Introduction to Amplitudes #1

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2019 TASI lectures by M.S.

Definition of particle. Wigner

particle = unitary, finite-dim rep of the Poincaré group

(4 translations, 3 rotations, 3 boosts)

- dont care it elementary or not
- eliminates the need to worry about ∞ redundancy that many operators create the same asymptotic (unlike in CFT!)

Consider a momentum 4-vector p^μ,

There are 6 orbits under (proper, orthochronous) Lorentz group $\quad (-1,1,1,1)$ signature

- $p^\mu = (0,0,0,0)$.
- $p^2 > 0$ tachyon unphysical
- $p^2 = 0, \; p^0 > 0$
- $p^2 = 0, \; p^0 < 0$

massless particles

- $p^2 < 0, \; p^0 > 0$
- $p^2 < 0, \; p^0 < 0$

massive particles.

Any massless 4-momentum p^μ can be

massive

brought into standard form $k^\mu = (k^0, 0, 0, k^0)$

$= (k^0, 0, 0, 0)$

by a suitable Lorentz transformation.

Suppose we have a basis

$$|k^\mu, \sigma\rangle$$ some labels

for all states with a given k^μ.

For any p^μ, choose a Lorentz transformation $L^\mu{}_\nu(p,k)$ such that $\boxed{p^\mu = L^\mu{}_\nu(p,k)\,k^\nu}$

(for given p & k, L is not unique!)

We assume we have a unitary, finite-dim rep $U(L)$ of the Lorentz group acting on Hilbert space.

Define $|p,\sigma\rangle = U(L(p,k))\,|k,\sigma\rangle$

How does this state $|p,\sigma\rangle$ transform?

$$U(L')\,|p,\sigma\rangle = U(L')\,U(L(p,k))\,|k,\sigma\rangle$$

rep: $U(a)\cdot U(b) = U(a\cdot b)$

$$= U\big(\underbrace{L'\cdot L(p,k)}\big)\,|k,\sigma\rangle$$

$$= U\big(\underbrace{L(L'p,k)\cdot L^{-1}(L'p,k)}_{=1}\ \square\big)\,|k,\sigma\rangle$$

$$= U(L(L'p,k))\;U(W)\,|k,\sigma\rangle$$

where $W = L^{-1}(L'p,k)\,\boxed{\cdot L'\cdot L(p,k)}$

Notice that W is a Lorentz transf. that does not change k!

$$\boxed{W^\mu{}_\nu k^\nu = k^\nu}$$

Therefore $U(W)\,|k,\sigma\rangle$ must be a lin comb of other states with the

..

same k!

$$U(L')|p,\sigma\rangle = U(L(L'p,k))\boxed{U(\omega)|k,\sigma\rangle}$$

$$= \underline{U(L(L'p,k))}\underbrace{\left(\sum_{\sigma'} D_{\sigma,\sigma'}(\omega)\,|k,\sigma'\rangle\right)}$$

some coefficients

Our one particle states lie in representations of the little group of k.
= the set of Lorentz transformations that preserve k.

massive particle little group $= SO(3)$

massless particle little group $= U(1) \times E_2$

2-dim Euclidean group

(noncompact = no nontrivial finite dim reps).

- massless $U(\text{rotation by } \theta \text{ around } z\text{-axis})|k^\mu\rangle = e^{ih\theta}|k^\mu\rangle$

where the eigenvalue h is called the helicity & it is quantized to be half-integer

- massive particles
  states have to lie in reps. of $SU(2)$.

Spinor Helicity

massless $p^2 = 0$

$$p_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} p_\mu = \begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}$$

$\underline{p^2 = 0} \Leftrightarrow \underline{\det p = 0}$

$\Rightarrow$ it is possible to express

$$\boxed{p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}} \leftarrow$$

The 2 2-component objects $\lambda, \tilde{\lambda}$ are called spinor helicity variables.

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} \lambda_1 \tilde{\lambda}_{\dot{1}} & \lambda_1 \tilde{\lambda}_{\dot{2}} \\ \lambda_2 \tilde{\lambda}_{\dot{1}} & \lambda_2 \lambda_{\dot{2}} \end{pmatrix} \Rightarrow \text{satisfies } p^2 = 0$$

given p, obviously λ & $\tilde{\lambda}$ are not unique. For example,

$$\left.\begin{aligned} \lambda_\alpha &\to e^{i\theta/2} \lambda_\alpha \\ \tilde{\lambda}_{\dot{\alpha}} &\to e^{-i\theta/2} \tilde{\lambda}_{\dot{\alpha}} \end{aligned}\right\} \text{leaves } p \text{ unchanged.}$$

If you want p^μ to be real in 3+1 dimensions, you should choose $\tilde{\lambda}$ to be the complex conj. of λ.

It's often useful to allow $\lambda, \tilde{\lambda}$ to be arbitrary, not nec. complex conjugate.

