A Survey of Quandle Theory

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Knots through Web, ICTS, August 24-28, 2020.

Overview of Knots and Motivation of Quandles

- A *knot* is the image of a smooth embeding $S^1 \to \mathbb{R}^3$ or S^3.

Knots and their diagrams

- Two knots K and K' are called isotopic if K' is obtained from K by continuous deformation with no self-intersection.

  Technically speaking, if there exists a smooth family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ for $t \in [0, 1]$ such that $h_0 = Id$ and $h_1(K) = K'$.

- One approach to study knot theory is combinatorial: using what's called *diagrams* (projection to the plane showing over- and under-crossings).

Knots and their diagrams

- Knot diagram = Image of a knot by a projection $\mathbb{R}^3 \to \mathbb{R}^2$ (finitely many transversal double points (crossings: over- and under-).

- For a well-known set S we call the map

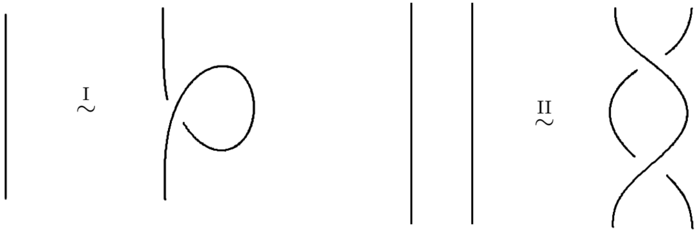
$$I : \{knots\} \to S$$

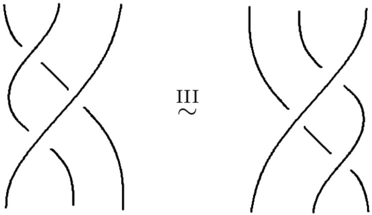
an isotopy invariant of knots, if $I(K) = I(K')$ for any two isotopic knots K and K'.

Knots and their diagrams

- *Reidemeister's theorem*: $\{$Knots$\}$/ isotopy of $\mathbb{R}^3$ = $\{$Knot Diagrams$\}$ / RI, RII, RIII and isotopy of $\mathbb{R}^2$.

Knots and their diagrams




III
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Bit of History of Quandles

•1940, Mituhisa Takasaki "Abstraction of symmetric transformations, Introduction to the theory of kei" (in Japanese).

• In 1982 Joyce "A classifying invariant of knots, the knot quandle".

• 1982 Matveev "Distributive groupoids in knot theory" (in Russian).

• 1988 Egbert V. Brieskorn "Automorphic sets and singularities".

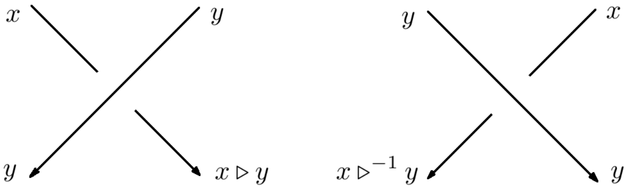
• Around 1990 L. Kauffman Louis introduced "Crystals"

A *Rack* $(X, \triangleright)$ is a set X with a binary operation $\triangleright$:

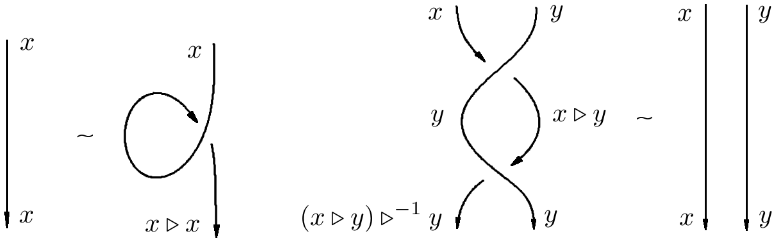
- $$R_x : y \longmapsto y \triangleright x \quad \text{is a bijective}$$
- $$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$$
- $$Quandle, if, further, x \triangleright x = x$$

Define an operation $\triangleright^{-1}$ by $x \triangleright^{-1} y = z \iff z \triangleright y = x$.

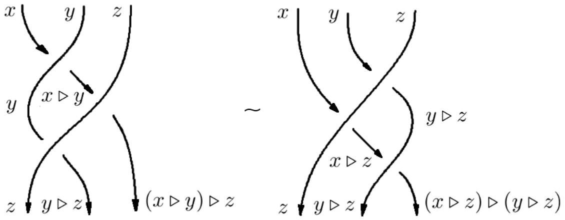
Coloring of arcs:



Now, the quandle axioms can be derived from knot diagrams:



Reidemeister move III and Right-Distributivity



Examples:

- $G =$ group and the binary operation is conjugation

$$x \triangleright y = yxy^{-1}$$

- $G =$ Core group.

$$x \triangleright y = yx^{-1}y$$

- The group $\mathbb{Z}_n$, $x \triangleright y = 2y - x$ (dihedral quandle, denoted R_n).
- *Takasaki* quandle: $G =$ any abelian group with $x \triangleright y = 2y - x$.
- Any $\mathbb{Z}[t, t^{-1}]$-module M is a quandle with

$$x \triangleright y = tx + (1 - t)y.$$

This is called *Alexander* quandle.

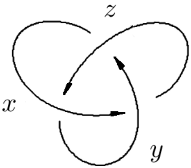
Knot Quandle

- The knot quandle (This is the reason of the whole theory)

  Consider a knot K and label the arcs

  At each crossing, one gets a relation like $x \triangleright y = z$.

