CLIMATE SCIENCE, WAVES, AND PDE'S for the Tropics:

Observations, Theory, and Numerics

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Tropics are Extremely Important in Globally Warming World

Rich Observed Phenomena

Hard to understand physically with Multi-Scale interaction

GCM Computer Models Fail to Represent

⇕

Challenge for Contemporary Appl. Math, Need New

Physical Theories

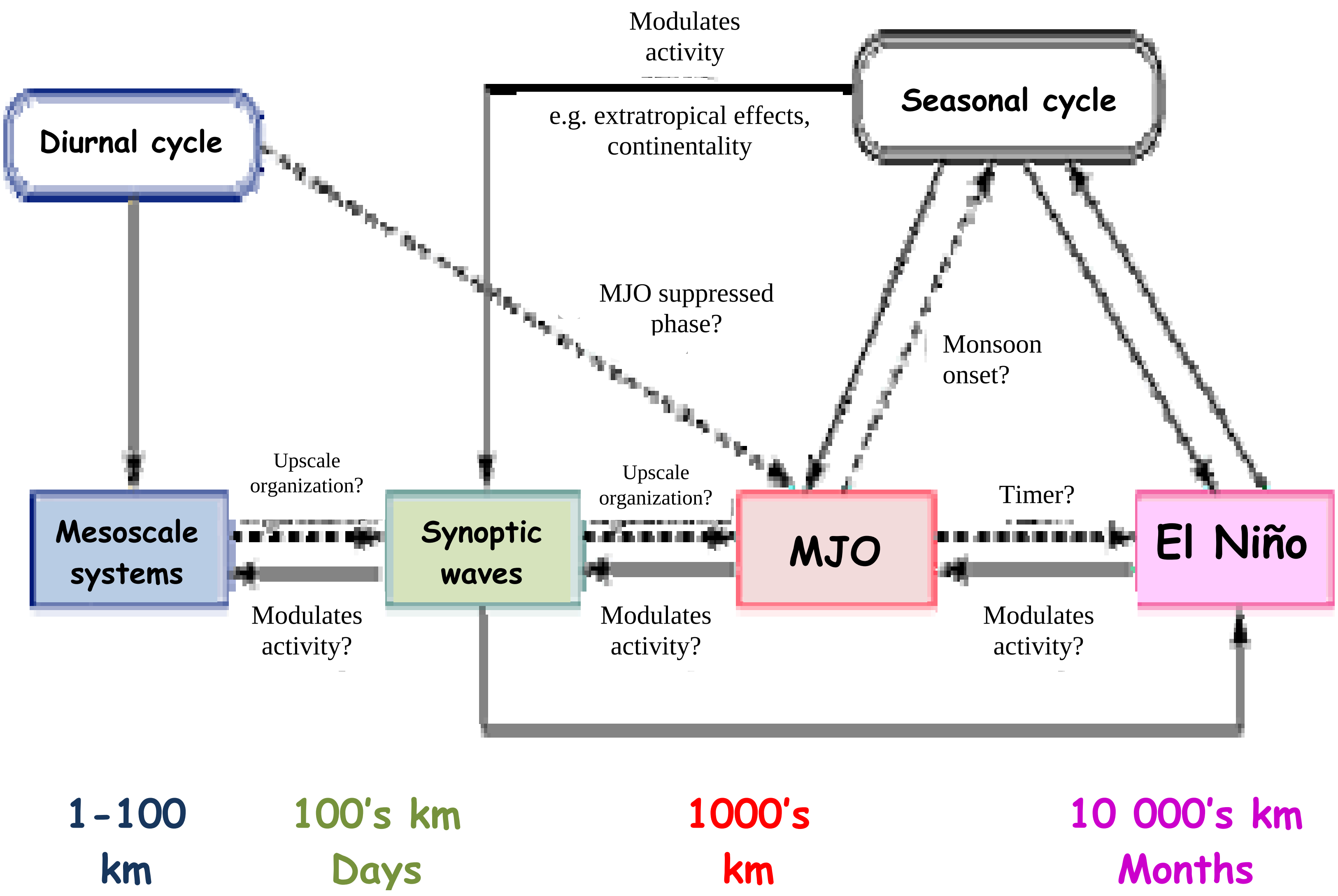
Multi-Scale Models

Numerics and Algorithms

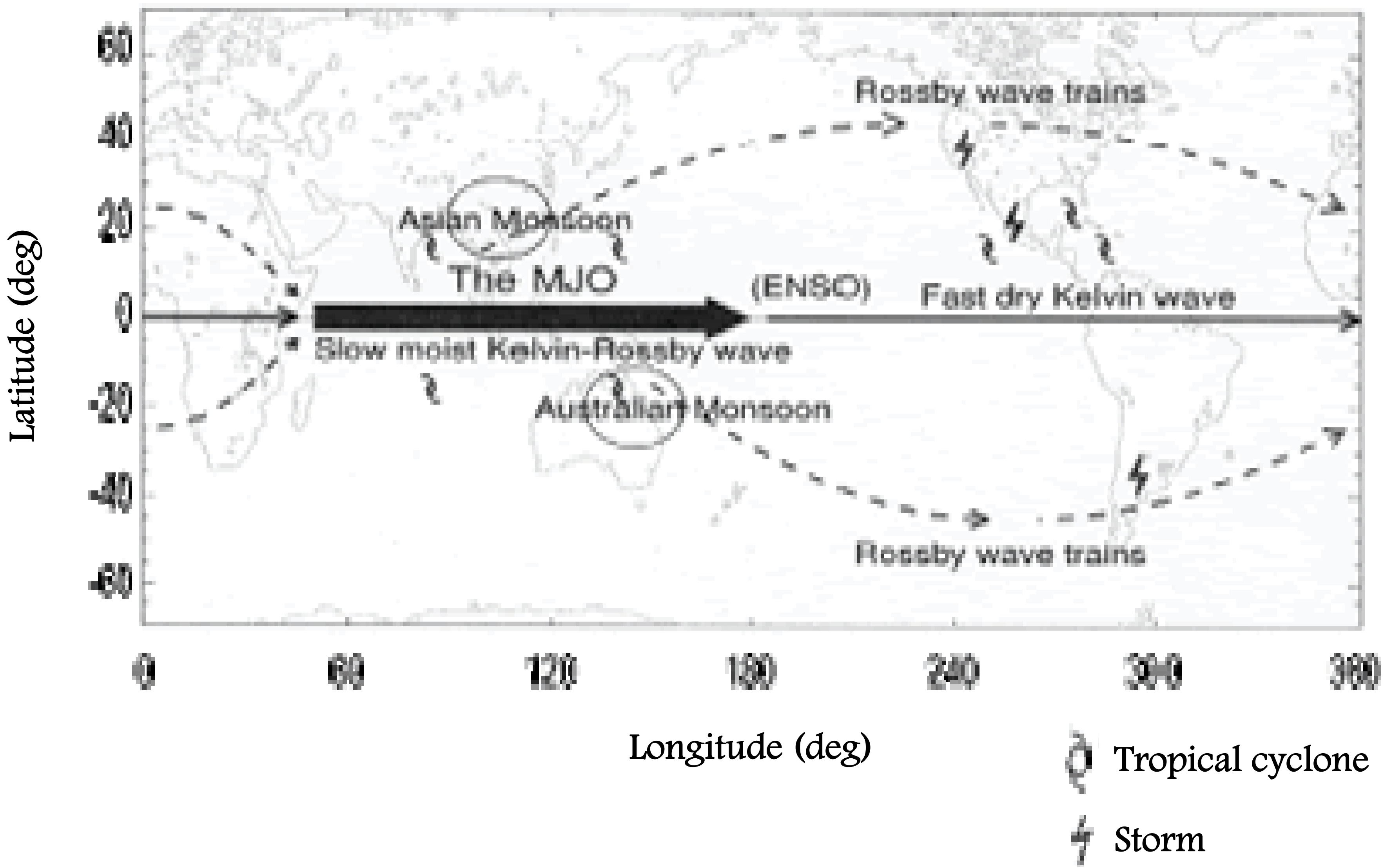
PDE Phenomena

And their symbiotic interaction

See Khouider, Majda, Stechmann, Nonlinearity (2013), Climate Science in the Tropics:
Also see Majda's NYU faculty website for many papers on all this over last ten years



Modulates activity
Seasonal cycle
Diurnal cycle
e.g. extratropical effects, continentality
MJO suppressed phase?
Monsoon onset?
Upscale organization?
Upscale organization?
Timer?
Mesoscale systems
Synoptic waves
MJO
El Niño
Modulates activity?
Modulates activity?
Modulates activity?
1-100 km
100's km Days
1000's km
10 000's km Months


Rossby wave trains
Asian Monsoon
The MJO
(ENSO)
Fast dry Kelvin wave
Slow moist Kelvin-Rossby wave
Australian Monsoon
Rossby wave trains
60
40
20
0
-20
-40
-60
Latitude (deg)
0
60
120
180
240
300
360
Longitude (deg)
Tropical cyclone
Storm

The Skeleton of

Tropical Intraseasonal Oscillations

Andrew Majda (Courant Institute, NYU)

Samuel N. Stechmann (U. Wisconsin)

In Proc. Natl. Acad. Sci., 2009

- New minimal dynamical model for the MJO
- Robustly recovers the MJO's fundamental features (i.e., its "skeleton") on intraseasonal/planetary scales:
  - slow phase speed of ≈ 5 m/s
  - peculiar dispersion relation of $d\omega/dk \approx 0$
  - horizontal quadrupole vortex structure

Multi-scale clouds and waves in the tropics

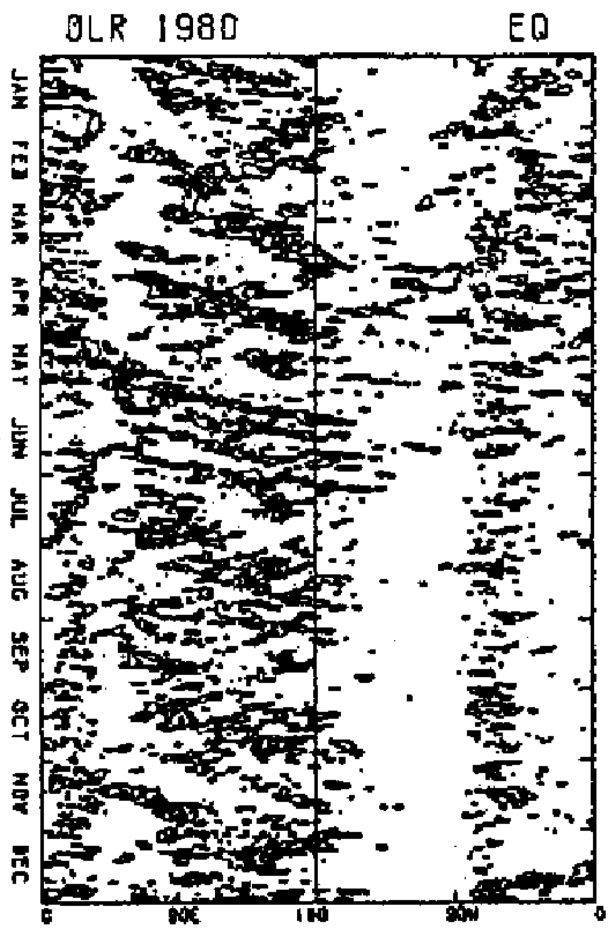
Clouds in the tropics are organized across vast length and time scales

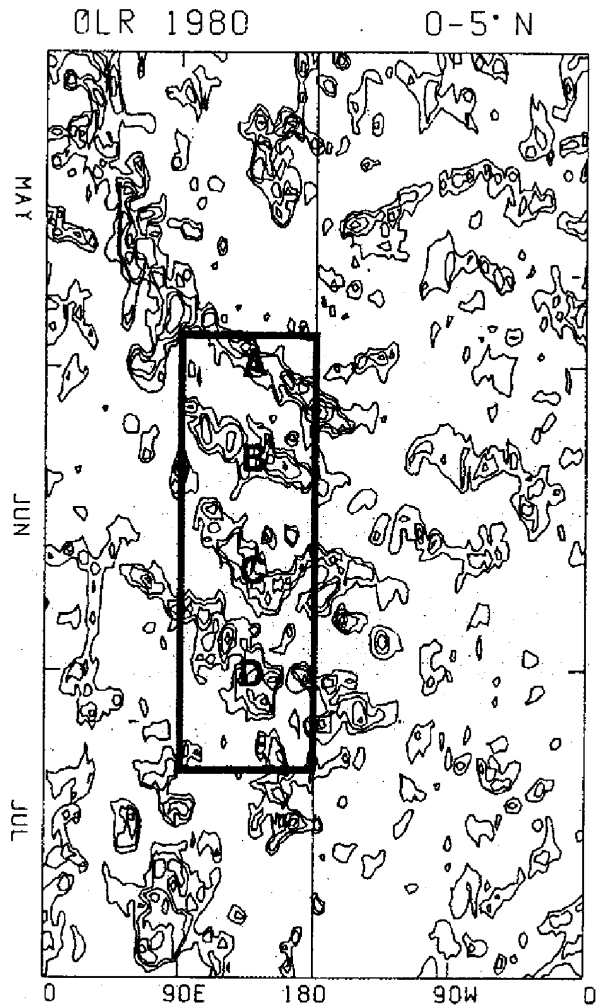
- Cloud systems
- Wave trains of cloud systems
  called convectively coupled waves
- Envelopes of convectively coupled waves
  called the Madden–Julian oscillation (MJO)

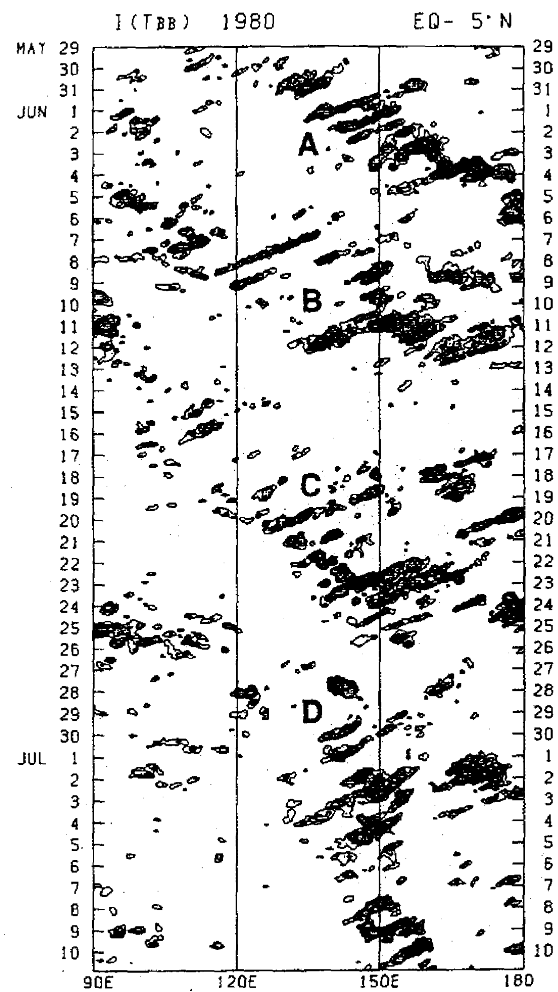
Global climate models (GCMs)

- use grid spacings of ≈ 100 km
- must represent clouds as a sub-grid scale process
- can hope to resolve convectively coupled waves
- but do not adequately capture convectively coupled waves or MJO