If you do a rotation by angle θ around the spatial 3-vector $\vec{p}$ of a massless particle,

$$\Longrightarrow \quad |e^{i\theta/2}\lambda, e^{-i\theta/2}\tilde{\lambda}\rangle = e^{-ih\theta}|\lambda,\tilde{\lambda}\rangle$$

Or, if we "complexify" (allow $\lambda, \tilde{\lambda}$ to be arbitrary, not necessarily conjugates)

$$\boxed{|t^{-1}\lambda, t\lambda\rangle = t^{2h}|\lambda,\tilde{\lambda}\rangle}$$

Note that the two states have the same $p = \lambda\tilde{\lambda}$

So this formula tells us how a massless 1-particle state should scale under a (complexified) little group transformation.

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fundamental assumption of S-matrix theory.

The Poincaré group acts on the S-matrix

★★ as it does on individual 1-particle states.

If we have an amplitude for n-particles (massless)

$$A \equiv \mathbf{A}\left(\lambda_1^{\alpha}, \tilde{\lambda}_1^{\dot{\alpha}}, \lambda_2^{\alpha}, \tilde{\lambda}_2^{\dot{\alpha}}, \ldots, \lambda_n^{\alpha}, \tilde{\lambda}_n^{\dot{\alpha}}\right)$$

having helicities $h_1, h_2, \ldots, h_n \in \frac{1}{2}\mathbb{Z}$

then A should scale like

$$\boxed{A\Big(\underbrace{\cdots\cdots}_{\text{unchanged}}, t^{-1}\lambda_i^{\alpha}, t\,\tilde{\lambda}_i^{\dot{\alpha}}, \overbrace{\cdots}^{\text{unchanged}}\Big) = t^{2h_i} \cdot A}$$

this is the letter "i"

Three particle (massless) amplitudes are completely determined!

p_1, p_2, p_3

$\Rightarrow p_1^\mu + p_2^\mu + p_3^\mu = 0 \qquad p_1^2 = p_2^2 = p_3^2 = 0$

$$\Rightarrow \boxed{(p_1+p_2)^2 = (p_1+p_3)^2 = (p_2+p_3)^2 = 0}$$

$$\Rightarrow (p_1+p_2)^2 = 2\,\boxed{\epsilon^{\alpha\beta}\lambda_{1\alpha}\lambda_{2\beta}}\,\boxed{\epsilon^{\dot\alpha\dot\beta}\tilde\lambda_{1\dot\alpha}\tilde\lambda_{2\dot\beta}}$$

$$\langle i,j\rangle = \epsilon^{\alpha\beta}\lambda_{i\alpha}\lambda_{j\beta}$$

$$[i,j] = \epsilon^{\dot\alpha\dot\beta}\tilde\lambda_{i\dot\alpha}\tilde\lambda_{j\dot\beta}$$

In 3-particle massless kinematics

$$\langle 12\rangle[12] = 0$$
$$\rightarrow \langle 13\rangle[13] = 0$$
$$\langle 23\rangle[23] = 0$$

With complex momenta, $\exists$ 2 nontrivial branches of solutions.

- either $[12] = [23] = [31] = 0$
- or $\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0$.

If $\langle 12\rangle = 0$, this implies $\lambda_1 \propto \lambda_2$

& $\langle 13\rangle = 0$, then $\lambda_1, \lambda_2, \lambda_3$ are all

$\propto \langle 13 \rangle$ ... that $\lambda_1, \lambda_2, \lambda_3$ are all prop. so $\Rightarrow \langle 23 \rangle = 0$ automatically.

So there are 2 types of massless 3-particle amplitudes

$$A(\langle 12 \rangle, \langle 23 \rangle, \langle 13 \rangle) \quad \text{or}$$

$$A([12], [23], [31])$$

Magic: there is a unique formula

nonperturbative result

$$\boxed{A = \alpha \langle 12 \rangle^{h_3 - h_1 - h_2} \cdot \langle 23 \rangle^{h_1 - h_2 - h_3} \cdot \langle 13 \rangle^{h_2 - h_3 - h_1}}$$

α: undetermined constant

that satisfies the required scaling property.

In order to avoid a singularity on Mink. space locus $(\langle 12 \rangle = \langle 23 \rangle = \langle 13 \rangle = 0)$ we require $\boxed{h_1 + h_2 + h_3 \leq 0.}$

(otherwise, $\alpha = 0$ is the only solution with a smooth Mink. space limit).

Setting $\langle 12\rangle = \langle 23\rangle = \langle 13\rangle = \epsilon$

$$A = \epsilon^{-h_1 - h_2 - h_3}$$

to have $\epsilon \to 0$ limit make sense,

$$h_1 + h_2 + h_3 \leq 0.$$

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lets consider spin-1 $(h = +1, -1)$.

$$\boxed{\begin{aligned} A(1^-_{h_1}, 2^-_{h_2}, 3^-_{h_3}) &= \alpha \langle 12\rangle\langle 23\rangle\langle 13\rangle \\ A(1^-, 2^-, 3^+) &= \beta \boxed{\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}} \end{aligned}}$$

must have dimen $(\text{energy})^{-2}$

1

dimensionless

dimension of energy $\frac{1}{2}$

For $h_1 + h_2 + h_3 \geq 0$ you can have the other type of amplitude.

$$A([12], [23], [13]) =$$

$$\alpha\, [12]^{h_1 + h_2 - h_3}\, [23]^{h_2 + h_3 - h_1}\, [13]^{h_1 + h_3 - h_2}$$

$$\boxed{\begin{aligned} A(1^-,2^+,3^+) &= \gamma\,\frac{[23]^3}{[31][12]} \\[1em] A(1^+,2^+,3^+) &= \delta\,[12][23][31] \end{aligned}}$$

3

4

[13] [32]

Note that each formula is fully antisymmetric under the exchange of any 2 particle labels.