  The quandle generated by the labeling of arcs with relations at each crossing is called the knot quandle and denoted $Q(K)$.



$$Q(K) = < x, y, z;\ x \triangleright^{-1} y = z,\ y \triangleright^{-1} z = x,\ z \triangleright^{-1} x = y >$$

which can be simplified to

$$Q(K) = < x, y;\ x \triangleright (y \triangleright x) = y,\ (y \triangleright x) \triangleright y = x >$$

Definition

Quandle homorphism:

$$f : (X, \triangleright) \longrightarrow (Y, *)$$

$$f(a \triangleright b) = f(a) * f(b).$$

If in addition f is bijective then we have quandle isomorphism.

Theorem (Joyce and Matveev, independently 1982)

Two knots K and L are (weakly) equivalent if and only if $Q(K)$ and $Q(L)$ are isomorphic as quandles.

Weakly equivalent means that $(\mathbb{R}^3, K)$ is homeomorphic to $(\mathbb{R}^3, L)$ when we ignore the orientations of $\mathbb{R}^3$ and the knots.

Let $m(K)$=mirror image of K and $r(K) = K$ with reverse orientation, then $Q(K) \cong Q(L) \iff L \equiv K$ or $L \equiv rm(K)$.

Notice that Q(K) doesn't distinguish between, for example, the trefoil knot and its mirror image.

Draw a right-handed trefoil and find its fundamental quandle then compare it with the fundamental quandle of left-handed trefoil.

The Theorem of Joyce and Matveev changes the problem of equivalence of knots (a topology problem) into a problem of isomorphism of quandles (algebra problem).

Table: Quandles up to isomorphism (M. Elhamdadi, J. Macquarrie and R. Restrepo, Automorphism groups of quandles. J. Algebra Appl. 11 (2012).

$n = cardinality \leq 9$	# of quandles
3	3
4	7
5	22
6	73
7	298
8	1581
9	11079

The following improvement was based on the classification of subgroups of small symmetric groups up to conjugation.

Table: Quandles up to isomorphism (P. Vojtěchovský and S. Yang, Enumeration of racks and quandles up to isomorphism. Math. Comp. 88 (2019), no. 319, 2523-2540

$9 < n \leq 13$	# of quandles
10	102771
11	1275419
12	21101335
13	469250886

Automomorphism groups of quandles

The set of all automorphisms of X is denoted $Aut(X)$.

The Inner automorphism group $Inn(X) :=< R_x, \ \ x \in X >$.

The equation $R_c \ R_b \ {R_c}^{-1} = R_{b \triangleright c}$ implies that the map $X \to Inn(X)$, $x \mapsto R_x$ is a quandle homorphism.

The transvection group $Transv(X) :=< R_x R_y^{-1}, \ x, y \in X >$.

Let $X = \mathbb{Z}_n$, $x \triangleright y = 2y - x$.

- $Aut(X) = \text{Aff}(\mathbb{Z}_n) := \{f_{a,b} : \mathbb{Z}_n \to \mathbb{Z}_n, \;\; f_{a,b}(x) = ax + b, \; a \in \mathbb{Z}_n^{\times}, \;\; b \in \mathbb{Z}\}$
- $Inn(X) = D_{\frac{m}{2}}$ where $m = l.c.m.(2, n)$.

Let $X = G_{conj}$ with $x \triangleright y = y^{-1}xy$

- $Inn(X) = G/Z(G)$
- $Aut(X) = Aut(G)$ iff $Z(G) = 1$

  (Bardakov-Nasybullov-Singh, 2017)

The Associated Group of a Quandle

Interpret the operation of a quandle X as a conjugation to get the Associated group $Assoc(X) := F(X)/N$ where N is the normal subgroup generated by $(x \triangleright y)yx^{-1}y^{-1}$.

Universal property: $\forall f : X \to G_{conj}, \exists! f_{\#} : Assoc(X) \to G$ s.t. the diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta} & Assoc(X) \\ f \downarrow & & \downarrow f_{\#} \\ G_{conj} & \xrightarrow{id} & G. \end{array}$$

$$Hom_{gr}(Assoc(X), G) \cong Hom_{qdle}(X, G_{conj})$$

Let $X = \mathbb{Z}_3$ with $x \triangleright y = 2y - x$

- $Assoc(X) = < x, y, \ \ xyx = yxy, \ \ x^2 = y^2 >$

  Furthermore, using, for connected quandle,

  $$Assoc(X) = [Assoc(X), Assoc(X)] \rtimes \mathbb{Z}$$

- $Assoc(X) = \mathbb{Z}_3 \rtimes \mathbb{Z}$

  (Bardakov-Nasybullov-Singh, 2017)

The group $Inn(X)$ acts on the quandle X. When the action is Transitive (only one orbit), the quandle is called Connected quandle.

- The dihedral quandles $\mathbb{Z}_{2n+1}$ are Connected.
- The dihedral quandles $\mathbb{Z}_{2n}$ have 2 orbits.

Some classes of quandles:

- A quandle is *medial (abelian)* if

$$(x \triangleright y) \triangleright (z \triangleright w) = (x \triangleright z) \triangleright (y \triangleright w).$$

  A quandle X is abelian iff $Transv(X)$ is abelian group (see Joyce's article)

- A quandle is *Latin* if its left multiplications are invertible maps.