Multi-scale clouds and waves in the tropics



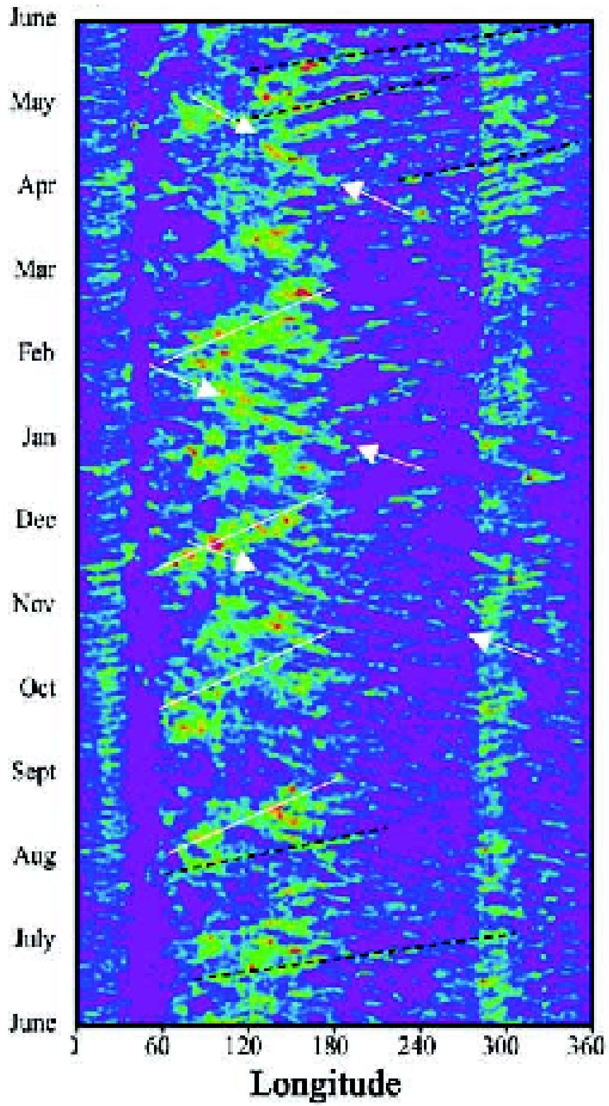




from Nakazawa (1988)

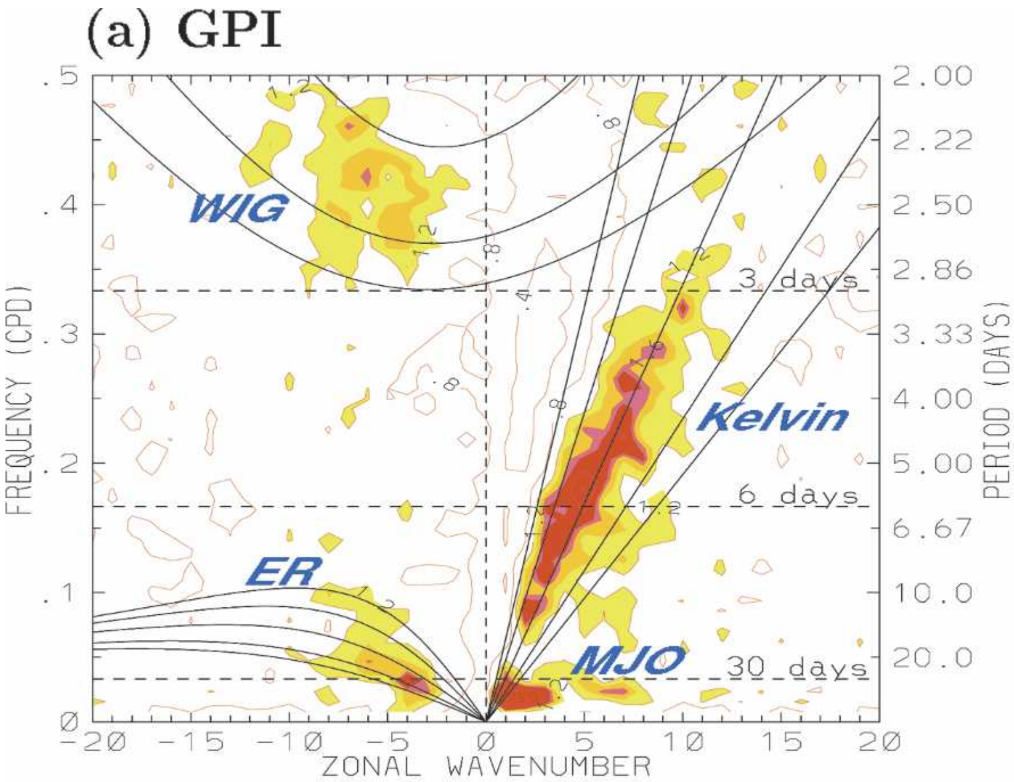
Multi-scale clouds and waves in the tropics

Precipitation



2000–2001 (from Zhang 2005)

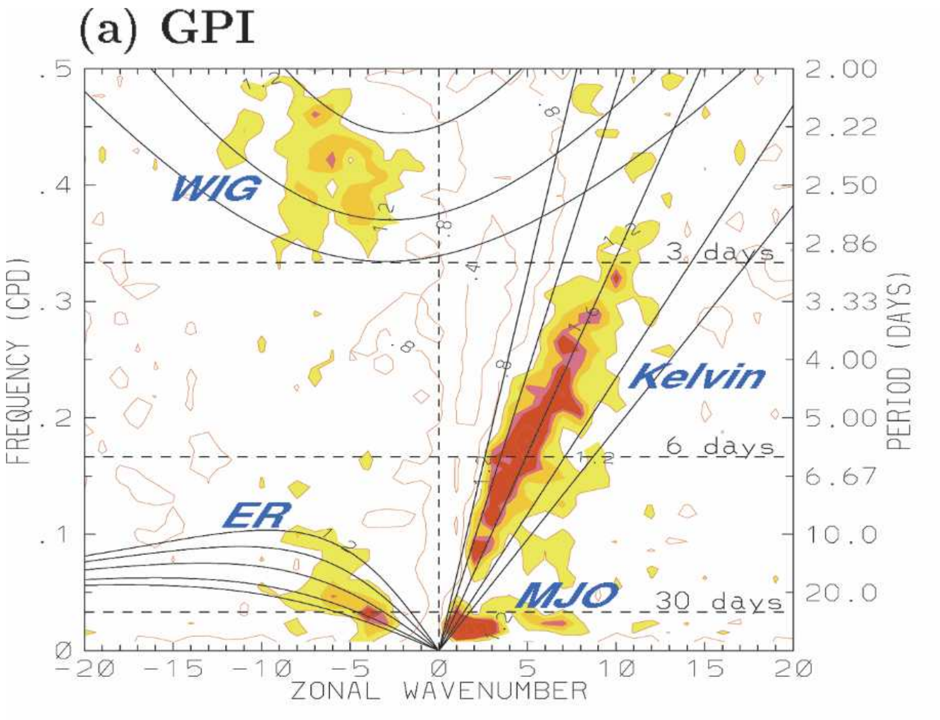
Spectral Power



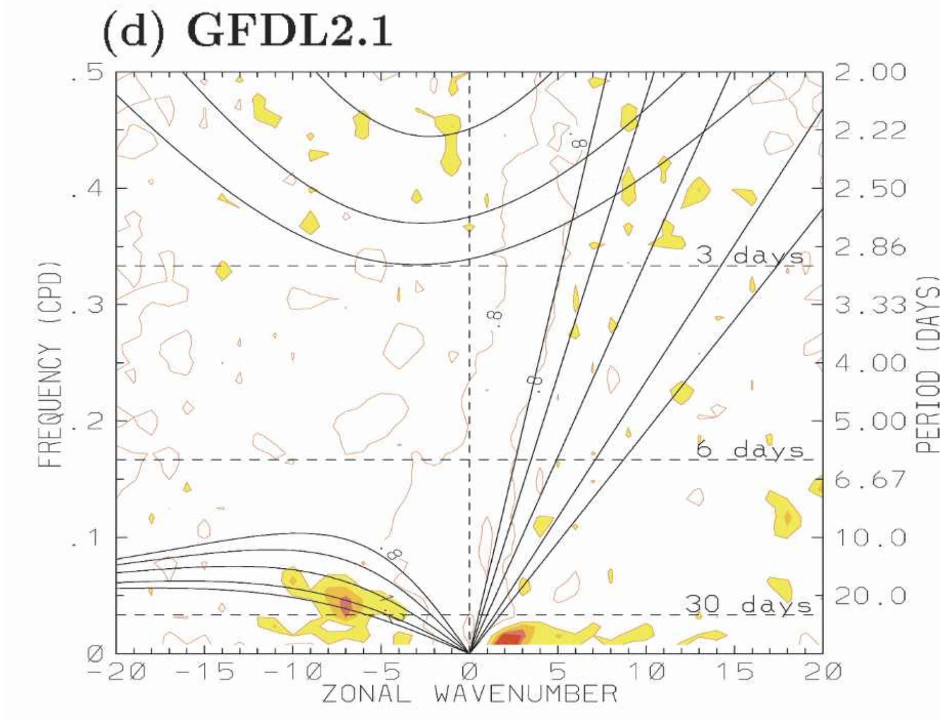
from Lin et al. 2006

Multi-scale clouds and waves in the tropics

Observations



Global Climate Model (GCM)



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Outline

1. Dry fluid dynamics of the tropical atmosphere

2. Observed features of MJO

3. A simple model for the MJO's "skeleton"

Dry fluid dynamics of the tropical atmosphere

$$\frac{Du}{Dt} - \beta y v = -\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + \beta y u = -\frac{\partial p}{\partial y}$$

$$0 = -\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_{ref}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{D\theta}{Dt} + w\frac{d\bar{\Theta}}{dz} = 0$$

$(u, v) =$ horizontal velocity

$w =$ vertical velocity

$p =$ pressure

$\theta =$ potential temp.

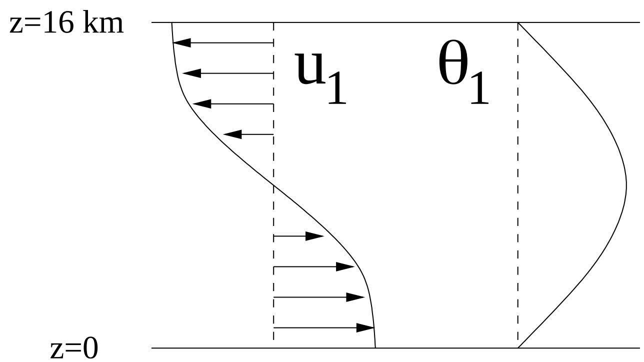
Vertical modes: Equatorial shallow water equations

Linear waves

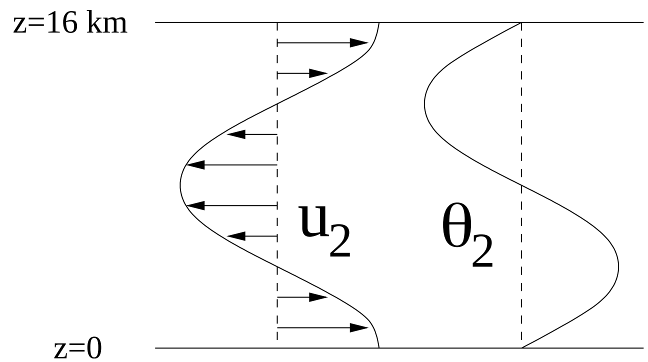
- Expand in vertical modes: $u(x,y,z,t) = \sum_j u_j(x,y,t)\cos jz$, etc.
- Equatorial shallow water system for each vertical mode j:

$$\frac{\partial u_j}{\partial t} - y v_j - \frac{\partial \theta_j}{\partial x} = 0$$
$$\frac{\partial v_j}{\partial t} + y u_j - \frac{\partial \theta_j}{\partial y} = 0$$
$$\frac{\partial \theta_j}{\partial t} - \frac{1}{j^2}\left(\frac{\partial u_j}{\partial x} + \frac{\partial v_j}{\partial y}\right) = 0$$

- Gravity wave speed $\propto 1/j$



50 m/s



25 m/s

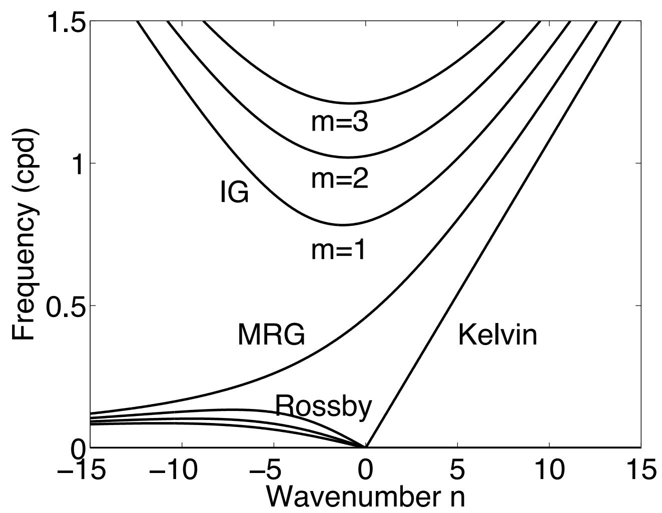
Meridional modes: Equatorially trapped waves

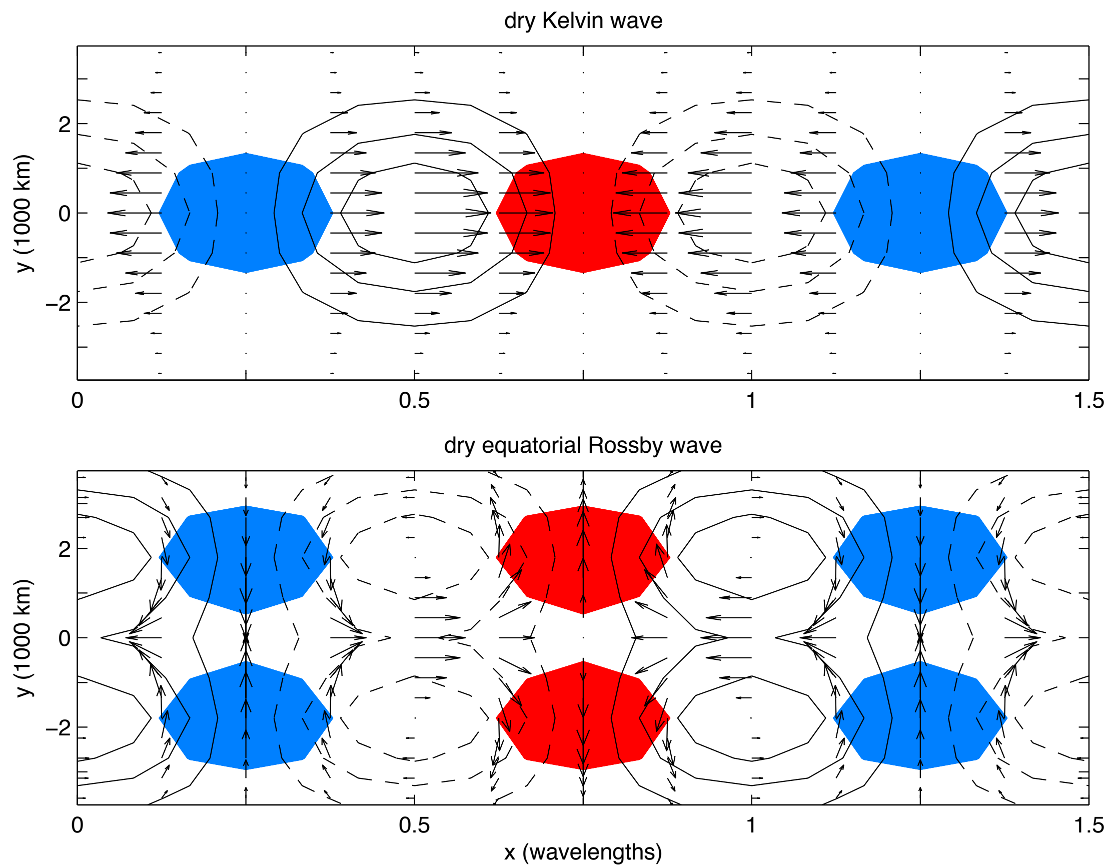
Expand in meridional modes: $u(x,y,t) = \sum_m u_m(x,t)\phi_m(y)$, etc.

$$\phi_0(y) \propto \exp\left(-\frac{y^2}{2}\right), \quad \phi_1(y) \propto y\exp\left(-\frac{y^2}{2}\right), \quad \phi_2(y) \propto (2y^2-1)\exp\left(-\frac{y^2}{2}\right)$$

Result: Zonally propagating waves $K(x,t)$, $R_m(x,t)$, etc.