However, spin-1 particles must be bosons so their amplitudes must be fully symmetric under exchange of any pair!

resolution $\rightarrow$ $\alpha=\beta=\gamma=\delta=0$.

$\Rightarrow$ A massless spin-1 particle (ie. photon) cannot have a self interaction (3-point).

To get interesting nonzero amplitudes we allow there to be N different species of particles

$$A(1_a^-, 2_b^-, 3_c^+) = \beta_{abc} \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle}$$

which of the N types you are scattering.

we require β_{abc} must be antisymmetric under exchange of any pair.

$$A(1_a^-, 2_b^-, 3_c^+) = \beta_{abc} \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}$$

$$A(1_a^-, 2_b^+, 3_c^-) = \beta_{abc} \frac{\langle 13\rangle^3}{\langle 12\rangle\langle 23\rangle}$$

Four-point amplitudes.

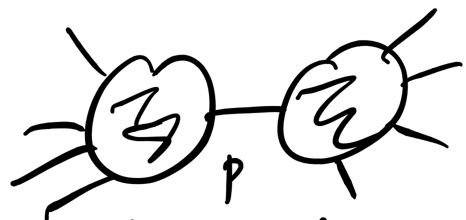
Here, for the first time, I will have to introduce perturbation theory.

<u>Tree level unitarity</u> looks like the following:

Factorization:

$$A \sim \frac{A_L \times A_R}{p^2}$$

take a limit where some intermediate p is going on shell $(p^2=0)$



A_L A_R.

(both A_L & A_R are tree level amplitudes).

Consider some

$$A(1_a^-, 2_b^-, 3_c^+, 4_d^+)$$

$$= \langle 12\rangle^2 [34]^2 / \ldots$$

$$= \underbrace{\langle 12\rangle\,[34]}_{\text{has correct weight.}}\ \boxed{F(s,t,u)}$$

- $\lambda_1 \to \frac{1}{t}\lambda_1 \quad \tilde{\lambda}_1 \to t\,\tilde{\lambda}_1 \quad A = t^{-2}A$ ✓
- $\lambda_3 \to \frac{1}{t}\lambda_3, \quad \lambda_3 \to t\,\tilde{\lambda}_3 \quad A = t^{+2}A$ ✓

$$s = (p_1+p_2)^2 \quad t = (p_2+p_3)^2 \quad u = (p_1+p_3)^2$$

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dim analysis tells us that

an n-particle amplitude must have

dimension $(\text{energy})^{4-n}$ —

The brackets $\langle\,\rangle$, $[\,]$ each have

dimension of energy.

$\Rightarrow$ our undetermined F should have dimension $1/\text{energy}^4$. ←

a simple ansatz with only simple

3 undetermined coeff

poles is icients.

$$F(s,t,u) = \frac{C_{st}}{s\cdot t} + \frac{C_{tu}}{t\cdot u} + \frac{C_{us}}{u\cdot s}$$

lets look @ the pole $\underset{s=}{(p_1+p_2)^2} \to 0.$

the residue @ that pole is

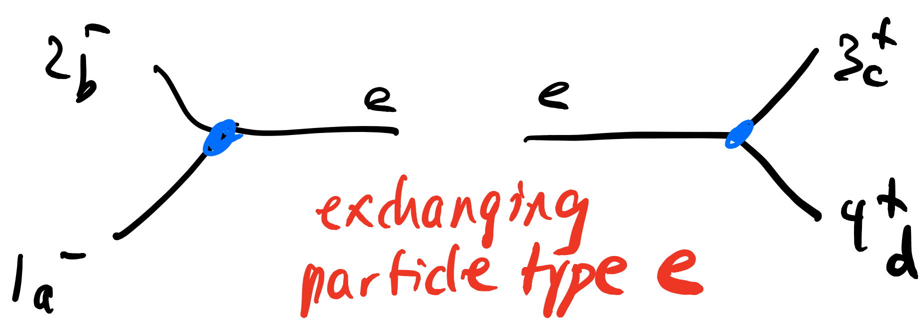
$$\lim_{s\to 0}(s\cdot A) = \frac{\langle 12\rangle^2 [34]^2}{t}\left(C_{st} - C_{us}\right)$$

$s+t+u=0 \Rightarrow$ when $s=0$, $u=-t$

by factorization, this must =

a product of 3-particle amplitudes

summed over intermediate states.