- A quandle is *semi-Latin* if its left multiplications are injective maps.

Examples:

- The quandle Core($\mathbb{Z}$) is semi-Latin but not Latin.
- For an abelian group G, the quandle Core(G) is semi-Latin iff G has no 2-torsion.

Number of Colorings is an invariant of knots

A coloring of a knot K by a quandle X is a homorphism

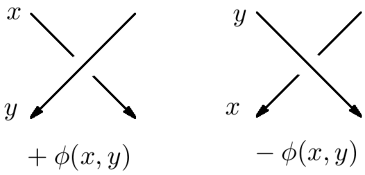
$$f : Q(K) \to X.$$

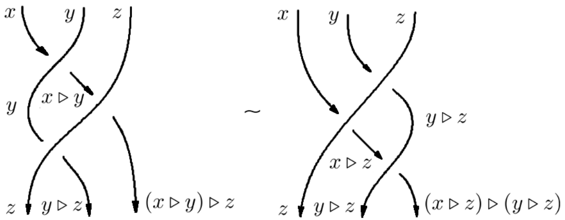
Theorem

Jozef Przytycki:
The number of colorings $Col_X(K)$ of a knot K by a quandle X is an invariant of knots.

Ref: 3-colorings and other elementary invariants of knots, Banach Center publications vol. 42, knot theory (1998), 275–295

2-cocycle




x
y
z
y
$x \triangleright y$
z
$y \triangleright z$
$(x \triangleright y) \triangleright z$
$\sim$
x
y
z
$y \triangleright z$
$x \triangleright z$
z
$y \triangleright z$
$(x \triangleright z) \triangleright (y \triangleright z)$

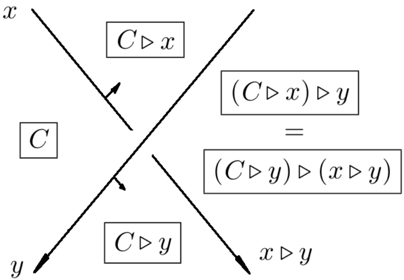
Let A be an abelian group.

A 2-cocycle is a function $\Phi : X \times X \to A$ such that $\Phi(x, x) = 0 \; \forall x \in X$, and

$$\Phi(x, y) + \Phi(x \triangleright y, z) = \Phi(x, z) + \Phi(x \triangleright z, y \triangleright z).$$

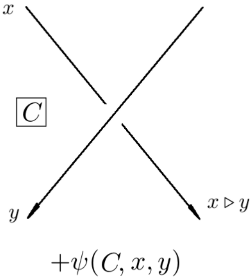
3-cocycle

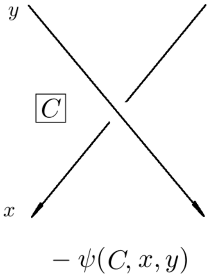
Region colorings:



3-cocycle

Weights and region colorings:





3-cocycle

3-cocycle: A function $\Psi : X \times X \times X \to A$ such that, $\forall x, y \in X$ $\Psi(x, x, y) = 0 = \Psi(x, y, y)$ and

$$\Psi(x, y, z) + \Psi(x, z, w) + \Psi(x \triangleright z, y \triangleright z, w) =$$
$$\Psi(x \triangleright y, z, w) + \Psi(x \triangleright w, y \triangleright w, z \triangleright w) + \Psi(x, y, w).$$

The chain complex

Let $C_n^R(X)$ = free abelian group generated by n-tuples $(x_1, ..., x_n) \in X^n$.

Define $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ by $\partial_n = 0$ for $n \leq 1$ and for $n \geq 2$,

$$\begin{aligned}\partial_n(x_1, x_2, ..., x_n) &= \sum_{i=2}^{n}(-1)^i[(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n)\\ &-(x_1 \triangleright x_i, x_2 \triangleright x_i, ..., x_{i-1} \triangleright x_i, x_{i+1}, ..., x_n)]\end{aligned}$$

This defines a chain complex $\{C_n^R(X), \partial_n\}$ ($\partial_n \circ \partial_{n+1} = 0$) which

gives rack homology theory (Fenn-Rourke-Sanderson)

The chain complex

Let $C_n^D(X)$ subset of $C_n^R(X)$ generated by n-tuples $(x_1, ..., x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, ..., n-1\}$ when $n \geq 2$.

For X quandle, $\partial_n(C_n^D(X)) \subset C_{n-1}^D(X)$ and define

$$C_n^Q(X) = C_n^R(X)/C_n^D(X).$$

For an abelian group A, define the chain and co-chain complexes $C_*^Q(X, A) = C_*^Q(X) \otimes A$ and $C_Q^*(X, A) = Hom(C_*^Q(X), A)$

$H_n^Q(X, A) := H_n(C_*^Q(X, A))$ and $H_Q^n(X, A) := H^n(C_*^Q(X, A))$

Examples

Some Computations for Dihedral quandles [CJKLS]

- $H^2_Q(R_3, A) \cong 0$, for any A and $H^3_Q(R_3, \mathbb{Z}_3) \cong \mathbb{Z}_3$
- $H^3_Q(R_3, A) \cong 0$ for any A without order 3 elements