Dispersion curves for equatorial shallow water eqns.





Dry fluid dynamics of the tropical atmosphere

Summary

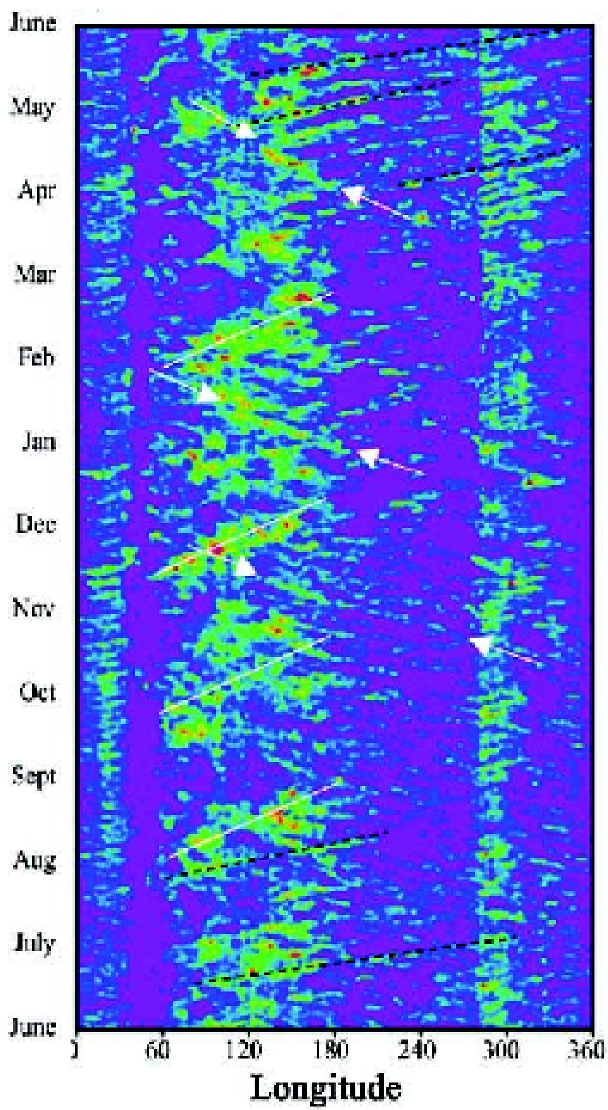
- Primitive equations $(x, y, z) \longrightarrow$ equatorial shallow water equations (x, y)
  - Expand in vertical modes $\cos(jz)$
- Equatorial shallow water equations $(x, y) \longrightarrow$ zonally propagating waves (x)
  - Expand in meridional modes $\phi_m(y)$

Result: Zonally propagating waves (Kelvin, Rossby, etc.)

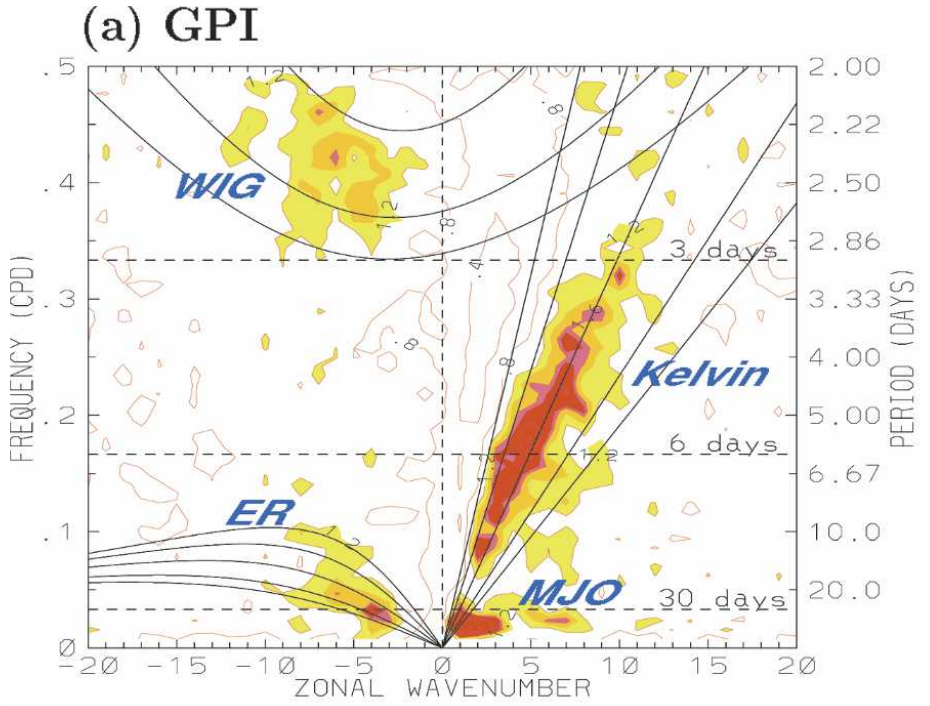
Observations of the MJO

Precipitation

2000–2001 (from Zhang 2005)



Spectral Power



from Lin et al. 2006

MJO: slow eastward propagation ≈ 5 m/s

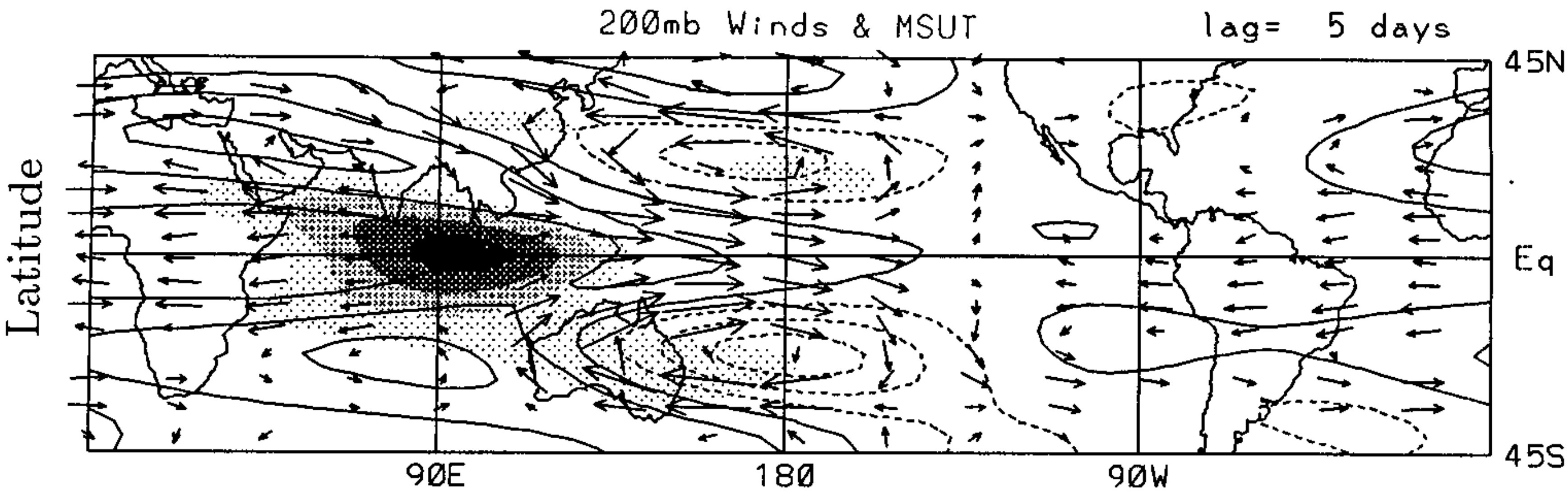
MJO: peculiar dispersion relation $\frac{d\omega}{dk} \approx 0$

MJO is envelope of smaller-scale convectively coupled waves

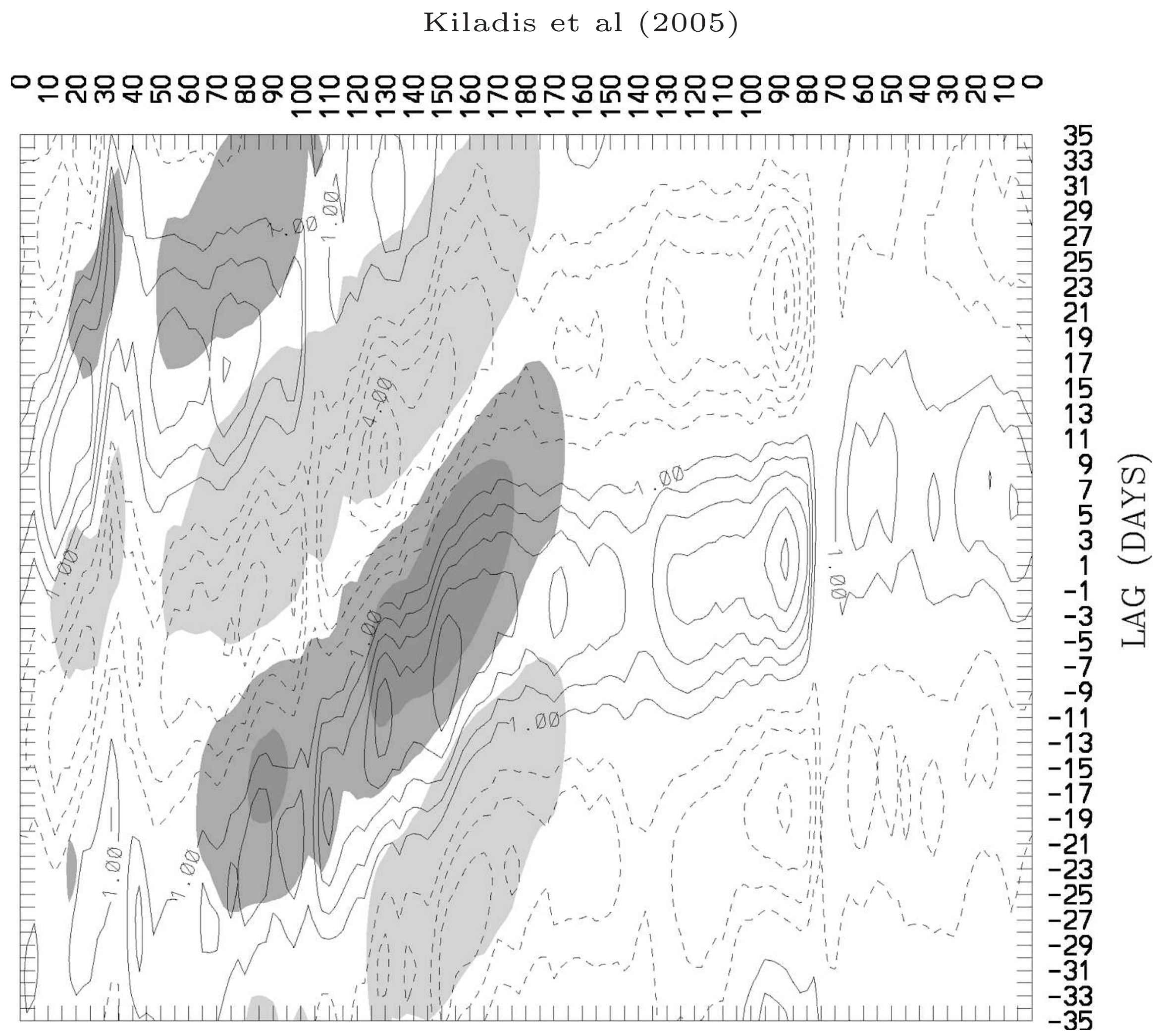
Horizontal structure of MJO

Quadrupole vortices:

Hendon and Salby (1994)



Moisture preconditioning in the MJO



Lower tropospheric moisture (contours) leads enhanced convection (dark shading)

Previous attempts at a theory for the MJO

MJO originally discovered in 1971

Previous theories emphasized planetary-scale instability mechanisms such as

- evaporation–wind feedback
- boundary layer frictional convective instability
- stochastic linearized convection
- radiation instability
- ...

But all these theories are at odds with observational record in various crucial ways

No theory for the MJO has yet been generally accepted

- "Search for the Holy Grail of tropical atmospheric dynamics"

The Skeleton of

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Fundamental mechanism proposed for MJO skeleton

Neutrally stable interactions between

1. planetary-scale, lower-tropospheric moisture
2. synoptic-scale, convectively-coupled-wave activity

- Tacit assumption: primary instabilities/damping occur on synoptic scales

- MJO "muscle" from other potential upscale transport effects from synoptic scales
  - convective momentum transports from synoptic-scale waves
  - variations in surface heat fluxes

Minimal dynamical model

$$u_t - yv = -p_x$$
$$yu = -p_y$$
$$0 = -p_z + \theta$$

$$u_x + v_y + w_z = 0$$

$$\theta_t + w = \bar{H}a$$
$$q_t - \tilde{Q}w = -\bar{H}a$$
$$a_t = \Gamma q(\bar{a} + a)$$

Momentum equations:

- Equatorial long-wave scaling
- Coriolis term: equatorial β-plane approx.
- Hydrostatic balance

Thermodynamic equations:

- q: lower tropospheric moisture
- a: amplitude of convective activity envelope

Key mechanism: positive q creates a tendency to enhance convective activity a

Minimal number of parameters: $\tilde{Q}, \Gamma, \bar{a}$

Minimal dynamical model

(vertical truncation)

$$
\begin{aligned}
u_t - yv - \theta_x &= 0 \\
yu - \theta_y &= 0 \\
\theta_t - u_x - v_y &= \bar{H} a \\
q_t + \tilde{Q}(u_x + v_y) &= -\bar{H} a \\
a_t &= \Gamma \bar{a} q
\end{aligned}
$$

- Truncate at first vertical baroclinic mode
- Matsuno–Gill-like model
  without dissipative mechanisms
  but with
  - lower tropospheric moisture, q
  - envelope of synoptic-scale wave activity, a, provides dynamic planetary-scale heating

Minimal dynamical model

(vertical and meridional truncation)

$$K_t + K_x = -\frac{1}{\sqrt{2}}\bar{H}A$$

$$R_t - \frac{1}{3}R_x = -\frac{2\sqrt{2}}{3}\bar{H}A$$

$$Q_t + \frac{1}{\sqrt{2}}\tilde{Q}K_x - \frac{1}{6\sqrt{2}}\tilde{Q}R_x = \left(-1 + \frac{1}{6}\tilde{Q}\right)\bar{H}A$$

$$A_t = \Gamma\bar{a}Q$$

Meridional structures:

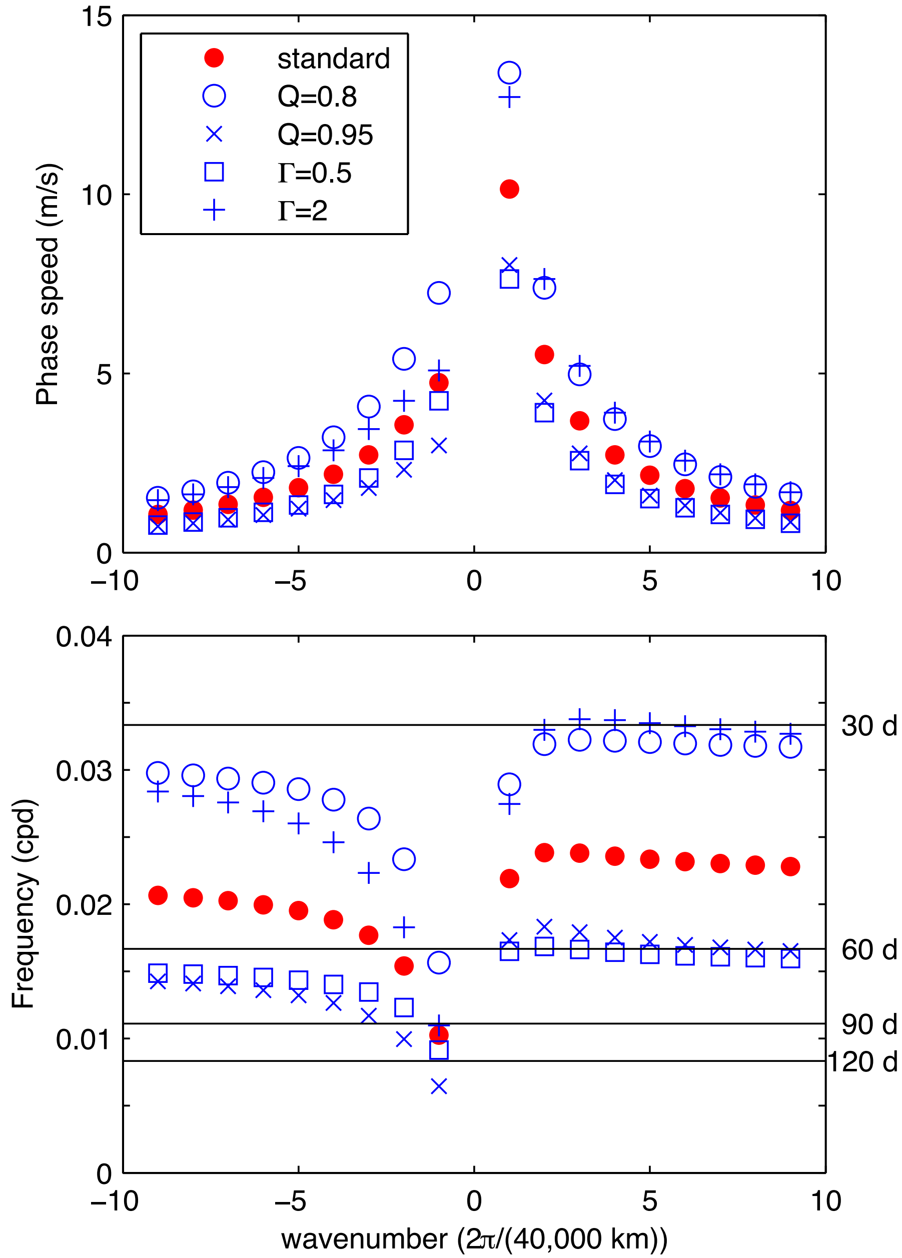
K: Kelvin wave

R: first symmetric equatorial Rossby wave

Q: $\exp(-y^2/2)$

A: $\exp(-y^2/2)$

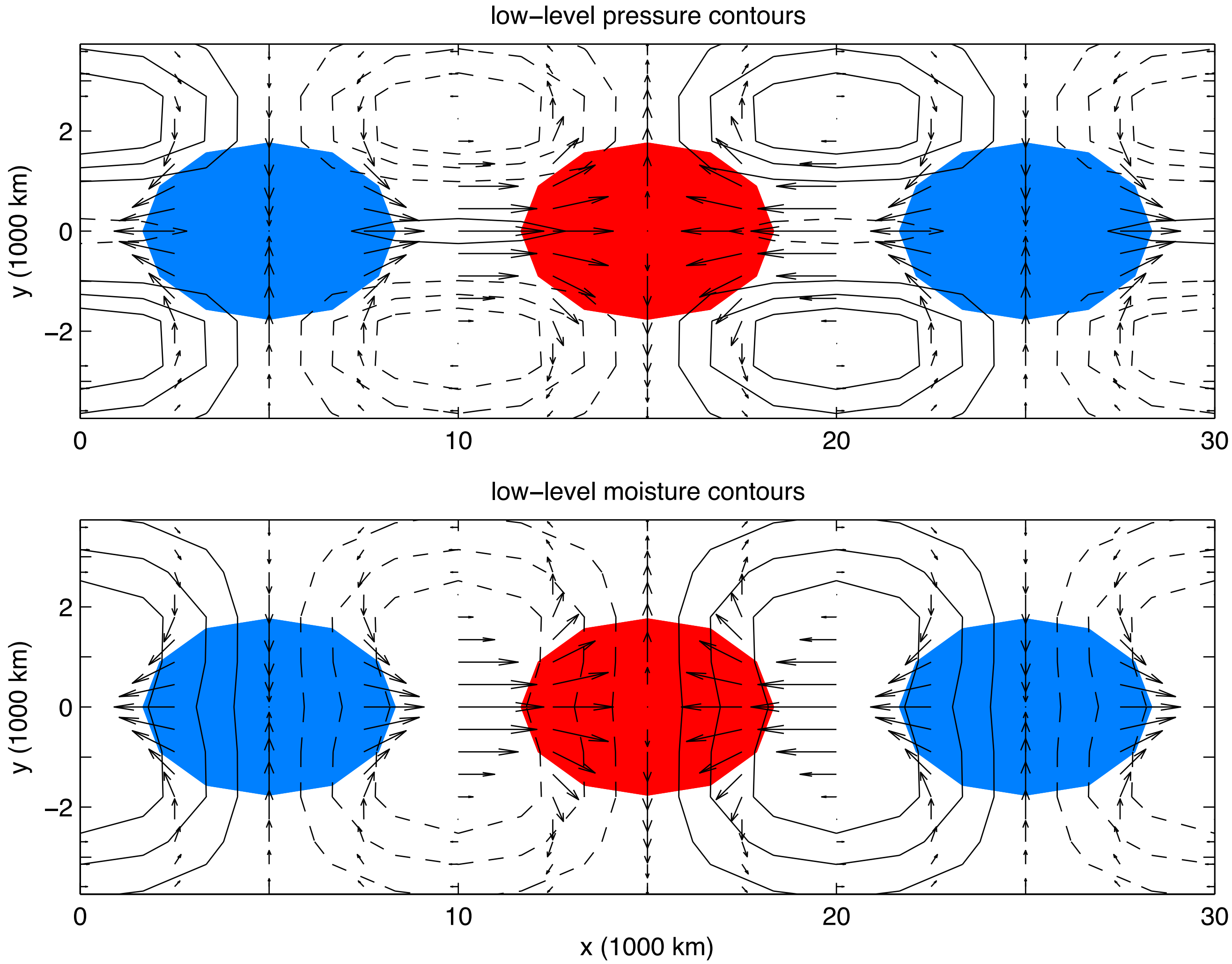
Phase speed and oscillation frequency



- Phase speeds of ≈ 5 m/s
- Results robust over parameter space

- Eastward MJO branch: $\frac{d\omega}{dk} \approx 0$ on *intraseasonal* time scales
- Westward branch: *seasonal* time scales for wavenumbers 1 and 2

Physical structure of MJO skeleton



suppressed convection ($A < 0$)

enhanced convection ($A > 0$)

- horizontal quadrupole vortices
- moisture leads convection
- Kelvin wave structure on equator
- off-equatorial quadrupole Rossby gyres

Formula for MJO frequency

Simplified case: 1D dynamics above the equator

- No rotation $\Rightarrow$ Perfect east–west symmetry
- Linear system in 4 variables (u, θ, q, a) + Perfect east–west symmetry $\Rightarrow$

Exact solution: $$2\omega^2 = \Gamma\bar{R} + k^2 \pm \sqrt{(\Gamma\bar{R} + k^2)^2 - 4\Gamma\bar{R}k^2(1 - \tilde{Q})}$$

Approx. solution: $$\omega \approx \sqrt{\Gamma\bar{R}(1 - \tilde{Q})}$$

- *Model recovers peculiar dispersion relation* $d\omega/dk \approx 0$
- *Simple formula for MJO frequency in terms of model parameters*

Summary

- New minimal dynamical model for the MJO
- Robustly recovers the MJO's fundamental features (i.e., its "skeleton") on intraseasonal/planetary scales:
  - slow phase speed of ≈ 5 m/s
  - peculiar dispersion relation of $d\omega/dk \approx 0$
  - horizontal quadrupole vortex structure
- Simple formula for MJO oscillation frequency: $\omega \approx \sqrt{\Gamma \bar{R}(1-\tilde{Q})}$
- Explanation of preferred eastward propagation of intraseasonal variability
- Neutrally stable model on planetary/intraseasonal scales
  - Tacit assumption: primary instabilities on synoptic scales
- "Muscle" of MJO provided by other upscale transports from synoptic scales

Majda and Stechmann (2009) *Proc. Natl. Acad. Sci.*

Precipitation Fronts in the Equatorial Atmosphere

Samuel N. Stechmann (NYU), Andrew J. Majda (NYU), and
Boualem Khouider (UVic)

Department of Mathematics and

Center for Atmosphere Ocean Science

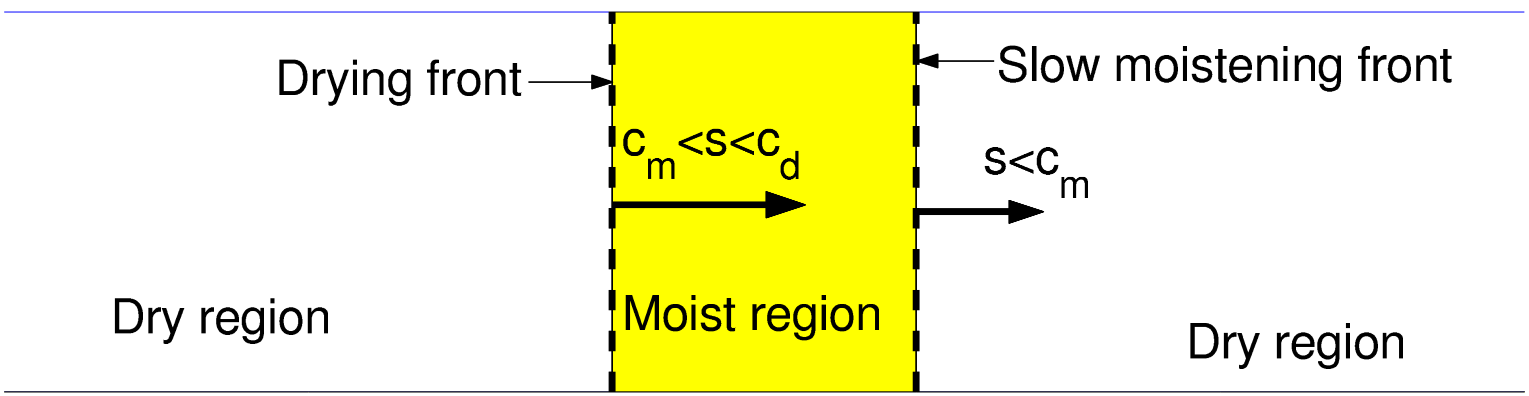
Courant Institute

New York University

May 15, 2006

Dying Rainy Region

From Khouider and Majda (2005) in *Theoretical and Computational Fluid Dynamics*

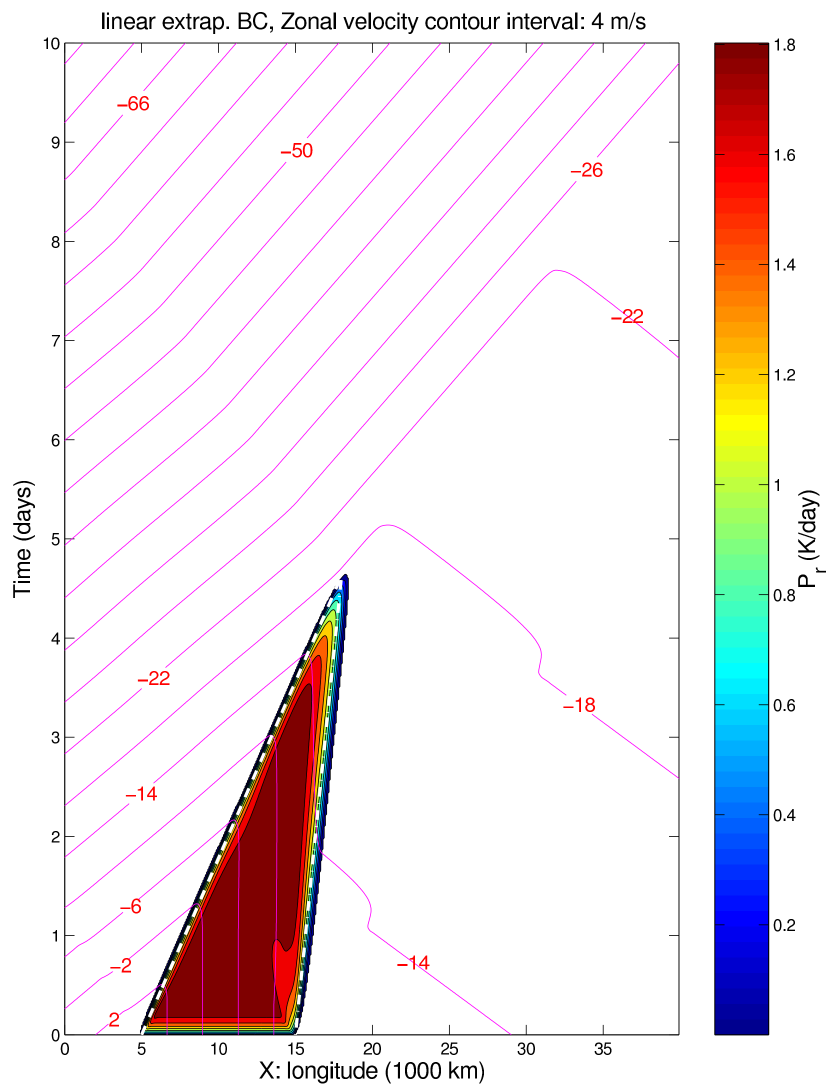


Moist region surrounded by two dry regions

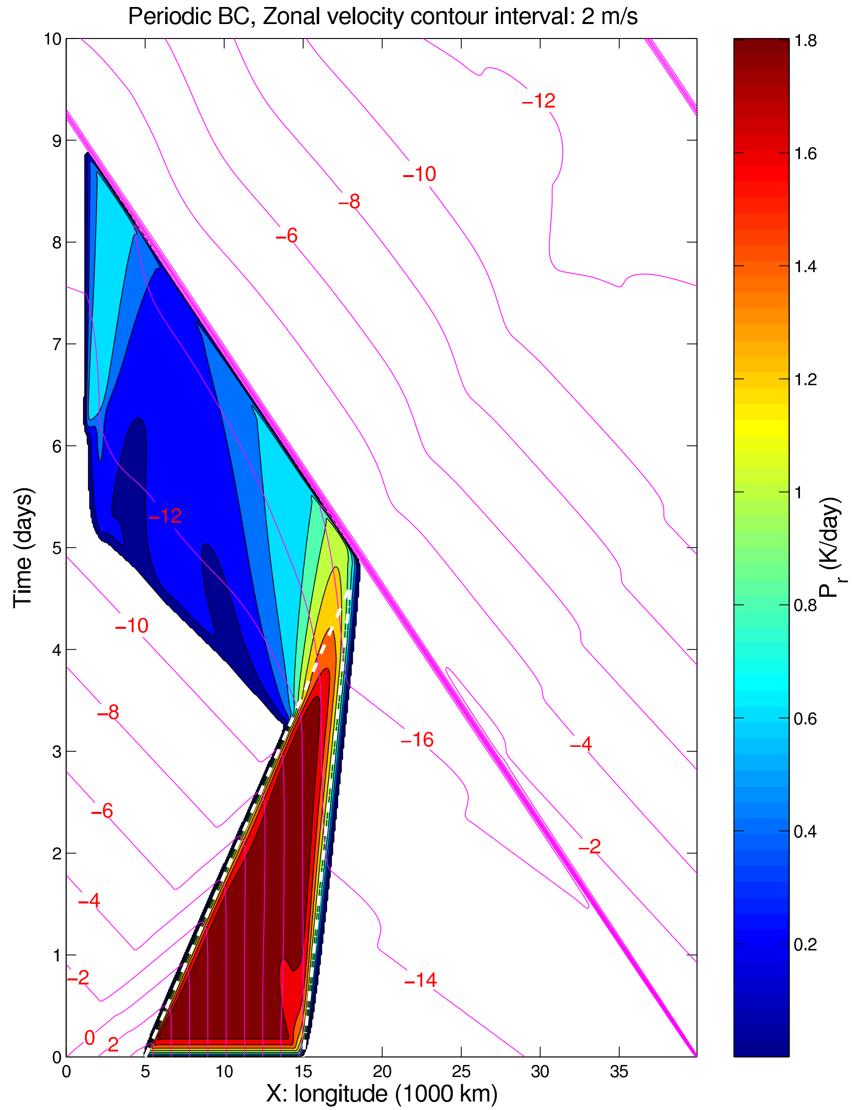
Moist region propagating eastward and shrinking