[If you plug in the actual amplitudes for A_L & A_R]

$\Rightarrow$
- $C_{st} - C_{us} = \sum_e \beta_{abe}\, \beta_{cde}$
- $C_{tu} - C_{st} = \sum_e \beta_{bce}\, \beta_{ade}$
- $C_{us} - C_{tu} = \sum_e \beta_{cae}\, \beta_{bde}$

A solution to the constraints only exists if

$$0 = \sum_e \beta_{abe}\,\beta_{cde} + \beta_{bce}\,\beta_{ade} + \beta_{cae}\,\beta_{bde}$$

$\Rightarrow$ Jacobi identity

Consistency requirement of tree-level unitarity $\Rightarrow$ demand the 3-point self coupling of a massless spin-1 particle must be structure coefficients of a Lie algebra. $\square$

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interactions of type

- $A(1^-,2^-,3^+)$ &
- $A(1^-,2^+,3^+)$

are the two you get from pure Yang-Mills theory. $\mathcal{L} = -\frac{1}{4g} \mathrm{Tr}(F^2)$.

dimensionless

the two

$A(1^+,2^+,3^+)$

& $A(1^-,2^-,3^-)$ come from

interactions of type $\mathrm{Tr}(F^3)$

with a coefficient that has dim

$(\text{energy})^{-2}$.

Introduction to Amplitudes #2

Massive spinor helicity 1709.04891

$p^2 = \pm m^2 \quad \Rightarrow \quad \det p_{2\times 2} = \pm m^2$

choice of signature

we need a pair of spinor helicity vars.

$\lambda^{\alpha}_{I}, \quad \tilde{\lambda}^{\dot{\alpha}}_{I} \qquad I = 1,2.$

$\Rightarrow \boxed{p_{\alpha\dot{\alpha}} = \lambda^{I}_{\alpha} \tilde{\lambda}_{\dot{\alpha} I}}$ with implied sum over I

such a choice of $\lambda, \tilde{\lambda}$ is not unique, there is an $GL(2)$ redundancy

$\lambda^{I}_{\alpha} \to W^{I}{}_{J} \lambda^{J}_{\alpha} \qquad \tilde{\lambda}_{\dot{\alpha}} \to (W^{-1})^{I}{}_{J} \tilde{\lambda}^{J}_{\dot{\alpha}}$

then p is unchanged.

★ Amplitudes must be symmetric rank $2S$ tensors in the indices $\{I_1, \dots, I_{2S}\}$ for a massive particle of spin S.

Example: Consider a 4-particle amplitude involving

#1	massive	spin-1/2
#2	massive	spin-2
#3	massless	$h = 3/2$
#4	massless	$h = -1$

A candidate amplitude (that means, one that satisfies ★★)

$$[2^{J_1}, 3][2^{J_2}, 3][2^{J_3}, 3]\langle 1^{I_1}, 2^{J_4}\rangle \cdot \langle 4| p_1 p_2 |4\rangle + \mathrm{sym}(J_1, J_2,$$

$J_3, J_4)$

means

$[\tilde\lambda_2^{J_1}, \tilde\lambda_3]$ $\quad \langle \lambda_4| \,(2\times 2 \text{ matrix } p_1)\cdot(2\times 2 \text{ matrix } p_2)\,|\lambda_4\rangle$

$$\epsilon^{\beta\gamma}\epsilon^{\delta\alpha}\epsilon^{\dot\alpha\dot\beta}(\lambda_4)_\delta (p_1)_{\alpha\dot\alpha}(p_2)_{\dot\beta\beta}(\lambda_4)_\gamma$$

There are other candidate amplitudes one can write down (a finite number of them – modulo the ability to multiply by arbitrary functions of Mandelstam variables).

Back to massless world, now $h = \pm 2$.

$$A(1^{--}, 2^{--}, 3^{++})$$

shorthand for $h=-2$

$$= \sqrt{G}\, \frac{\langle 12\rangle^6}{\langle 23\rangle^2 \langle 13\rangle^2}$$

coefficient with dimenion $1/\text{energy}$.

G = Newton's constant.

parity conjugate

$$A(1^{++},2^{++},3^{--}) = \sqrt{G}\,\frac{[12]^6}{[23]^2[13]^2}$$

$$A(1^{--},2^{--},3^{--}) = \square\left(\langle 12\rangle\langle 23\rangle\langle 13\rangle\right)^2$$

the coef. here must have dimension $1/\text{energy}^5$.

Factorization of 4-particle amplitude. $A(1^{--},2^{--},3^{++},4^{++})$

- must be proportional to G
- must have an overall factor of

  $\langle 12\rangle^4[34]^4$ to account

  for little group scaling.

so far $A = \underline{G \cdot \langle 12\rangle^4 [34]^4} \cdot F(s,t,u)$

undetermined so far

- crossing symmetry $\Rightarrow$ fully symmetric in s, t, u.

- the function F should have dim. $1/(\text{energy})^6$

- in perturbation theory — at tree level, F must be rational, with only simple poles in s, t, u

$\Rightarrow$ determine $F = \dfrac{1}{s \cdot t \cdot u}$.