- $H^2_Q(R_4, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^4$
- $H^2_Q(R_4, A) \cong A \times A$ for any A without order 2 elements

- $H^2_Q(R_5, A) \cong 0$, for any A and $H^3_Q(R_5, \mathbb{Z}_5) \cong \mathbb{Z}_5$
- $H^3_Q(R_5, A) \cong 0$ for any A without order 5 elements

H^3 of Dihedral quandles

In fact Mochizuki, in 2003, proved that for p prime

$$H^3_Q(R_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$$

and gave an explicit expression of generating 3-cocycle

$$\begin{aligned}\theta_p(x,y,z) &= 4(x-y)(y-z)z^{p-1} + (x-y)^2[(2z-y)^{p-1} - y^{p-1}] \\ &+(x-y)\frac{(2z-y)^p + y^p - 2z^p}{p}\end{aligned}$$

Note that the coefficients of $(2z-y)^p + y^p - 2z^p$ are divisible by p.

H_4 of Dihedral quandles

Maciej Niebrzydowski and Josef Przytycki proved, in 2008: For p odd prime $H_4^Q(R_p)$ contains $\mathbb{Z}_p$.

Conjecture:

$$H_n^Q(R_p) \cong \mathbb{Z}_p^{f_n}$$

Where f_n are "Delayed" Fibbonacci numbers:

$$f_n = f_{n-1} + f_{n-3}, \text{ and } f_1 = f_2 = 0,\ f_3 = 1.$$

Quandle Cocycle Invariant of Knots[CJKLS]

Let X be a finite quandle, A be an abelian group and ψ be a 2-cocycle $\psi : X \times X \to A$.

Definition

The $State - Sum$ invariant of a knot K is

$$\Phi(K) = \sum_{\mathcal{C}} \prod_{\tau} \psi(x,y)^{\epsilon(\tau)} \in \mathbb{Z}[A]$$

Where the product is taken over all crossings, the sum is over all possible colorings and $\epsilon(\tau)$ is the sign of the crossing τ.

$\Phi(K)$ is a knot invariant.

Notice that this invariant $\Phi(K)$ reduces to the number of coloring invariant $Col_X(K)$, when $\prod_\tau \psi(x,y)^{\epsilon(\tau)} = 1 \in A$, (this is the case when the 2-cocycle is a coboundary).

Thus $\Phi(K)$ can be seen as an *enhancement* of $Col_X(K)$. For example given two colorable knots K and K' by a quandle X such that $Col_X(K) = 9 = Col_X(K')$, $\Phi(K) = 6 + 3u$ and $\Phi(K') = 3 + 6u$ then K and K' are distinguished by Φ but not by Col.

Explicit calculations

- The torus link T(4,2):
  Let $X = R_4 = \{0, 1, 2, 3\}$ where $x \triangleright y = 2y - x \pmod 4$

  $A =$ group of integers $\mathbb{Z} =< t >$ (multiplicative notation).

  with the 2-cocycle $\psi = t^{\chi_{0,1}\chi_{0,3}}$.

  Any pair (a, b) in $R_4 \times R_4$ colors K. The 8 pairs (a, b) with $a + b$ being *odd*, each contributes t.
  All other pairs each contributes 1 so that the state-sum for this link is

  $$\Phi(K) = 8 + 8t$$

Generalized Homology Theory (Andruskiewitsch and Graña)

Let $(X, \triangleright)$ be a quandle, a *quandle module* is an abelian group A with a collection of automorphisms $\eta_{x,y} : A \to A$ and a collection of endomorphisms $\tau_{x,y} : A \to A$.

$$\eta_{x\triangleright y,z}\eta_{x,y} = \eta_{x\triangleright z,y\triangleright z}\eta_{x,z} \quad (1)$$

$$\eta_{x\triangleright y,z}\tau_{x,y} = \tau_{x\triangleright z,y\triangleright z}\eta_{y,z} \quad (2)$$

$$\tau_{x\triangleright y,z} = \eta_{x\triangleright z,y\triangleright z}\tau_{x,z} + \tau_{x\triangleright z,y\triangleright z}\tau_{y,z} \quad (3)$$

$$\tau_{x,x} + \eta_{x,x} = Id, \quad (4)$$

$(x, a) * (y, b) := (x \triangleright y, \eta_{x,y}(a) + \tau_{x,y}(b))$ gives a quandle on $X \times A$.

Let $A =$ abelian group and f be a group automorphism of A.

$$x \triangleright y := f(x) + (Id - f)(y)$$

Then A becomes a quandle module by $\eta_{x,y} := f$ and $\tau_{x,y} := Id - f$.