Left and right boundaries move at different speeds

Boundary Conditions: Linear Extrapolation

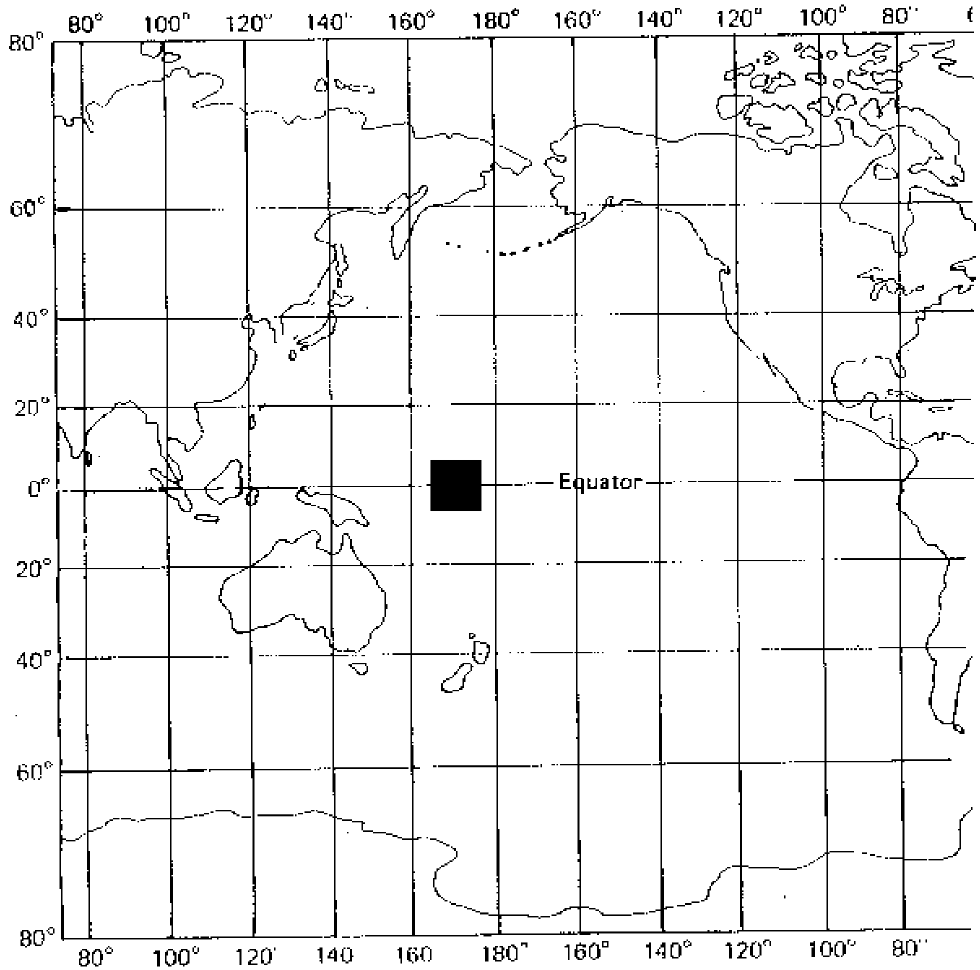


Periodic Domain



Typical Scales for Large Scale Dynamics in Tropics

c	50 m/s	Velocity scale
L	1500 km	Length scale
T	8 hours	Time scale
H	16 km	Tropopause height
W_0	0.2 m/s	Vertical velocity scale



1500 km × 1500 km black square shown

Motivation

General circulation models (GCMs) use parameterizations for interactions between moisture and large scale dynamics

What kinds of waves with moisture does the parameterization support? Are they physically meaningful?

Use a simplified tropical climate model to investigate: shallow water equations + water vapor equation, with a parameterization for convection (i.e., storms, rain)

Outline

- 1st baroclinic mode equations with moisture
- Discontinuous fronts for vanishing convective adjustment time ($\tau_c \to 0$) (Frierson, Majda, and Pauluis 2004)
- Front structure for nonzero convective adjustment time ($\tau_c \neq 0$)

1st Baroclinic Mode Equations

- Start with hydrostatic Boussinesq equations
- Choose vertical structure for velocity, temperature (i.e., choose the 1st baroclinic mode)
- Project hydrostatic Boussinesq equations onto this vertical structure
- Result is the 1st Baroclinic Mode Equations

Hydrostatic Boussinesq Equations

$$\frac{D\mathbf{U}}{Dt} + y\mathbf{U}^{\perp} + \nabla\Phi = S_{\mathbf{U}} \quad \text{Conservation of momentum}$$

$$\frac{D\Theta}{Dt} + W\frac{d\bar{\Theta}}{dz} = S_{\Theta} \quad \text{Conservation of energy}$$

$$\frac{\partial\Phi}{\partial z} = \Theta \quad \text{Hydrostatic balance}$$

$$\nabla\cdot\mathbf{U} + \frac{\partial W}{\partial z} = 0 \quad \text{Incompressibility constraint}$$

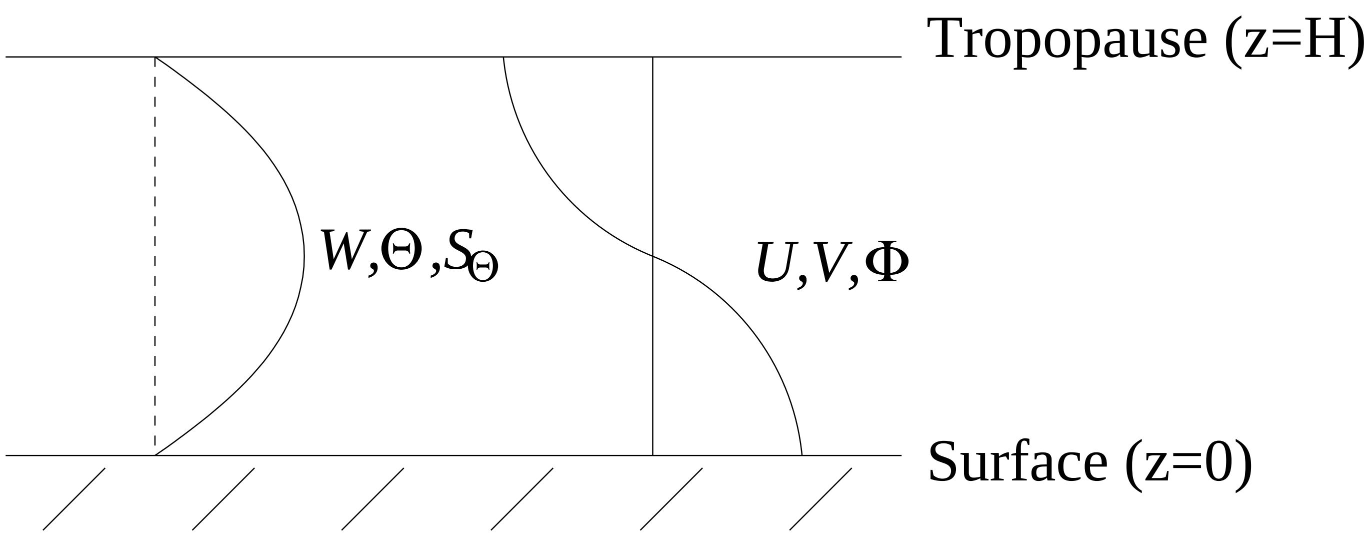
$$\mathbf{U} = (U, V)$$

$$\nabla = (\partial_x, \partial_y)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla + W\frac{\partial}{\partial z}$$

$$W\Big|_{z=0,H} = 0$$

Vertical Structure



Θ : Potential temperature

S_Θ : Source of potential temperature (precipitation)

Φ : Pressure

Vertical Projection

$$\begin{pmatrix} W \\ \Theta \\ S_\Theta \end{pmatrix}(x,y,z,t) = \begin{pmatrix} w \\ \theta \\ P \end{pmatrix}(x,y,t)\sin\left(\frac{\pi z}{H}\right)$$

$$\begin{pmatrix} \mathbf{U} \\ \Phi \end{pmatrix}(x,y,z,t) = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}(x,y,t)\cos\left(\frac{\pi z}{H}\right)$$

Project using the inner product

$$\langle F, G\rangle = \frac{1}{H}\int_0^H F(z)G(z)\,dz$$

Result of projection is ...

1st Baroclinic Mode Equations with Moisture

1st Baroclinic:
$$\boxed{\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + y\mathbf{u}^{\perp} - \nabla\theta &= 0 \\ \frac{\partial \theta}{\partial t} - \nabla\cdot\mathbf{u} &= P \end{aligned}}$$

Moisture:
$$\boxed{\frac{\partial q}{\partial t} + \bar{Q}\nabla\cdot\mathbf{u} = -P}$$

$$\boxed{\begin{aligned} w &= -\nabla\cdot\mathbf{u} \\ P &: \text{Precipitation} \end{aligned}}$$

Those are linear shallow water equations with rotation, coupled to moisture through the nonlinear precipitation P

See Frierson et al (2004) or Khouider and Majda (2005) for derivation of moisture equation

Precipitation Parameterization

Betts–Miller Scheme

$$P = \frac{1}{\tau_c}(q - \tilde{q}(\theta))^+$$

τ_c : convective adjustment time

$\tilde{q}$: moisture saturation profile

Choose linear form for moisture saturation $\tilde{q}$:

$$\tilde{q}(\theta) = \hat{q} + \alpha\theta$$

Conservation Principles

Equivalent Potential Temperature $\theta_e = q + \theta$

$$\frac{\partial \theta_e}{\partial t} = (1 - \bar{Q})\nabla \cdot \mathbf{u}$$

$Z = q + \bar{Q}\theta$

$$\frac{\partial Z}{\partial t} = -(1 - \bar{Q})P \leq 0$$

Total Energy Density

$$\epsilon = \frac{1}{2}(|\mathbf{u}|^2 + \theta^2) + \frac{1}{2}\frac{(q + \bar{Q}\theta)^2}{(1 - \bar{Q})(\alpha + \bar{Q})}$$

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$$\int \epsilon(t)dxdy = \int \epsilon(0)dxdy - \int_0^t \int \frac{q - \alpha\theta}{\alpha + \bar{Q}} P dxdydt$$
$$\leq \int \epsilon(0)dxdy$$

Energy estimate is independent of relaxation time τ_c

Gradient Equations

$$
\begin{aligned}
\frac{\partial \nabla u}{\partial t} &= y\nabla v + v\hat{\mathbf{y}} + \frac{\partial \nabla \theta}{\partial x} \\
\frac{\partial \nabla v}{\partial t} &= -y\nabla u - u\hat{\mathbf{y}} + \frac{\partial \nabla \theta}{\partial y} \\
\frac{\partial \nabla \theta}{\partial t} &= \nabla(\nabla \cdot \mathbf{u}) + \nabla P \\
\frac{\partial \nabla q}{\partial t} &= -\bar{Q}\nabla(\nabla \cdot \mathbf{u}) - \nabla P
\end{aligned}
$$

Use to show gradient energy estimate

Use to find travelling wave solutions for $\tau_c \neq 0$

Gradient Energy Estimate

$$\epsilon_{grad} = \frac{1}{2}\left(|\nabla \mathbf{u}|^2 + |\nabla\theta|^2 + \frac{(\nabla q + \bar{Q}\nabla\theta)^2}{(1-\bar{Q})(\alpha+\bar{Q})}\right)$$

$$\begin{aligned}\int \epsilon_{grad}(t)dxdy &= \int \epsilon_{grad}(0)dxdy + \int_0^t \int \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) dxdydt \\ &\quad - \int_0^t \int \frac{|\nabla(q-\alpha\theta)|^2}{\alpha+\bar{Q}} P' dxdydt \\ &\leq \int \epsilon_{grad}(0)dxdy + \int_0^t \int \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) dxdydt\end{aligned}$$

For 1D fronts, u, θ, and q are continuous if continuous initially by Sobolev's Lemma

This result is independent of relaxation time τ_c

Simplified Situation for Analytical Study

One dimension: $u = u(x,t)$, $v = 0$

No rotation (i.e., motion on equator, $y = 0$)

$$\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial \theta}{\partial x} &= 0 \\
\frac{\partial \theta}{\partial t} - \frac{\partial u}{\partial x} &= \frac{1}{\tau_c}(q - \tilde{q})^+ \\
\frac{\partial q}{\partial t} + \bar{Q}\frac{\partial u}{\partial x} &= -\frac{1}{\tau_c}(q - \tilde{q})^+
\end{aligned}$$

Linear shallow water equations in 1D

With a linear moisture equation

Coupled by a nonlinear precipitation term

Ignore all other sources and sinks (evaporation, radiation, etc.)