We've determined 4-point tree amplitude with little work..

BCFW Recursion (Tree level)

Pick 2 particles, say #1 & #2, and deform their spinor helicity variables by a (complex) parameter z.

original amplitude

$$A(\lambda_1, \tilde{\lambda}_1, \lambda_2, \tilde{\lambda}_2, \dots, \lambda_n, \tilde{\lambda}_n) = A(0)$$

$$\longrightarrow A(z) = (\lambda_1, \hat{\tilde{\lambda}}_1, \hat{\lambda}_2, \tilde{\lambda}_2, \dots, \lambda_n, \tilde{\lambda}_n]$$

where

$$\hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 + z\tilde{\lambda}_2$$

$$\lambda_2 = \lambda_2 - z\lambda_1$$

Nice feature: preserves momentum conservation:

$$0 = \underset{\sim}{p_1} + \underset{\sim}{p_2} + \cdots\cdots + \underset{\sim}{p_n}$$

$$= \lambda_1\tilde\lambda_1 + \lambda_2\tilde\lambda_2 + \cdots + \lambda_n\tilde\lambda_n$$

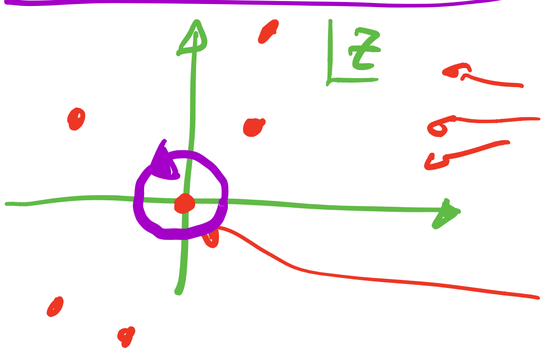
$$\to \lambda_1(\tilde\lambda_1 + z\tilde\lambda_2) + (\lambda_2 - z\lambda_1)\tilde\lambda_2 + \cdots + \lambda_n\tilde\lambda_n$$

$$\cancel{+z\lambda_1\tilde\lambda_2} \qquad \cancel{-z\lambda_1\tilde\lambda_2}$$

$= 0$ still.

Since (at tree level) amplitudes are rational functions

$$\boxed{A(0) = \frac{1}{2\pi i}\oint dz\, \frac{A(z)}{z}}$$



other poles of $\frac{A(z)}{z}$

$\frac{A(z)}{z}$ has a pole with residue $A(0)$

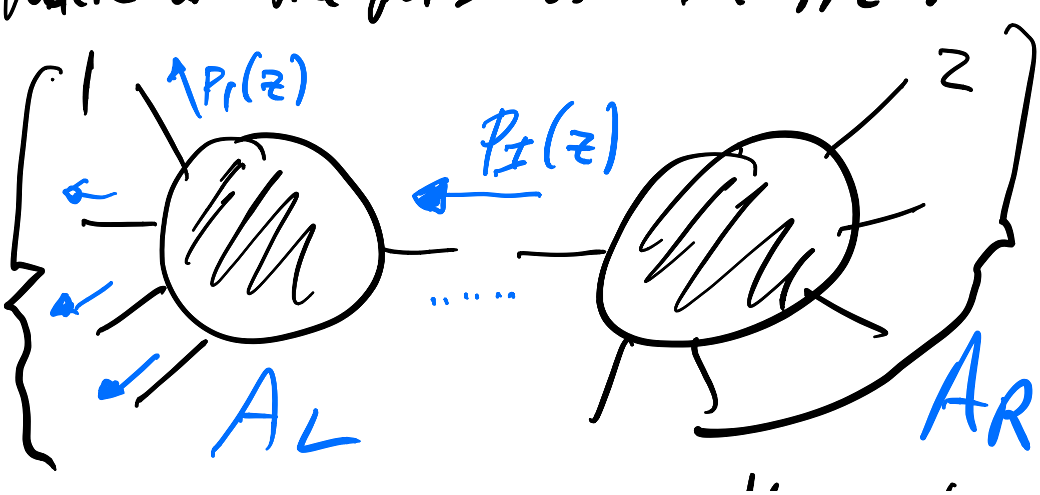
If $A(z) \to 0$ as $z \to \infty$, then there is no pole at ∞, so we can push the contour out to ∞ and find

$$\boxed{A(0) = -\sum_{\substack{z_I \in \text{other poles} \\ \text{of } A(z)/z}} \operatorname*{Res}_{z=z_I}\left(\frac{A(z)}{z}\right)} \quad \bigstar$$

It was proven by BCFW that pure Yang-Mills theory and pure Einstein-Hilbert gravity have the property that $A(z) \to 0$ as $z \to \infty$ so there is no boundary term.

Why is this useful?

Where are the poles of $A(z)/z$?



some collection of particles containing 1

Let's call this set I

the complementary set

$p_I = \Sigma$ of momenta in set I.

There will be some value of z, call it z_I, such that $p_I^2(z_I) = 0$.