Now we define (Co)homology. Let $(X, \triangleright)$ be a quandle and A be an X-quandle module. Define

$$[x_1, x_2, \dots, x_n] = ((\cdots(x_1 \triangleright x_2) \triangleright x_3) \triangleright \cdots) \triangleright x_n.$$

Boundary operators $\partial = \partial_n : C_{n+1}(X) \to C_n(X)$ are defined by

$$\begin{aligned}\partial(x_1, \ldots, x_{n+1}) &= \sum_{i=2}^{n+1} (-1)^i \eta_{[x_1,\ldots,\widehat{x_i},\ldots,x_{n+1}],[x_i,\ldots,x_{n+1}]}(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}\\ &\quad - \sum_{i=2}^{n+1} (-1)^i (x_1 \triangleright x_i, \ldots, x_{i-1} \triangleright x_i, x_{i+1}, \ldots, x_{n+1})\\ &\quad + \tau_{[x_1,x_3,\ldots,x_{n+1}],[x_2,x_3,\ldots,x_{n+1}]}(x_2, \ldots, x_{n+1}),\end{aligned}$$

In particular, we have:

- $$\partial(x, y) = \eta_{x,y}(x) - (x \triangleright y) + \tau_{x,y}(y)$$
- $\partial(x, y, z) = \eta_{x \triangleright z, y \triangleright z}(x, z) - (x \triangleright y, z) + \tau_{x \triangleright z, y \triangleright z}(y, z) - \eta_{x \triangleright y, z}(x, y) + (x * z, y * z)$.

Topological quandles

A *topological quandle* is a topological space with a binary operation $\triangleright$:

(right) rack $(X, \triangleright)$,

- $$(x, y) \longmapsto x \triangleright y \quad \text{is a continuous}$$

- $$R_x : y \longmapsto y \triangleright x \quad \text{is a homeomorphism}$$

- Fix $\mathbb{R} \ni t \neq 0$, $x * y = tx + (1-t)y$.
- $G =$ topological group.

$$x \triangleright y = yxy^{-1}$$

- $G =$ topological group.

$$x \triangleright y = yx^{-1}y$$

- Riemannian manifold M and an isometry $i_x : M \circlearrowleft$.

$$x \triangleright y := i_y(x).$$

- S^n = the unit sphere

$$x \triangleright y = 2(x \cdot y)y - x.$$

- $$\lambda x \triangleright \mu y = \lambda[2\mu^2(x \cdot y)y - x].$$

$$\pm x \triangleright \pm y = \pm(x \triangleright y),$$

$$\implies \mathbb{RP}^n = \text{quandle}.$$

- Let $M \neq I_2 \in GL_2(\mathbb{Z})$. On $\mathbb{R}^2$, define

$$\vec{x} \triangleright \vec{y} := M\vec{x} + (I_2 - M)\vec{y}.$$

Notice that for $\vec{m}, \vec{n} \in \mathbb{Z}^2$,

$$(\vec{x} + \vec{m}) \triangleright (\vec{y} + \vec{n}) = \vec{x} \triangleright \vec{y} + \vec{m} \triangleright \vec{n}$$

Since $\vec{m} \triangleright \vec{n} \in \mathbb{Z}^2$ thus a quandle structure on $T^2 = S^1 \times S^1$.

Observe that this example can be generalized to an n-torus for $n \geq 2$.

Definition

Quandle Homorphism:

$$f : X \longrightarrow Y \text{continous}$$

$$f(x \triangleright y) = f(x) \triangleright f(y).$$

If furthermore f is homeomorphism then it is called isomorphism.

$X = \{1, 2, 3\}$, with quandle operations

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix}.$$

Let $\tau_1 = \{\emptyset, \{1\}, \{1, 2, 3\}\}$ and
$\tau_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$ be two topologies on X.

(X, τ_1) and (X, τ_2) are Non-isomorphic topological quandles since they are not even homeomorphic as topological spaces.

Definition

Let $(G, +)$ be a topological abelian group and let f be a continuous automorphism of G. The continuous binary operation on G given by $x \triangleright y = f(x) + (Id - f)(y), \forall x, y \in G$, makes $(G, \triangleright)$ a topological quandle called *topological Alexander quandle*.

In particular, if $G = \mathbb{R}$ and $\sigma(x) = tx$ for $t \neq 0$, we have a topological Alexander structure on $\mathbb{R}$ given by
$x \triangleright y = tx + (1 - t)y$.

Classification of Topological Alexander quandles on $\mathbb{R}$:

For non-zero $a, b \in \mathbb{R}$ define:

$x *_1 y := ax + (1-a)y$ and $x *_2 y := bx + (1-b)y$.

Theorem

Cheng Z., E.M., Shekhtman B., 2018:
The topological quandles $(\mathbb{R}, *_1)$ *and* $(\mathbb{R}, *_1)$ *are isomorphic iff* $a = b$.

First easy case:

Lemma

If $t_1 = 1$ and $t_2 \neq 1$, then $(\mathbb{R}, *_1)$ and $(\mathbb{R}, *_2)$ are different topological quandles.

Proof.

If there exists a homeomorphism f of $\mathbb{R}$ which induces an isomorphism on the quandle structure, then

$$f(x) = f(x *_1 y) = f(x) *_2 f(y) = t_2 f(x) + (1 - t_2) f(y),$$

which implies f is a constant function. Contradiction. □

Lemma

If $t_1 > 1$ *and* $0 < t_2 < 1$, *then* $(\mathbb{R}, *_1)$ *and* $(\mathbb{R}, *_2)$ *are different topological quandles.*

Proof.

Let f be a homeomorphism of $\mathbb{R}$ such that:

$$f(t_1 x + (1 - t_1)y) = t_2 f(x) + (1 - t_2)f(y).$$

Note that since $f(x) + b$ also gives an isomorphism from $(\mathbb{R}, *_1)$ to $(\mathbb{R}, *_2)$, we assume that $f(0) = 0$. Thus $f(t_1 x) = t_2 f(x)$. Contradiction since f is monotonic. □

Lemma

If $t_1 > t_2 > 1$ then $(\mathbb{R}, *_1)$ and $(\mathbb{R}, *_2)$ are different topological quandles.