Dry Wave Speed

Within dry region, $P = 0$ and

$$\frac{\partial u}{\partial t} - \frac{\partial \theta}{\partial x} = 0$$
$$\frac{\partial \theta}{\partial t} - \frac{\partial u}{\partial x} = 0$$

Dry wave speed $c_d = 1$ (50 m/s)

Moist Wave Speed

Within moist region, formally for $\tau_c \to 0$,

$$q = \tilde{q}(\theta) = \hat{q} + \alpha\theta$$

and

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial \theta}{\partial x} &= 0 \\ \frac{\partial \theta}{\partial t} - \frac{1-\bar{Q}}{1+\alpha}\frac{\partial u}{\partial x} &= 0 \end{aligned}$$

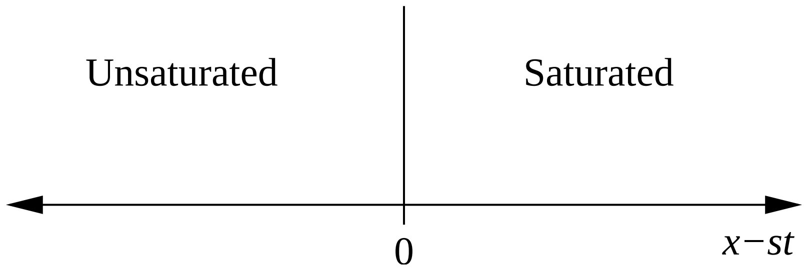
Moist wave speed $c_m = \sqrt{\frac{1-\bar{Q}}{1+\alpha}} < 1$

Dry wave speed $c_d = 1$

Outline

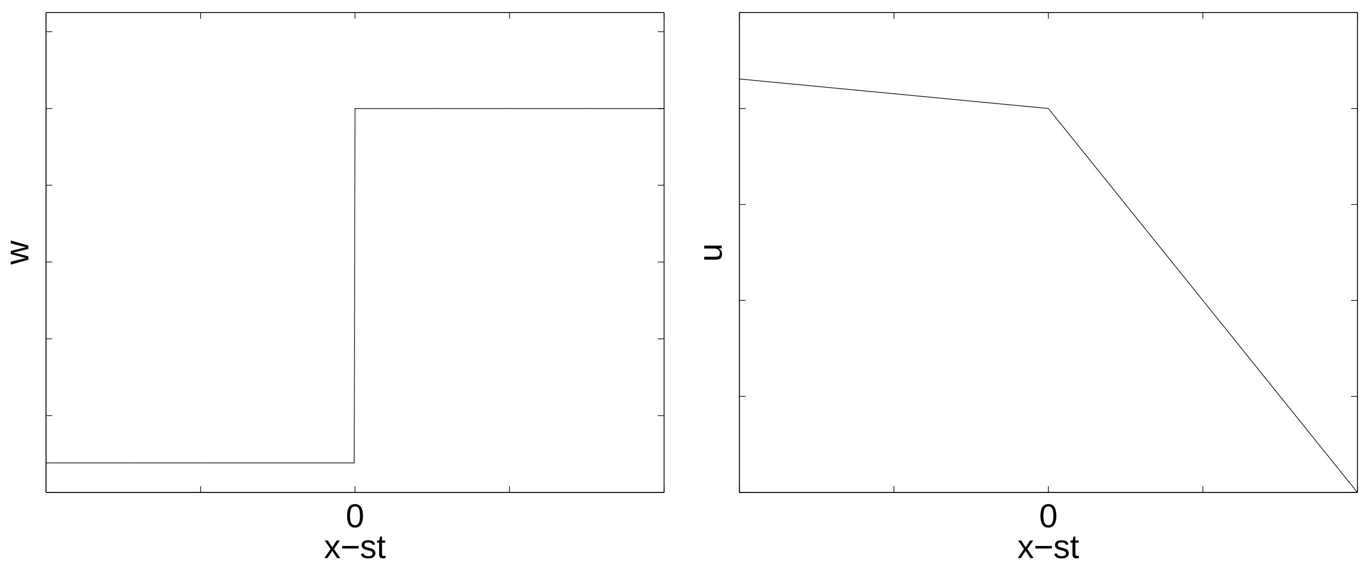
- 1st baroclinic mode equations with moisture
- Discontinuous fronts for vanishing convective adjustment time ($\tau_c \to 0$) (Frierson, Majda, and Pauluis 2004)
- Front structure for nonzero convective adjustment time ($\tau_c \neq 0$)

Free Boundary Problem for Limit $\tau_c \to 0$



Gradient energy estimate $\Rightarrow$

discontinuous w, θ_x, q_x but piecewise linear u, θ, q:



Jump Conditions

Solve the jump conditions for the gradient equations to get the front velocity s:

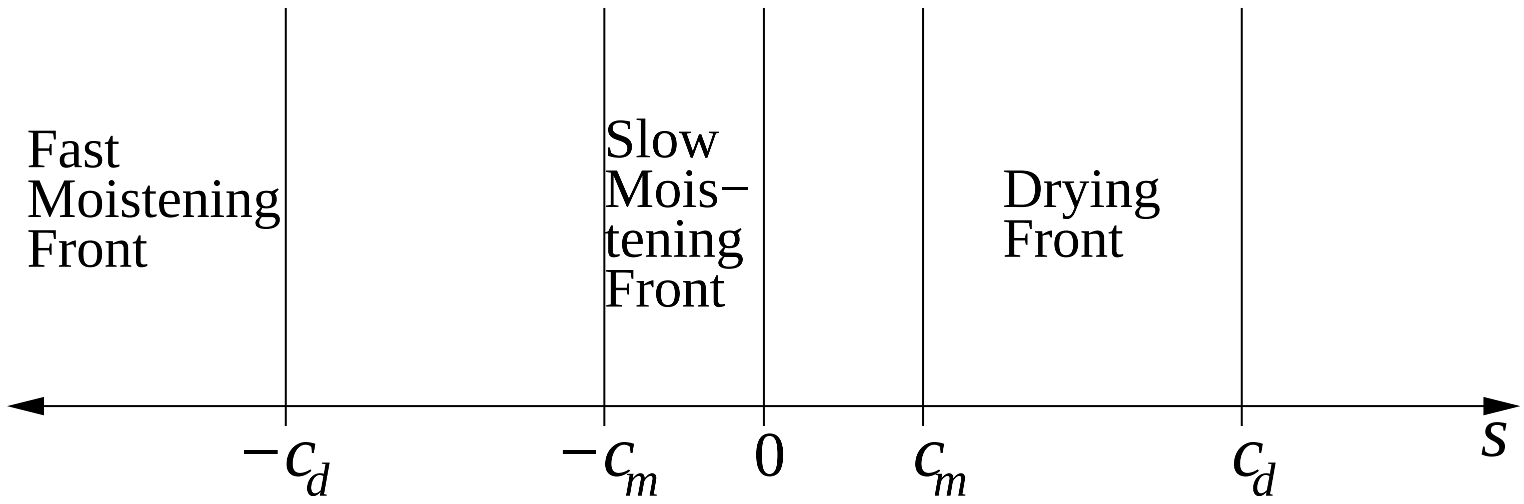
$$-s[w] + [\theta_x] = 0$$
$$-s[\theta_x] + [w] = [P]$$
$$-s[q_x] - \bar{Q}[w] = -[P],$$

where $[w] = w_+ - w_-$ is the jump in w across the free boundary.

The solution for s is real only for certain cases ...

3 Branches of Precipitation Fronts

Frierson, Majda, and Pauluis (2004) find the real values of s to give



$15 \text{ m/s} < s < 50 \text{ m/s}$	Drying Front
$-15 \text{ m/s} < s < 0 \text{ m/s}$	Slow Moistening Front
$s < -50 \text{ m/s}$	Fast Moistening Front

Analogy between Precipitation Fronts and Reacting Gas Fronts

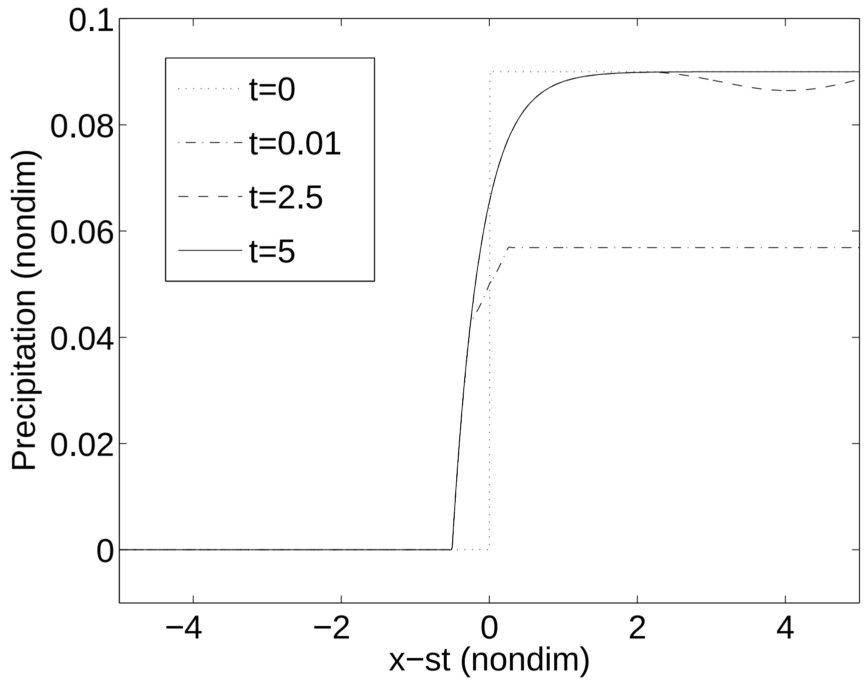
Precipitation front	*Slow moistening front*	*Fast moistening front*	*Drying front*
Reacting gas front	*Flame front*	*Weak detonation*	*Strong detonation*
Lax's shock inequalities	Violated	Violated	Satisfied
Front speed	Subsonic	Supersonic	Subsonic from one side, supersonic from the other
Realizability of precipitation front	Always	Always	Always
Realizability of reacting gas front	Unique speed arises as nonlinear eigenvalue problem	Only for special coefficient values of viscosity and heat conduction	Always

Lax Stability Criterion and Numerical Solutions

Only drying fronts satisfy Lax's stability criterion: $c_m < s < c_d$

But all 3 types of fronts realizable numerically in robust fashion?!

Precipitation for fast moistening front:



Similar results for slow moistening front and drying front

Outline

- 1st baroclinic mode equations with moisture
- Discontinuous fronts for vanishing convective adjustment time ($\tau_c \to 0$) (Frierson, Majda, and Pauluis 2004)
- Front structure for nonzero convective adjustment time ($\tau_c \neq 0$)

Travelling Wave Solution Form

$$\tilde{x} = x - st$$

In dry region, exact linear structure:

$u(\tilde{x}, t) = -w_- \tilde{x} + (\theta_{x-} - sw_-)t$

In moist region, linear structure + correction:
$u(\tilde{x}, t) = -w_+ \tilde{x} + (\theta_{x-} - sw_-)t + \frac{\tau_c}{a}[w](1 - e^{-a\tilde{x}/\tau_c})$

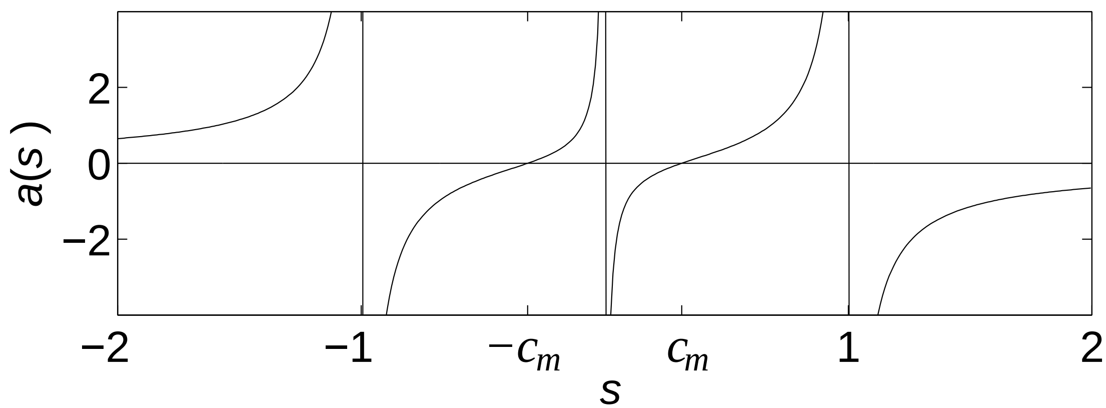
Similar form for θ and q

What is a above?

3 Branches of Precipitation Fronts

$$w = w_+ - [w]e^{-a\tilde{x}/\tau_c} \quad \text{in moist region,} \qquad a = -\frac{1+\alpha}{s}\frac{c_m^2 - s^2}{1-s^2}$$

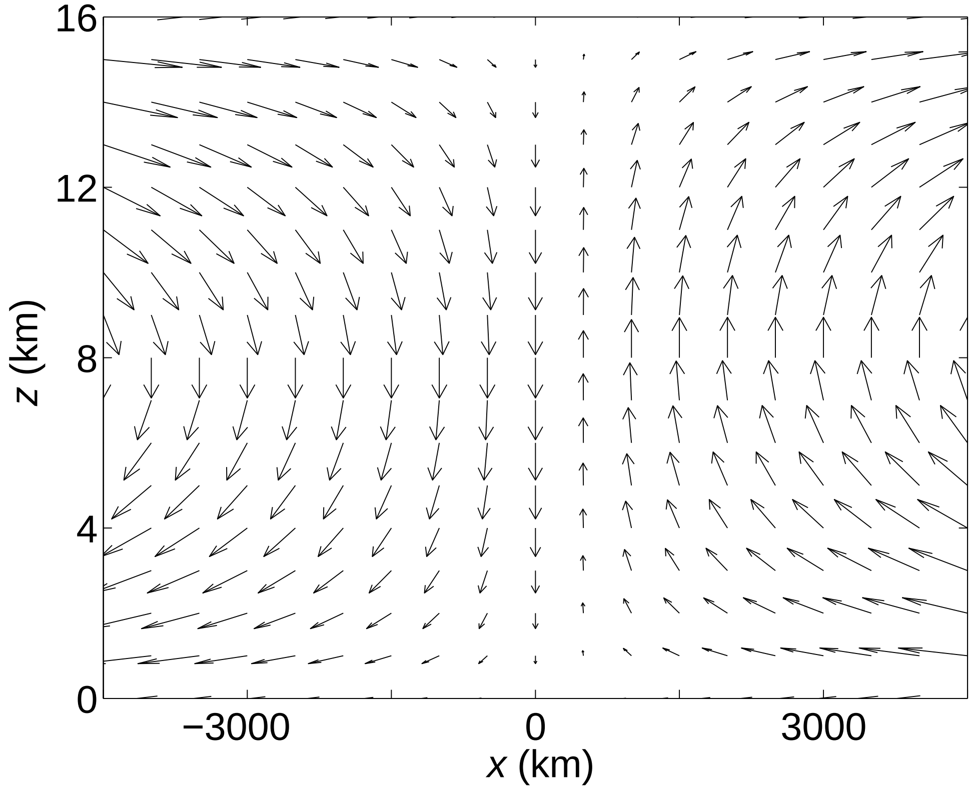
Qualitatively, $a(s)$ looks like



Require exponential decay not growth: $a > 0$ for

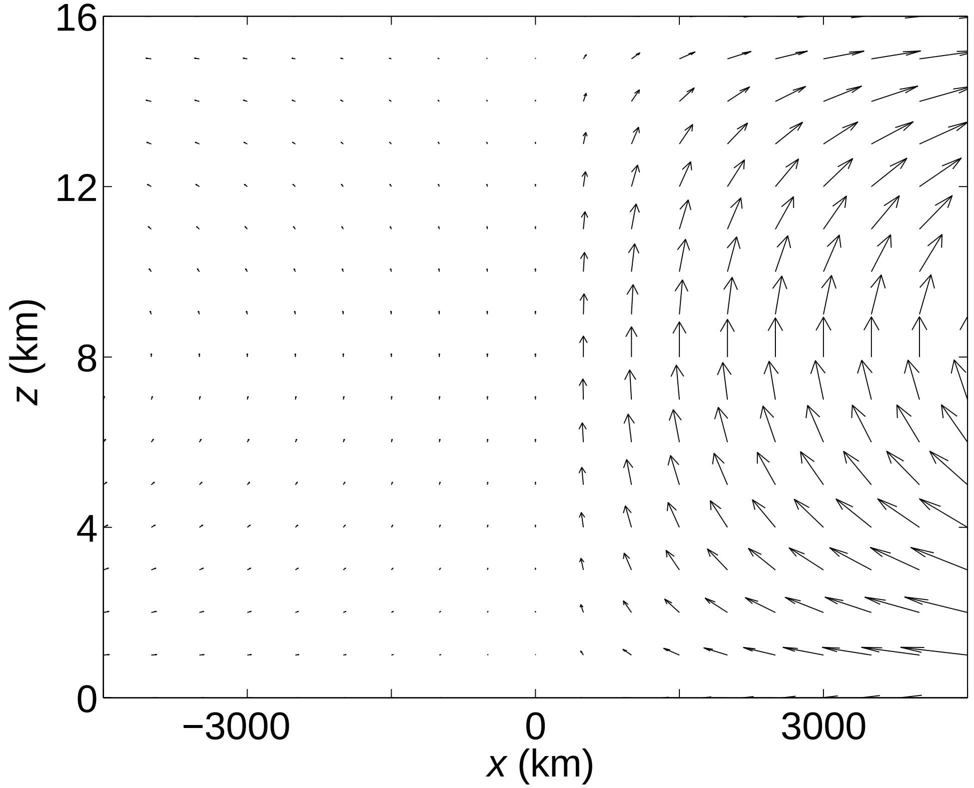
$c_m < s < c_d$	Drying Front
$-c_m < s < 0$	Slow Moistening Front
$s < -c_d$	Fast Moistening Front