So, $A(z)$ has a pole at z_I whose residue is $A_L(z_I) \cdot A_R(z_I)$.

$$p_I(z) = p_I(0) + z \underbrace{\lambda_1 \tilde{\lambda}_2}_{\text{a null vector}}$$

$$p_I^2(z) = p_I^2(0) + 2z \cdot p_I(0) \cdot \lambda_1 \tilde{\lambda}_2 + 0$$

$$= p_I^2(0) - 2z \cdot \langle 1 | p_I | 2]$$

(define this notation)

the value of z that makes this $= 0$ is

$$\boxed{z_I = -\frac{P_I^2(0)}{\langle 1|P_I|2]}}$$

The residue we need is not $\mathrm{Res}(A(z))$ but $\mathrm{Res}(A(z)/z)$

$$\frac{A(z)}{z} \sim \frac{1}{z}\cdot\underbrace{\frac{A_L\cdot A_R}{P_I^2(z)}}_{\text{from tree level factorization}}$$

$$= \frac{1}{z}\cdot\frac{\overbrace{A_L\cdot A_R}^{F}}{\underbrace{P_I^2(0)}_{x} - 2z\,\underbrace{\langle 1|P_I|2]}_{y}}$$

Exercise: Check that

$$\underset{z=\frac{x}{2y}}{\mathrm{Res}}\left(\frac{1}{z}\cdot\frac{F}{x-2zy}\right) = -\frac{1}{x}.$$

Therefore: we have

$$\boxed{A(0) = \sum_{\substack{\text{sets } I \\ \text{that contain particle 1} \\ \text{but not particle 2}}} \frac{A_L(z_I)\cdot A_R(z_I)}{P_I^2(0)}}$$

BCFW

Right-hand side only involves on-shell amplitudes, so it is a recursion!

Try some examples: Use the recursion to compute for example

$A(1^-, 2^-, 3^+, 4^+)$

You'll find $\dfrac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$

More generally you can check that the tree level amplitude for n gluons in the "MHV" helicity configuration

is $\begin{matrix} n-2 & \text{positive helicity} \\ 2 & \text{negative helicity} \end{matrix}$

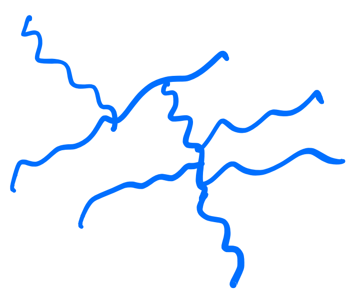
$$A(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$

$$= \frac{\langle i\, j \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n 1 \rangle}$$

Parke–Taylor formula.

Supersymmetry

At tree level, gluon amplitudes are insensitive to susy.



So we might as well use maximal ($\mathcal{N}=4$) supersymmetry. This is a multiplet with

$-1 \quad -1/2 \quad 0 \quad +1/2 \quad +1$

1	4	6	4	1
gluon	gluinos	scalar	gluinos	gluon

A notational device to handle the bookkeeping is to introduce anticommuting Grassmann variables η_i^A $\quad A = 1,2,3,4$ $\quad i = 1,\dots,n$

Recall that functions of such variables have finite taylor series

$$f(\eta) = f(0) + f'(0)\eta + 0$$

because $\eta^2 = 0$

We define a superamplitude

$\mathcal{A}(1,\dots,n) =$ Taylor series expansion in the 4n Grassmann variables

$$= A(1^+, 2^+, \ldots, n^+) \qquad (\text{in fact this} = 0).$$

$$+ \sum_{i=1}^{n} \sum_{A=1}^{4} \eta_i^A \, A(1^+, \ldots, i, \ldots, n^+).$$

not + helicity gluon, but $\lambda = +1/2$ gluino.

$$+ \ldots\ldots$$

$$+ \prod_{i=1}^{n} \prod_{A=1}^{4} \eta_i^A \, A(1^-, 2^-, \ldots, n^-)$$

Altogether there are 16^n terms.

How can we make susy manifest?

$$p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$$

$$\boxed{q^{\alpha A} = \lambda^\alpha \eta^A} \quad \boxed{\bar q^{\dot\alpha A} = \tilde\lambda^{\dot\alpha} \frac{\partial}{\partial \eta^A}} \, ?$$

Momentum conservation is made manifest by an overall factor $\delta^4(\sum_i \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha})$

in each amplitude.

Similarly, the q-supersymmetry invariance can be made manifest by demanding that every amplitude has a factor of

$$\delta^8(q^{\alpha A}) = \prod_{\alpha=1}^{2}\prod_{A=1}^{4}\left(\sum_{i=1}^{n}\lambda_i^{\alpha}\eta_i^{A}\right)$$

(Recall for a fermion $\delta(\eta)=\eta$).

Note: This implies that the Taylor expansion of any superamplitude is zero until you reach terms of order $(\text{fermion})^8$!