Proof.

Idea of Proof: Assume $\phi(t_1 x + (1-t_1)y) = t_2\phi(x) + (1-t_2)\phi(y)$. Assume $\phi(0) = 0$ then $\phi(1) \neq 0$. Now assume $\phi(1) = 1$. Then $\phi(t_1 x) = t_2\phi(x)$ and $\phi((1-t_1)x) = (1-t_2)\phi(x)$. Thus $\forall m, n \in \mathbb{Z}$ we have $\phi(\frac{t_1^m}{(1-t_1)^{2n}}x) = \frac{t_2^m}{(1-t_2)^{2n}}\phi(x)$ thus
$\phi(\frac{t_1^m}{(1-t_1)^{2n}}) = \frac{t_2^m}{(1-t_2)^{2n}}$. □

Proof.

Assume $\frac{\ln(1-t_1)^2}{\ln(t_1)}$ is a irrational number. Now we can choose a sequence $\{\frac{m_i}{n_i}\}_{i\in\mathbb{N}}$ which converges to $\frac{\ln(1-t_1)^2}{\ln(t_1)}$ (if $\frac{\ln(1-t_1)^2}{\ln(t_1)}$ equals a rational number $\frac{m}{n}$, we just choose $\frac{m_i}{n_i} = \frac{m}{n}$ for any $i \in \mathbb{N}$), which means that $\phi(\frac{t_1^{m_i}}{(1-t_1)^{2n_i}})$ converges to $\phi(1) = 1$. On the righthand side we have $\lim_{i\to\infty} \frac{t_2^{m_i}}{(1-t_2)^{2n_i}} = \lim_{i\to\infty} \exp(m_i \ln(t_2) - n_i \ln(1-t_2)^2)$.
To get a contradiction one needs to show that $\lim_{i\to\infty}(m_i \ln(t_2) - n_i \ln(1-t_2)^2) \neq 0$. One computes $\lim_{i\to\infty}(m_i \ln(t_2) - n_i \ln(1-t_2)^2) = \lim_{i\to\infty}(\frac{m_i}{n_i} \ln(t_2) n_i - n_i \ln(1-t_2)^2) = \lim \ n_i(\frac{\ln(t_1-1)^2 \ln(t_2) - \ln(t_2-1)^2 \ln(t_1)}{\ln(t_1)}) \neq 0.$ □

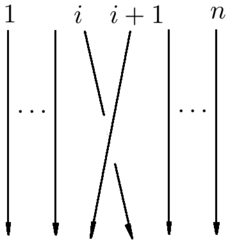
Action of the Braid group B_n on X^n

The braid group B_n on n strings is given by generators $\sigma_1, \dots, \sigma_{n-1}$ with relations:

(I) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$,

(II) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

$\sigma_i =$



$\sigma_i^{-1} =$

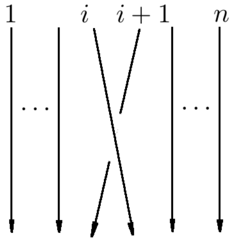
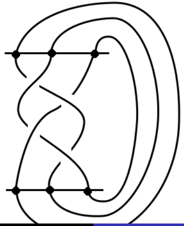


Figure eight knot. Its braid index=3 and its braid form $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$



The braid group B_n acts on X^n by

$$\sigma_i(x_1, \dots, x_n) := (x_1, \dots x_{i-1}, x_{i+1}, x_i \triangleright x_{i+1}, x_{i+2} \dots, x_n).$$

The action of B_n on X^n gives a continous map $\sigma : X^n \longrightarrow X^n$.

Definition

R.L. Rubinsztein
Let $K = cl(w)$ be a knot that is the closure of a braid $w \in B_n$. The space of *fixed points* of σ is the space of Colorings $Col_X(K)$ (well defined up to homeomorphism).

Here are few Examples taken from R.L. Rubinsztein's paper

Example

- $Col_{S^2}(3_1) = S^2 \sqcup \mathbb{RP}^3$.
- $Col_{S^2}(4_1) = S^2 \sqcup \mathbb{RP}^3 \sqcup \mathbb{RP}^3$.
- $Col_{S^2}(5_1) = S^2 \sqcup S^2 \sqcup \mathbb{RP}^3 \sqcup \mathbb{RP}^3$.
- $Col_{S^2}(5_2) = S^2 \sqcup \mathbb{RP}^3 \sqcup \mathbb{RP}^3$.

Note: There are some analogies between these computations and the computations of Khonaov homology of these knots.

Group Algebras: Important because of their applications in the theory of group representations.

R.D. Brauer; E. Noether, I. Schur, I. Kaplansky, ...

Book by: Passman, D.S.: The algebraic Structure of Group Rings, 1977, Pure Applied Math Wiley.

Book by Passi, Inder Biir S.: Group rings and their augmentation ideals. LNM, 715. Springer, 1979.

Conjecture

Kaplansky's zero divisor conjecture:
For a torsion-free group G and a field $\mathbb{F}$, the group ring $\mathbb{F}[G]$ has no zero divisors.

Conjecture

Kaplansky's idempotent conjecture:
For a torsion-free group G and a field $\mathbb{F}$, the group ring $\mathbb{F}[G]$ has no non-trivial idempotent.

