is in $H^1(\omega)$ for any ω bounded open subset of $\mathbb{R}^N$.

Periodic boundary conditions

In the sequel, we will make use of the subspace

$$W_{\text{per}}(Y) = \left\{ v \mid v \in H^1_{\text{per}}(Y),\ \mathcal{M}_Y(v) = 0 \right\}.$$

Proposition. *The space $W_{per}(Y)$ is a Banach space for the norm*

$$\|u\|_{W_{per}(Y)} = \|\nabla u\|_{L^2(Y)}, \qquad \textit{for any } u \in W_{per}(Y).$$

☛ We introduce now a class of problems which plays a crucial role in homogenization. For $A \in M(\alpha, \beta, Y)$ and h a given function in $L^2(Y)^N$, consider the problem

$$\begin{cases} \text{- div } (A\nabla u) = \text{- div } h \quad \text{in } Y, \\ u \ Y\text{- periodic}, \\ \mathcal{M}_Y(u) = 0, \end{cases}$$

Its variational formulation is:

$$\begin{cases} \text{Find } u \in W_{per}(Y) \text{ such that} \\ \displaystyle\int_Y A(y)\nabla u \, \nabla v \, dy = \int_Y h\nabla v \, dy, \quad \forall v \in W_{per}(Y) \end{cases}$$

Theorem *The problem above admit a unique solution u and the following a priori estimate holds*

$$\|u\|_{W_{per}(Y)} \leq \frac{1}{\alpha}\|f\|_{L^2(Y)}.$$

Moreover, its periodic extension to $\mathbb{R}^N$ *(still denoted* u*) is the unique solution of the variational problem*

$$\int_{\mathbb{R}^N} A(y)\nabla\phi \, \nabla v \, dy = \int_{\mathbb{R}^N} h\nabla\phi, dy, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^N),$$

☛ By density, one can take $\phi \in H_0^1(\Omega)$.

A particular case

For $h = A(y)\lambda$, where $\lambda \in \mathbb{R}^N$, one has the problem arising in homogenization

$$\begin{cases} \text{- div } (A\nabla\widehat{\chi}_\lambda) = \text{-div } (A\lambda) \quad \text{in } Y, \\ \widehat{\chi}_\lambda \ Y\text{- periodic.} \\ \mathcal{M}_Y(\widehat{\chi}_\lambda) = 0, \end{cases}$$

whose variational formulation is

$$\int_Y A(y)\nabla\widehat{\chi}_\lambda \, \nabla v \, dy = \int_Y A(y) \, \lambda \, \nabla v \, dy, \quad \forall \phi \in H^1_{per}(Y),$$

By linearity, $\widehat{\chi}_\lambda = \sum_{i=1}^N \lambda_i \widehat{\chi}_{e_j}$, where $(e_i)_{i=1,\dots,n}$ is the canonical basis of $\mathbb{R}^N$.

☛ The functions $\widehat{w}_\lambda = \lambda \cdot x - \widehat{\chi}_\lambda$ play also an important role.

Other boundary conditions

One can adapt these ideas to study:

- Dirichlet-Neumann conditions.
- Robin conditions.
- other situations ...

☛ We can discuss them in the afternoons !!

Thanks for your attention!