The $n=3$ MHV superamplitude is

$$\mathcal{A}(1,2,3) = \frac{\delta^8(\sum \lambda \eta)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

but there is also, due to the special features of 3-particle kinematics, an allowed "$\overline{\text{MHV}}$" superamplitude

$$\mathcal{A}(1,2,3) = \frac{\delta^4([12]\eta_3 + [23]\eta_1 + [31]\eta_2)}{[12][23][31]}$$

exercise: check that this is annihilated by $Q = \sum_{i=1}^{3} \lambda_i^a \eta_i^A$.

$\bar{Q}$ susy places additional, very

powerful constraints on amplitudes

MHV 3-point amplitudes that are nonzero

- $A(1^-, 2^-, 3^+)$

let's look at other amplitudes with the same total helicity $-1-1+1=-1$

- $A(1^{-1/2}, 2^{-1}, 3^{+1/2})$
- $A(1^0, 2^{-1}, 3^0)$

etc.

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in general the n-particle superamplitude in $\mathcal{N}=4$ SYM theory has the form

$$\mathcal{A} = \frac{\delta^8(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\Big[1 + \mathcal{O}(\eta^4)$$

NMHV superamplitude

N^2MHV superamplitude

$\underbrace{}_{\text{MHV superamplitude}} + \mathcal{O}(\eta^8)$

$+ \text{etc} \Big].$

In fact, due to SU(4) R-symmetry, you only get terms with powers of fermions that are multiples of 4 → the indices must be contracted by ϵ_{ABCD}.

Momentum Twistors.

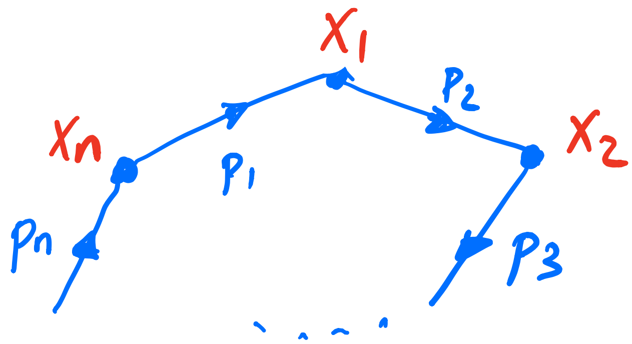
We have seen that writing $p = \lambda\tilde{\lambda}$ trivializes $p^2 = 0$ constraint.

Amplitudes are still subject to a nonlinear constraint $\sum \lambda\tilde{\lambda} = 0$

Let's consider "dual variables"

$$\boxed{p_i = x_{i+1} - x_i}$$

given a collection of p_i, the x's



are not uniquely determined because we can arbitrarily translate in x space.

$\sum p_i = 0$ is automatic if we choose to parameterize $p_i = x_{i+1} - x_i$ (with it understood that $x_{n+1} = x_1$)

$$(p_i)_{\alpha\dot\beta} = (x_{i+1})_{\alpha\dot\beta} - (x_i)_{\alpha\dot\beta}$$

Contract both sides with λ_i^{β}

$$\epsilon^{\alpha\beta}\underbrace{(p_i)_{\alpha\dot\beta}}_{\epsilon^{\alpha\beta}\lambda_{i\alpha}\tilde\lambda_{i\dot\beta}\lambda_{i\beta}}\lambda_{i\beta} = \cdots$$

$$\epsilon^{\alpha\beta}\lambda_{i\alpha}\tilde\lambda_{i\dot\beta}\lambda_{i\beta}$$

$$\|$$

$$0 = (x_{i+1}\lambda_i)_{\dot\beta} - (x_i\lambda_i)_{\dot\beta}$$

So lets call this thing

$$\boxed{\mu_i^{\dot\alpha} = (x_{i+1}\lambda_i)^{\dot\alpha} = (x_i\lambda_i)^{\alpha}}$$

then lets assemble the two λ's and the 2 μ's into

this is 1 dot

this is 2 dot

$$Z_i^I = (\lambda_i^1, \lambda_i^2, \mu_i^{\dot 1}, \mu_i^{\dot 2})$$

runs over n particles

$I = 1,2,3,4$

the scaling $(\lambda, \tilde\lambda) \to (\frac{1}{t}\lambda, t\tilde\lambda)$ evidently transforms μ in the same way as λ! So all four components of Z transform homogenously.

Therefore Z_i^I is properly understood as a point in projective space $\mathbb{P}^3$:

$$(a_1, a_2, a_3, a_4) \sim (ta_1, ta_2, ta_3, ta_4)$$

for any nonzero t.

---

collection of p_i satisfying $\sum p_i = 0$

$\Rightarrow$ a collection of n points in $\mathbb{P}^3$

---

other way: suppose you give me

n points in $\mathbb{P}^3$. $\quad Z_i = (\lambda_{1i}, \lambda_{2i}, \mu_{1i}, \mu_{2i})$

consider the four equations

$$\boxed{\begin{array}{ll} i & \mu_i^{\dot\alpha} = (x_i \lambda_i)^{\dot\alpha} \\ i-1 & \mu_{i-1}^{\dot\alpha} = (x_i \lambda_{i-1})^{\dot\alpha} \end{array}} \qquad \begin{array}{l} \dot\alpha = 1, 2. \\ \dot\alpha = 1, 2. \end{array}$$

This is a collection of 4 equations

for the 4 components of x_i.