Bardakov, Passi and Singh investigated an anologue of Kaplansky's conjecture over quandle rings.

Quandle Rings

Let $\mathbf{k}[X] \ni \sum_{x\in X} a_x e_x$, where $a_x = 0$ for almost all x.

Then $\mathbf{k}[X]$ becomes a ring with:

$$(\sum_{x\in X} a_x e_x) + (\sum_{x\in X} b_x e_x) = \sum_{x\in X} (a_x + b_x) e_x.$$

$$(\sum_{x\in X} a_x e_x) \cdot (\sum_{y\in X} b_y e_y) = \sum_{x,y\in X} a_x b_y e_{x\triangleright y}.$$

The quandle operation of $(X, \triangleright)$ is associative if and only if

$$x \triangleright y = x, \forall x, y \in X.$$

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright z \implies x = x \triangleright z, \forall x, z \in X.$$

Thus if $(X, \triangleright)$ is non-trivial quandle then the quandle ring $\mathbf{k}[X]$ is a non-associative ring.

Left identity in a quandle:

If $e \triangleright x = x, \forall x \in X$ then $X = \{e\}$.

Thus if the X is not a singleton then the quandle ring $\mathbf{k}[X]$ doesn't have a unity.

Augmentation ideal:

The augmentation map is $\epsilon : \mathbf{k}[X] \to \mathbf{k}$

$$\epsilon(\sum_{x \in X} a_x e_x) = \sum_{x \in X} a_x$$

$\Delta_{\mathbf{k}}(X) := ker(\epsilon)$ is a 2-sided ideal called Augmentation ideal.

Thus a ring isomorphism:

$$\mathbf{k}[X]/\Delta_{\mathbf{k}}(X) \cong \mathbf{k}.$$

Fix $x_0 \in X$, then $\Delta_{\mathbf{k}}(X) = < e_x - e_{x_0}, x \in X >$.

Characterization of trivial quandles in term of their augmentation ideals

Theorem

Bardakon-Passi-Singh, 2017:

The quandle $(X, \triangleright)$ *is trivial if and only if* $\Delta^2_{\mathbf{k}}(X) = \{0\}$.

Power associativity of Quandle Rings

There is a notion slightly weaker than associativity, called, power associativity.

But first let's recall the following definition from an article of A. A. Albert "Power Associative rings" Trans. AMS 1948.

Definition

A ring $\mathbf{R}$ in which every element generates an associative subring is called a *power-associative* ring.

Example

Any alternative algebra is power associative. Recall that an algebra A is called *alternative* if $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ and $x \cdot (y \cdot y) = (x \cdot y) \cdot y, \forall x, y \in A$.
The 8-dim algebra of Octonions is an example of non-associative alternative algebra.

It is well known (A. A. Albert) that a ring **R** of characteristic zero is power-associative if and only if, for all $x \in \mathbf{R}$,

$$(x \cdot x) \cdot x = x \cdot (x \cdot x) \textit{ and } (x \cdot x) \cdot (x \cdot x) = [(x \cdot x) \cdot x] \cdot x$$

Theorem (Bardakov-Passi-Singh, 2017)

Let $\mathbf{k}$ *be a ring of characteristic not equal 2,3 or 5 and let* $(X = \mathbb{Z}_3, \triangleright)$ *be the dihedral quandle. Then the quandle ring* $\mathbf{k}[X]$ *is not power associative.*

Theorem (Bardakov-Passi-Singh, 2017)

Let $\mathbf{k}$ be a ring of characteristic not equal 2 and let $(X = \mathbb{Z}_n, \triangleright)$ be the dihedral quandle where $n > 3$. Then the quandle ring $\mathbf{k}[X]$ is not power associative.

Quandle rings are, in general, not power associative when the ring has characteristic zero.

Theorem (E., Fernando, Tsvelikhovskiy, 2018)

Let $\mathbf{k}$ *be a ring of characteristic zero and let* $(X, \triangleright)$ *be a non-trivial quandle. Then the quandle ring* $\mathbf{k}[X]$ *is not power associative.*

Proof.

Proof by contradiction is based on 2 cases:
(a)$\exists\, x \neq y, x \triangleright y = y \triangleright x$.
Let $u = ax + by$. $u \cdot u = a^2 x + 2ab(x \triangleright y) + b^2 y$ then
$(u \cdot u) \cdot u = u \cdot (u \cdot u)$.

$$((u \cdot u) \cdot u) \cdot u = (u \cdot u) \cdot (u \cdot u) \implies x \triangleright y = (x \triangleright y) \triangleright y$$

Thus $x = x \triangleright y = y \triangleright x$. Contradiction. □

Proof.

Proof by contradiction on second case:
(b) $\forall x, y$ s.t. $x \neq y$, $x \triangleright y \neq y \triangleright x$.

Let $u = ax + by$. $u \cdot u = a^2x + ab(x \triangleright y) + ab(y \triangleright x) + b^2y$ then

$$(u \cdot u) \cdot u = u \cdot (u \cdot u) \implies x \triangleright y + (y \triangleright x) \triangleright x = y \triangleright x + x \triangleright (x \triangleright y)$$

Thus $x \triangleright (x \triangleright y) = x \triangleright y$. Contradiction. □

Bardakov, Passi and Singh raised the question:

$$\mathbf{k}[X] \cong \mathbf{k}[Y] \implies X \cong Y?$$

Now we give two examples, when the quandle rings $\mathbf{k}[X]$ and $\mathbf{k}[Y]$ are isomorphic, but the quandles X and Y are not.