Drying Front:
A Sample Velocity Field



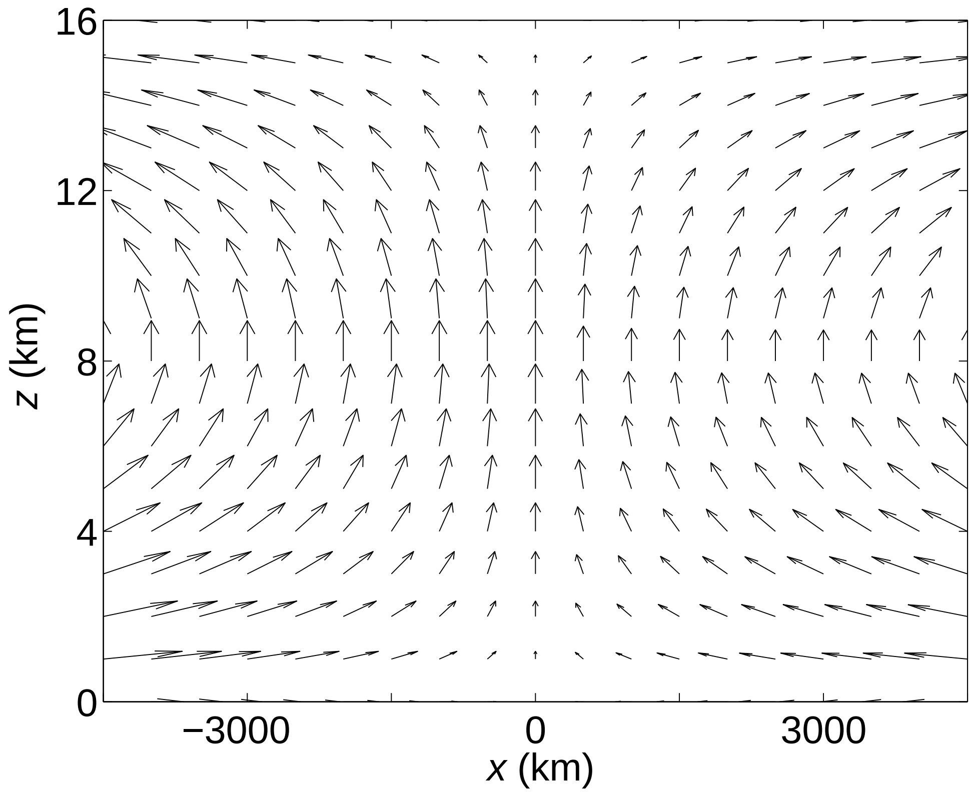
$$c_m < s < c_d, \quad w_- < 0 < w_+$$

Slow Moistening Front



$$-c_m < s < 0, \quad 0 < w_- < c_m^2 w_+$$

Fast Moistening Front



$$s < -c_d, \quad 0 < w_+ < w_-$$

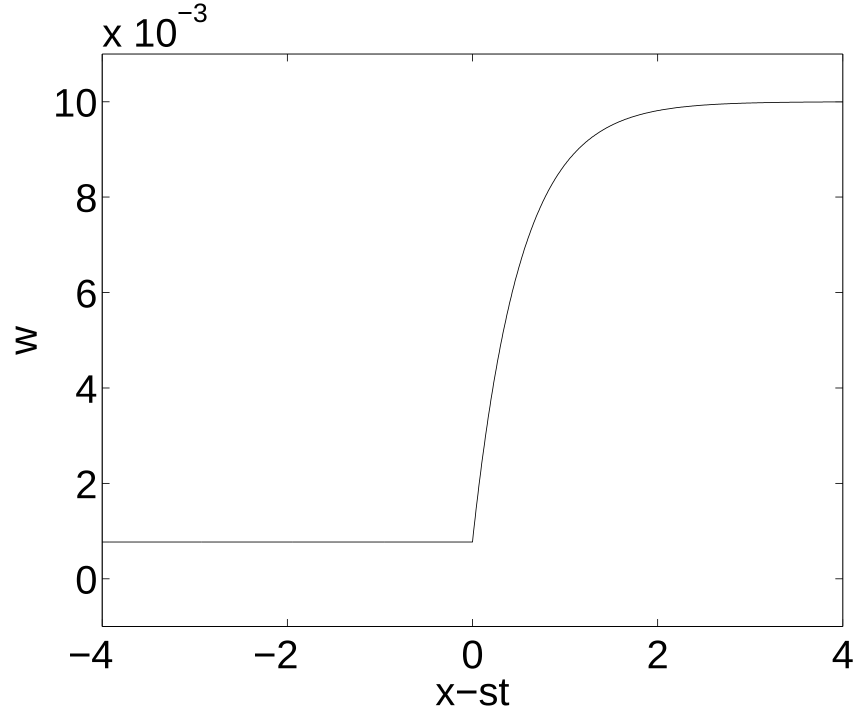
Summary

- Studied a convective parameterization in a simplified setting
- 3 branches of precipitation fronts arise: drying fronts, slow moistening fronts, fast moistening fronts
- Even though Lax stability criterion is met only by drying fronts, all 3 branches are robustly realizable
- Fronts have exponential structure for finite convective adjustment time ($\tau_c \neq 0$) and become discontinuous for instantaneous convective adjustment time ($\tau_c \to 0$)
- Slope of front depends on front speed s and convective adjustment time τ_c

Precipitation Fronts for $\tau_c \neq 0$

Look for solutions of form $w = w\left(\frac{x-st}{\tau_c}\right)$

Result:



The Existence and Uniqueness of Weak Solutions for Precipitation Fronts: A novel Hyperbolic Free Boundary Value Problem in Several Space Variables

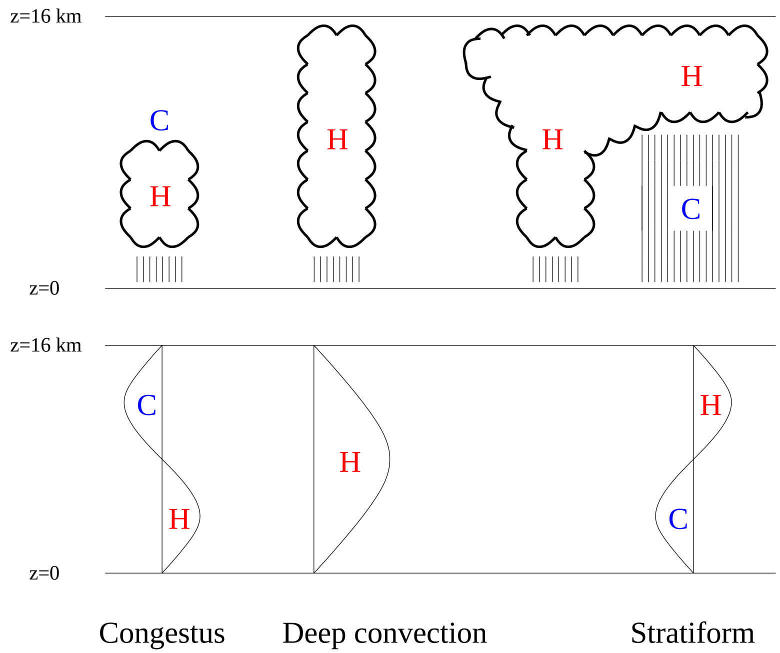
Andrew J. Majda and Takis Souganidis

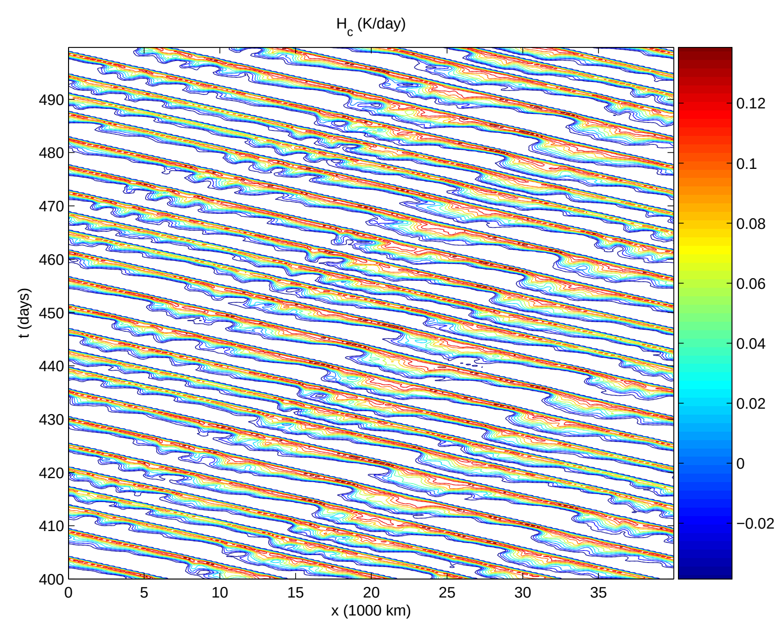
To appear in Communication Pure Applied Mathematics, 2010

Besides Existence and Uniqueness of Weak Solves infinite relaxation limit, also established L^2 - contraction in suitable moist energy metric.

Part II: Models with 2 Vertical Modes

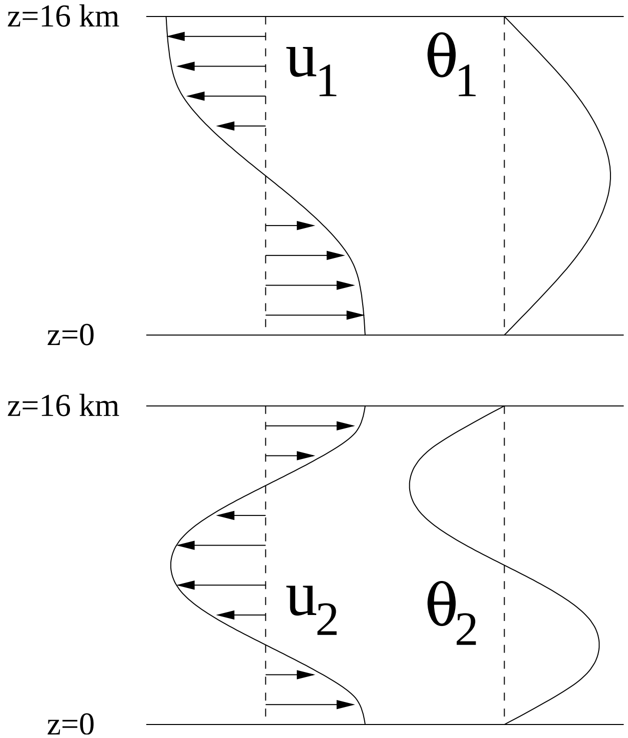
- Convective parameterizations based on only 1 vertical mode (and on only 1 cloud type) do not adequately represent CCWs
- Observations suggest that two other cloud types are also important: stratiform and congestus
- Stratiform, congestus clouds project strongly onto the 2nd baroclinic mode
- Khouider and Majda (2006) designed a model for CCWs that includes these effects (called "the multicloud model")





Equations of the multicloud model

Two linear shallow water systems, coupled through nonlinear source terms:



$$\left\{\begin{aligned} \frac{\partial u_1}{\partial t} - \frac{\partial \theta_1}{\partial x} &= -\frac{1}{\tau_u} u_1 \\ \frac{\partial \theta_1}{\partial t} - \frac{\partial u_1}{\partial x} &= H_d - R_1 \end{aligned}\right.$$

$$\left\{\begin{aligned} \frac{\partial u_2}{\partial t} - \frac{\partial \theta_2}{\partial x} &= -\frac{1}{\tau_u} u_2 \\ \frac{\partial \theta_2}{\partial t} - \frac{1}{4}\frac{\partial u_2}{\partial x} &= H_c - H_s - R_2 \end{aligned}\right.$$

H_d = Deep convective heating

R = Radiative cooling

H_c = Congestus heating

H_s = Stratiform heating

+ 4 more prognostic equations for θ_{eb}, q, H_s, H_c

+ diagnostic equations for some source terms

How can shear effects be added to the multicloud model?

Project nonlinear equations

$$\partial_t U + U\partial_x U + W\partial_z U + \partial_x P = 0$$

onto vertical modes

$$U(x,z,t) = u_1(x,t)\sqrt{2}\cos\frac{\pi z}{H} + u_2(x,t)\sqrt{2}\cos\frac{2\pi z}{H}$$

using the inner product

$$\langle f, g\rangle = \frac{1}{H}\int_0^H f(z)g(z)\,dz$$

The result is ...

2-Mode Shallow Water Equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \dfrac{\partial \theta_1}{\partial x} = -\dfrac{3}{\sqrt{2}}\left[u_2 \dfrac{\partial u_1}{\partial x} + \dfrac{1}{2} u_1 \dfrac{\partial u_2}{\partial x}\right] \\ \dfrac{\partial \theta_1}{\partial t} - \dfrac{\partial u_1}{\partial x} = -\dfrac{1}{\sqrt{2}}\left[2u_1 \dfrac{\partial \theta_2}{\partial x} + 4\theta_2 \dfrac{\partial u_1}{\partial x} - u_2 \dfrac{\partial \theta_1}{\partial x} - \dfrac{1}{2}\theta_1 \dfrac{\partial u_2}{\partial x}\right] \end{cases}$$

$$\begin{cases} \dfrac{\partial u_2}{\partial t} - \dfrac{\partial \theta_2}{\partial x} = 0 \\ \dfrac{\partial \theta_2}{\partial t} - \dfrac{1}{4}\dfrac{\partial u_2}{\partial x} = -\dfrac{1}{2\sqrt{2}}\left[u_1 \dfrac{\partial \theta_1}{\partial x} - \theta_1 \dfrac{\partial u_1}{\partial x}\right] \end{cases}$$

- Nonlinear, hydrostatic internal gravity waves with effect of background shear
- Stechmann, Majda, and Khouider (2008) Theor. Comp. Fluid Dyn.