$\Rightarrow$ solve for x_i.

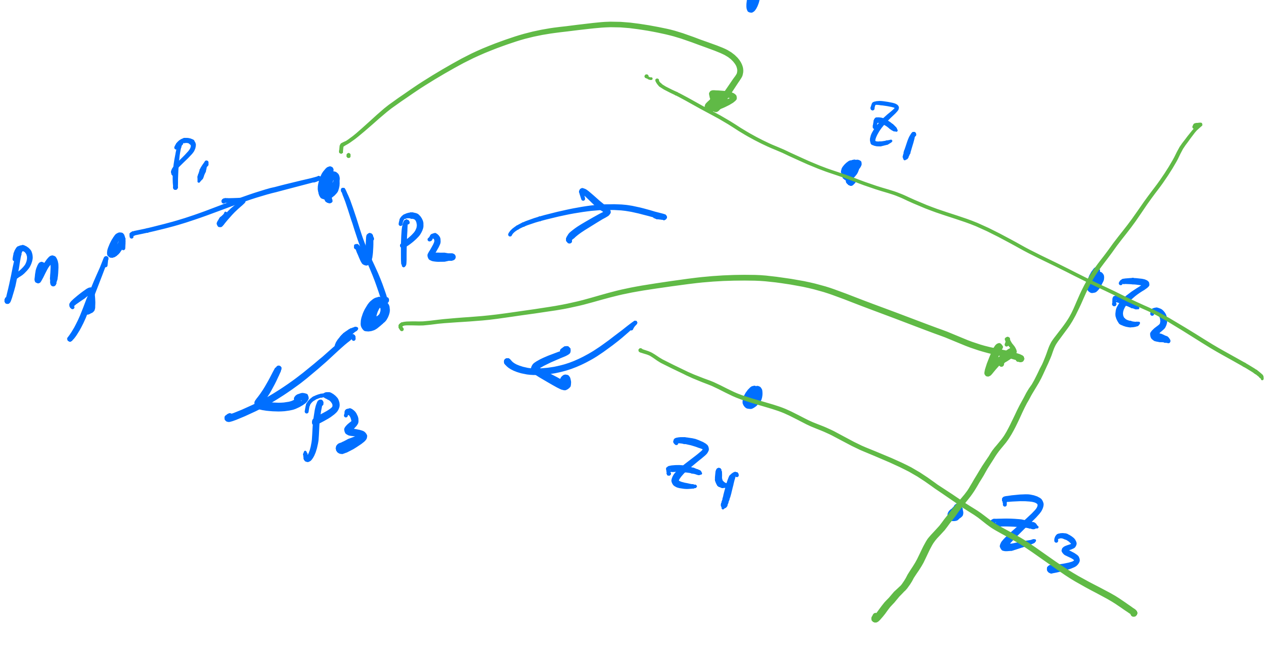
Note that by construction they will satisfy

$$0 = \underbrace{(x_i - x_{i+1})_{\dot{\alpha}\alpha}}_{2\times 2 \text{ matrix}} \underbrace{\lambda_{i\beta}\, \epsilon^{\alpha\beta}}_{2 \text{ component vector}}$$

$\Rightarrow$ therefore

$$\det(x_i - x_{i+1}) = 0$$

$\Rightarrow$ x_i, x_{i+1} are null separated.



The Z_i are called momentum twistors.

They are particularly useful in $\mathcal{N}=4$ SYM theory which has a remarkable symmetry called dual (super)conformal invariance = conformal invariance in x space ←

← the dual space to momentum

What is the dim of kinematic space of $\mathcal{N}=4$ SYM?

$3n$ degrees of freedom for n null momenta

-10 constraints of Poincaré invariance.

-5 1 dilatation in x space

4 special conformal transformations
in x-space.

$3n-15$ dimensional.

At the level of momentum twistors,
assemble the momentum twistor vars
into a $4 \times n$ matrix

$$\begin{pmatrix} | & & | \\ Z_1 & \cdots\cdots & Z_n \\ | & & | \end{pmatrix}$$

The set of symmetry transformations ✱
corresponds to left-multiplication
by $SL(4)$ on the above $4 \times n$
matrix. Therefore you'll usually see
amplitudes in $\mathcal{N}=4$ written in terms

of the $SL(4)$-invariant inner product

$$\langle ijkl \rangle = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L$$

$$= \det \begin{pmatrix} | & | & | & | \\ Z_i & Z_j & Z_k & Z_l \\ | & | & | & | \end{pmatrix} ! \quad \square$$

---

you can also introduce

$$\lambda_i \eta_i = \theta_{i+1} - \theta_i$$

(analog to $p_i = x_{i+1} - x_i$)

the $Q, \bar{Q}$ SUSY generators are joined by partners $S, \bar{S}$.

altogether, the closure of all of the aforementioned symmetries $= \infty$-dimensional "Yangian" algebra.