Example

Let $\mathbf{k}$ be a field with $char(\mathbf{k}) = 3$ and quandles X and Y:

$X =$

$\triangleright$	e_1	e_2	e_3	e_4
e_1	e_1	e_1	e_2	e_2
e_2	e_2	e_2	e_1	e_1
e_3	e_3	e_3	e_3	e_3
e_4	e_4	e_4	e_4	e_4

$Y =$

$\triangleright$	e'_1	e'_2	e'_3	e'_4
e'_1	e'_1	e'_1	e'_2	e'_1
e'_2	e'_2	e'_2	e'_1	e'_2
e'_3	e'_3	e'_3	e'_3	e'_3
e'_4	e'_4	e'_4	e'_4	e'_4

The isomorphism is given $\varphi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} : \mathbf{k}[X] \xrightarrow{\sim} \mathbf{k}[Y]$

Generalization from the previous Example:

$\tilde{X} = X \amalg \{e_5\} \amalg \{e_6\} \amalg \ldots \amalg \{e_n\}$ and
$\tilde{Y} = Y \amalg \{e_5\} \amalg \{e_6\} \amalg \ldots \amalg \{e_n\}$.

Clearly the quandles $\tilde{X}$ and $\tilde{Y}$ are not isomorphic. Let $\mathbf{k}$ be a field with $char(\mathbf{k}) = p$ with $p \mid n-1$. Then $\varphi : \mathbf{k}[X] \xrightarrow{\sim} \mathbf{k}[Y]$ given by $\varphi(e_i) = e'_i$ for $i \neq 4$ and $\varphi(e_4) = \sum_{j=1}^{n} e'_j$ is a ring isomorphism.

Example

Let $\mathbf{k}$ be a field with $char(\mathbf{k}) = 5$ and quandles X and Y:

$\triangleright$	e_1	e_2	e_3	e_4	e_5	e_6
e_1	e_1	e_1	e_1	e_1	e_1	e_2
e_2	e_2	e_2	e_2	e_2	e_2	e_1
e_3	e_3	e_3	e_3	e_3	e_4	e_4
e_4	e_4	e_4	e_4	e_4	e_3	e_3
e_5	e_5	e_5	e_5	e_5	e_5	e_5
e_6	e_6	e_6	e_6	e_6	e_6	e_6

,

$\triangleright$	e'_1	e'_2	e'_3	e'_4	e'_5	e'_6
e'_1	e'_1	e'_1	e'_1	e'_1	e'_1	e'_2
e'_2	e'_2	e'_2	e'_2	e'_2	e'_2	e'_1
e'_3	e'_3	e'_3	e'_3	e'_3	e'_4	e'_3
e'_4	e'_4	e'_4	e'_4	e'_4	e'_3	e'_4
e'_5	e'_5	e'_5	e'_5	e'_5	e'_5	e'_5
e'_6	e'_6	e'_6	e'_6	e'_6	e'_6	e'_6

The isomorphism: $\phi(e_i) = e'_i$ for $1 \leq i \leq 5$, $\phi(e_6) = \sum^{6} e'$

Proposition

Let $X = X_1 \amalg X_2$ be a quandle with $|X_1| < \infty$. Then $\mathbf{k}[X]$ is not an integral domain.

Proof.

Indeed, it follows from property (II) of Quandle Definition that

$$(\sum_{z \in X_1} z) \cdot (x - y) = 0,$$

where x and y are any two distinct elements of X. □

Are there other quandles for which the ring $\mathbf{k}[X]$ is a domain?

Bardakov, Passi and Singh introduced the notion of *up-quandles* (unique product quandle) a class for which $\mathbf{k}[X]$ has no zero divisors.

Definition

Bardakov, Passi and Singh, (January 2020):
A quandle X is *up-quandle* if $\forall\ A, B \neq \emptyset$ finite subsets of X, $\exists x \in X$ that has a unique representation $x = ab$ for some $a \in A$ and $b \in B$.

Proposition

Bardakov, Passi and Singh, (January 2020):
Assume that $\mathbf{k}$ is a domain and that X is an up-quandle then $\mathbf{k}[X]$ has no zero divisors.

Proof.

Let $u = \sum_{i=1}^{n} a_i x_i$ and $v = \sum_{j=1}^{n} b_j y_j$ then $u \cdot v = a_1 b_1 x_1 \triangleright y_1 + w$.

Choose $A = \{x_i\}$ and $B = \{y_i\}$.

Since X is an *up-quandle* then $z = x_1 y_1$ is unique and thus $u \cdot v \neq 0$. □

In order to state an analogue of Kaplansky's zero-divisors conjecture

Definition

Bardakov, Passi and Singh:
A quandle X (with $|X| > 1$) is called inert if there exists a finite subset $A = \{a_1, \ldots, a_n\}$ and 2 distinct elements x and y such that $Ax = Ay$.

Conjecture

Bardakov, Passi and Singh:
Let $\mathbf{k}$ *be an integral domain and* X *be a non-inert semi-Latin quandle. Then the quandle ring* $\mathbf{k}[X]$ *has no-zero divisors.*

THANK YOU!

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