DRY DYNAMICS

Nonlinear dynamics of hydrostatic internal gravity waves

Interesting properties of the 2MSWE:

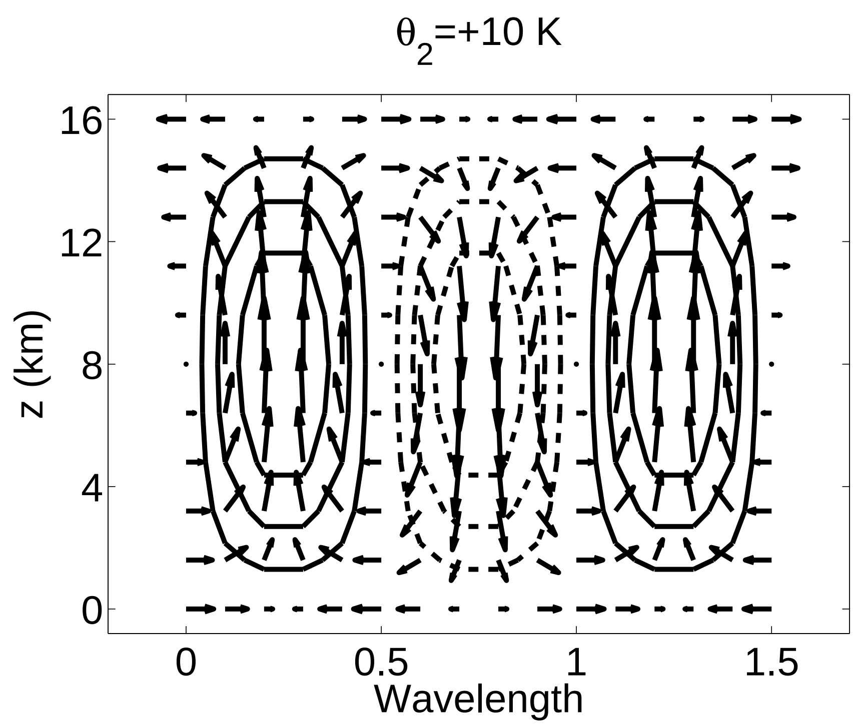
- Non-conservative

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = 0, \qquad A(\mathbf{u}) \neq \frac{\partial \mathbf{f}}{\partial \mathbf{u}}, \qquad \mathbf{u} = (u_1, \theta_1, u_2, \theta_2)$$

- Energy is conserved: $(u_1^2 + u_2^2 + \theta_1^2 + 4\theta_2^2)/2$
- Conditionally hyperbolic
- Neither genuinely nonlinear nor linearly degenerate
- Background shear can affect propagating waves

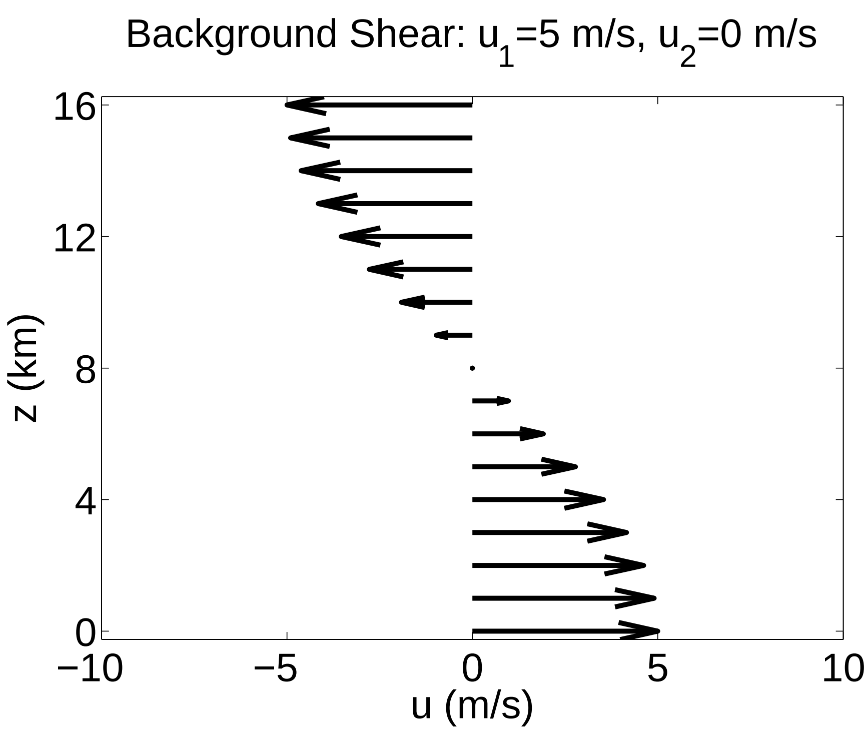
2MSWE are conditionally hyperbolic

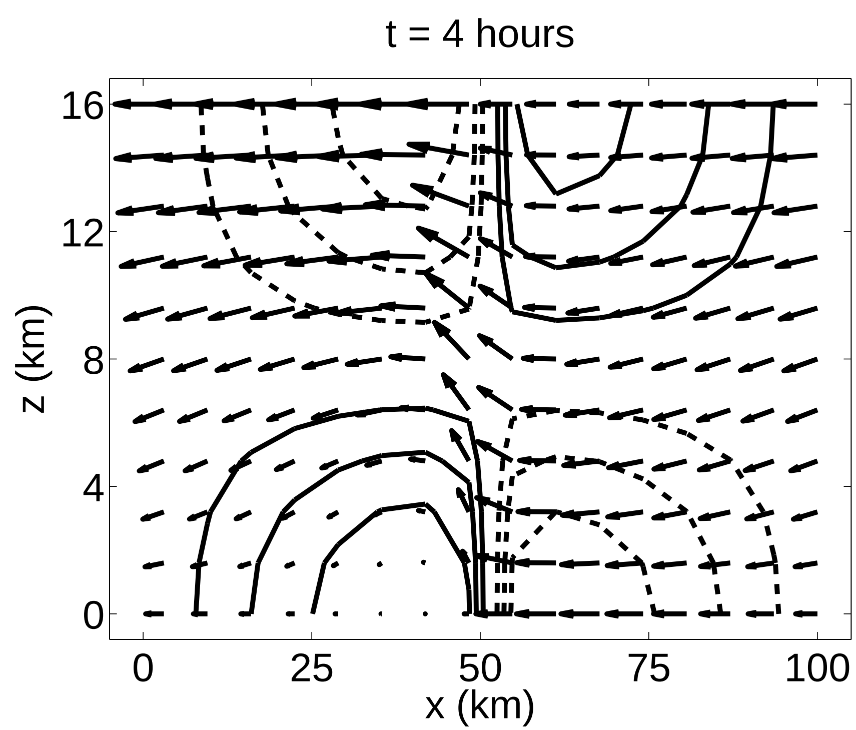
- Hyperbolic for lower values of u and θ
  - For $\bar{u}_1 = \bar{u}_2 = \bar{\theta}_1 = \bar{\theta}_2 = 0$, wavespeeds are ± 50 m/s and ± 25 m/s
- Not hyperbolic for larger shears or temperatures
  - Richardson number criterion for instability
  - For instance, unstable for $\theta_2 > 5$ K
  - Unstable waves have overturning circulation to stabilize



Smooth waves can break sometimes, but not always

- Without shear (and without a background θ), the linear waves are exact solutions to the nonlinear equations
- With u_1 background shear, smooth waves break to form bore-like waves





Numerical Methods

Numerical methods are a challenge for non-conservative PDE

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = 0$$

Our approach: split A into conservative and non-conservative parts:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}) = -A_{nc}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x}, \qquad \text{where } \frac{\partial \mathbf{F}}{\partial \mathbf{u}} = A_c$$

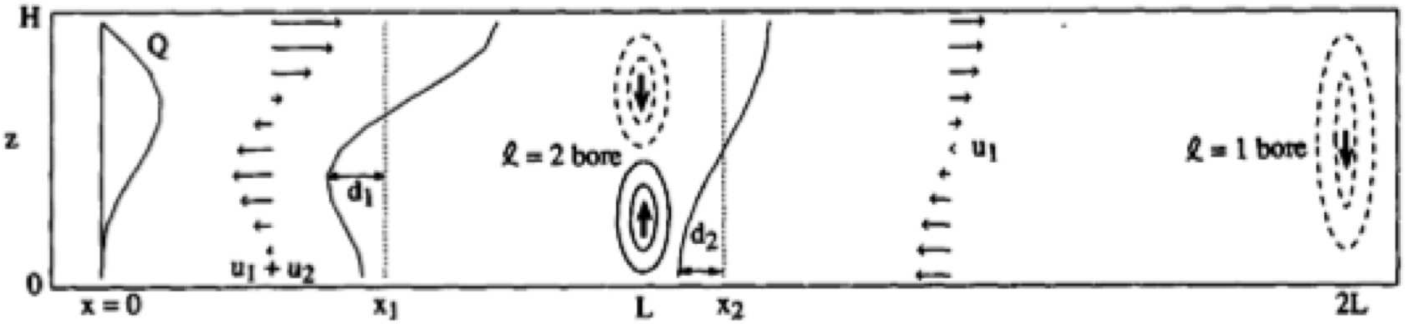
Operator splitting:

1. Non-oscillatory central scheme of Nessyahu and Tadmor (1990) for $\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}) = 0$
2. Centered spatial differences with 2nd order Runge–Kutta for $\frac{\partial \mathbf{u}}{\partial t} = -A_{nc}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x}$
   (Note: eigenvalues of A_{nc} are all zero)

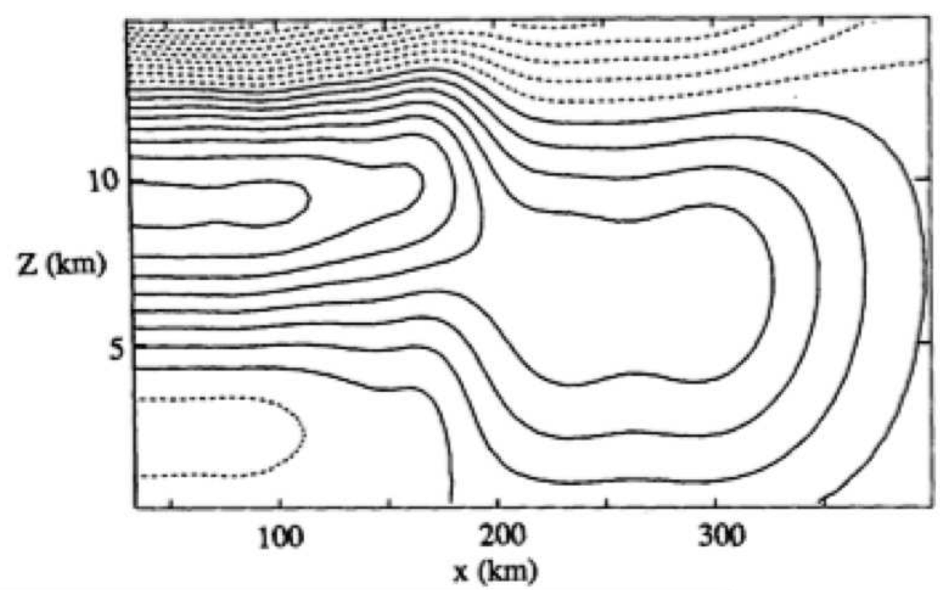
Application of 2MSWE: Gravity waves and organized convection

Gravity waves excited by convection can favor/trigger new nearby convection

From Mapes (1993):



Buoyancy contours



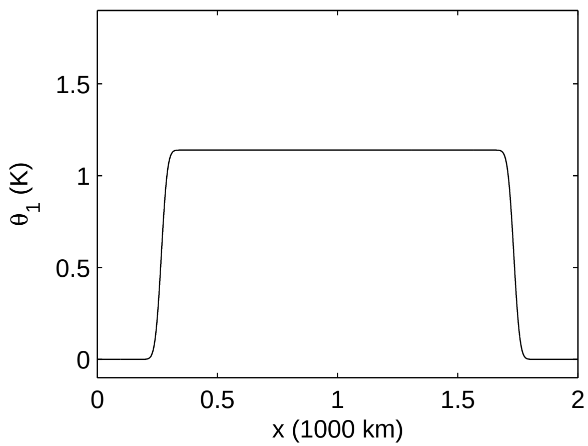
Wind shear ⇒ asymmetries in waves

- Apply an imposed localized heating to generate "buoyancy bores"

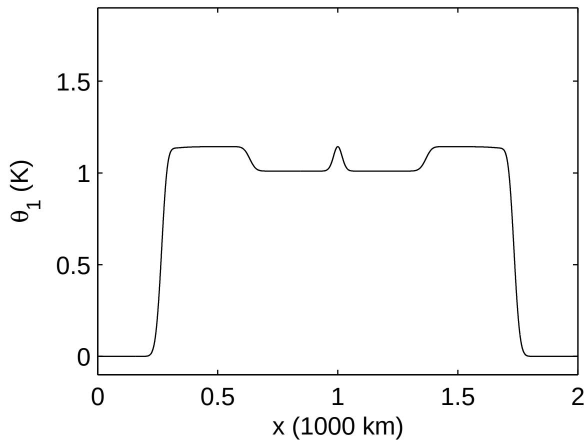
  Bretherton and Smolarkiewicz 1989, Nicholls et al 1991, Mapes 1993, Liu and Moncrieff 2004

  – all of these references ignore wind shear for simplicity

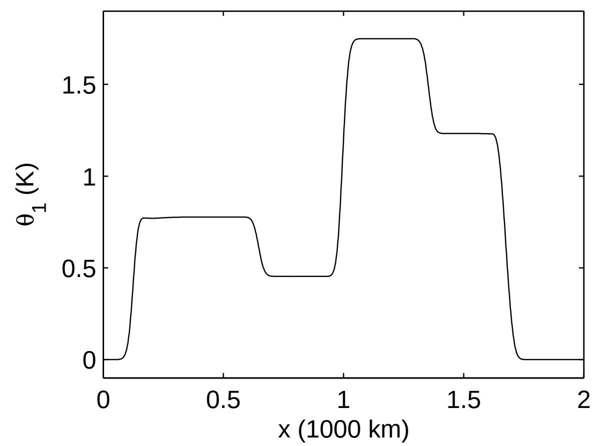
Linear, no shear

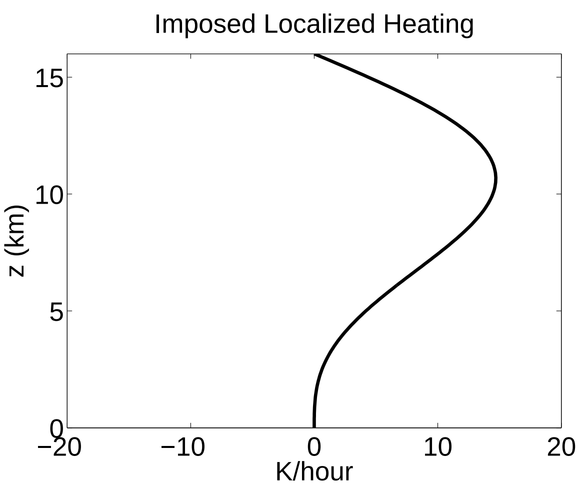


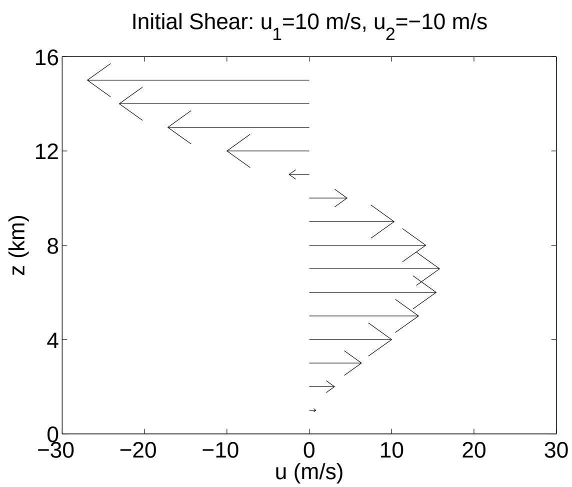
Nonlinear, no shear



Nonlinear, with shear







PDE's and Tropics: Other Topics

I. <u>Regorous Derivation of Multi-Scale PDE's</u>

Equatorial SWE and Singular Limits with Fast Variable Coefficients: Special Structure and Algebra of Hermite Operators

Dutrifoy & Majda, Comm. Math. Sci, 2006, 2007
Dutrifoy , Majda & Schochet, CPAM 2009

<u>Open Problems</u>: Primitive Equations with Minimal Dissipative Mechanisms Include Active moisture

II. <u>Tropical /Extratroprical Interactaions</u>

Novel Coupled KdV-like Equations with Energy Conserving nonlinear Coupled Interaction

Majda & Biello, Journal of Atmospheric Sciences, 2003
Biello & Majda, GAFD, 2004, Stud. Applied Math, 2004
Biello, Chinese Annals Math (B) 2009

III. New Multi-Scale Models: Rich Source of PDE's

A. MJO, 1,500km to 10,000km
Majda & Klein, JAS, 2003
Majda & Biello, PNAS, 2004
Biello & Majda, JAS, 2005
Biello, Majda & Moncrieff, JAS, 2007

B. Self-Similarity Across Scales and Superparameterization
Majda, JAS, (2007 A, B)
Xing, Majda & Grabowski, MWR, (2009)
Majda & Grote, PNAS, (2009)

C. Squall Lines, Hurricane Embryo, Deep Moist Convection; 1km to 100km
Klein & Majda, TCFD, 2006
Majda & Xing, Comm. Math. Sci., 2010
Majda, Mohammedian, & Xing, GAFD, 2008
Majda, Xing, Mohammedian, JFM, 2010
Ruprecht, Klein, Majda, JAS, 2010

Nice Review: Rupert Klein, Scale Dependent Models for Atmospheric Flows Ann. Rev. Fluid Mech, 2010, 42, pp. 249-274