The Fibonacci family of non-equilibrium universality classes

Gunter M. Schütz

Institute of Complex Systems II, Forschungszentrum Jülich, 52425 Jülich, Germany

and

Interdisziplinäres Zentrum für Komplexe Systeme, Universität Bonn

- Introduction
- Nonlinear fluctuating hydrodynamics
- Mode coupling theory
- Simulation results
- Conclusions

Joint work with V.Popkov, A. Schadschneider, J. Schmidt
[PRL (2014), JSP (2015), PNAS (2015)]

1. Introduction

A) Fibonacci numbers and the Golden Mean:

- Fibonacci numbers: $F_i = F_{i-1} + F_{i-2}$, $F_1 = F_2 = 1$

  $F_i = 1, 1, 2, 3, 5, 8, 13, \ldots$

- Kepler ratios $K_i = F_{i+1} / F_i$

  $K_i = 1, 2, 3/2, 5/3, 8/5, 13/8, \ldots \rightarrow \varphi = (1+\sqrt{5})/2 \approx 1.618$ (Golden Mean)

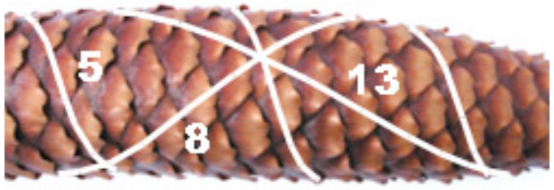
- Continued Fraction:

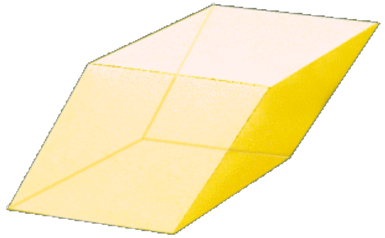
$$\varphi = [1; 1, 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

Finite truncation yields Kepler ratios

Significance:

Patterns in Biology, Physics, Architecture, Arts,… Mathematics



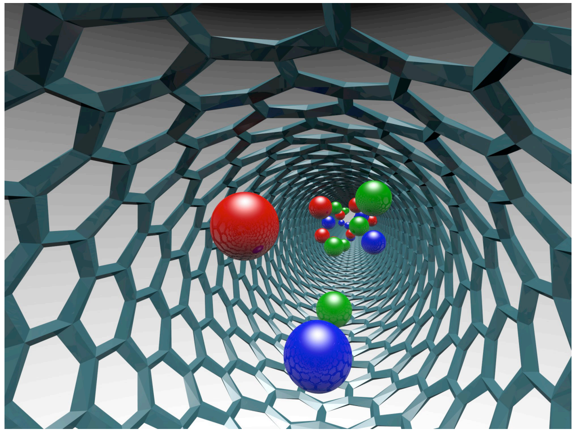
Some of the greatest mathematical minds of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa and the Renaissance astronomer Johannes Kepler, to present-day scientific figures such as Oxford physicist Roger Penrose, have spent endless hours over this simple ratio and its properties. But the fascination with the Golden Ratio is not confined just to mathematicians. Biologists, artists, musicians, historians, architects, psychologists, and even mystics have pondered and debated the basis of its ubiquity and appeal. In fact, it is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics.

Mario Livio,*The Golden Ratio: The Story of Phi, The World's Most Astonishing Number*, p.6

B) Bulk-driven particle systems with several conservation laws:

- 1-D diffusive interacting particle systems with driving field
- Many Applications:

  (1) Diffusion in carbon nanotubes, zeolites, …
  (2) Interface growth
  (3) Molecular Motors
  (4) Automobile traffic flow
  …

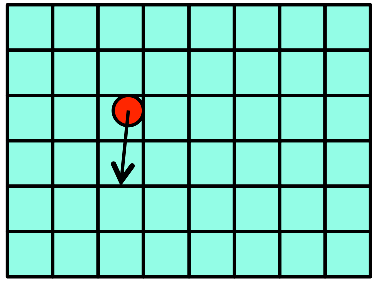


- Rich behaviour, e.g.,

  - Phase transitions
  - Hysteresis, phase separation, spontaneous symmetry breaking
  - Nonlinear hydrodynamic equations
  - Shocks and rarefaction waves
  - Universal fluctuations: $z=2$ (Diffusive), $z=3/2$ (KPZ)

Lattice gas models for driven diffusive systems:

Imagine video imaging of a Brownian particle:

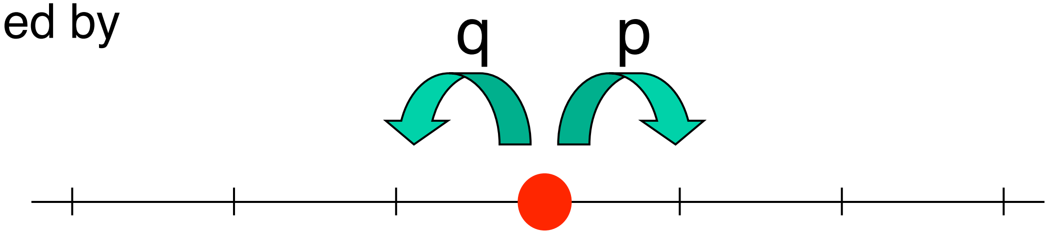


- Pixels form a grid (lattice)

➔ Continuous random motion appears as random jumps between neighboring pixels

➔ Brownian motion modeled by

(biased) random walk



- Many particles: Interacting random walks, Particle species $\alpha = 1, \ldots, n$
  Particle densities $\rho_\alpha = N_\alpha / L$
- Randomness: Expected particle occupation $\rho_\alpha(k,\tau) = \langle \xi_\alpha(k,\tau) \rangle$
- Driving force: Relaxation to non-equilibrium distribution $P \neq \exp(-\beta E) / Z$

Non-equilibrium steady states:

Assume translation invariance and ergodicity for fixed values of conserved particle numbers N_α (periodic boundary conditions)

➔ Translation invariance: Stationary density $\rho_\alpha(k) = \rho_\alpha$

➔ “Canonical” non-equilibrium distribution P ($\{\rho\}$) is unique

Construct “grand-canonical” distribution P($\{\mu\}$) with chemical potentials μ_α

- Densities of particles of type α: $\rho_\alpha(\{\mu\}) = 1/L \langle N_\alpha \rangle$

- Stationary Fluctuations: Compressibility matrix K

$$(K)_{\alpha\beta} = \partial\rho_\alpha / \partial\mu_\beta = 1/L \langle (N_\alpha - L\rho_\alpha)(N_\beta - L\rho_\beta) \rangle$$

- Stationary currents of particles of type α: $j_\alpha(\{\mu\})$

Onsager-type current symmetry $\partial j_\alpha/\partial\mu_\beta = \partial j_\beta/\partial\mu_\alpha$ [Grisi, GMS (2011)]

Current Jacobian A: $(A)_{\alpha\beta} = \partial j_\alpha/\partial\rho_\beta$

➔ $\Upsilon := AK = (AK)^T$ is symmetric (far from equilibrium!)

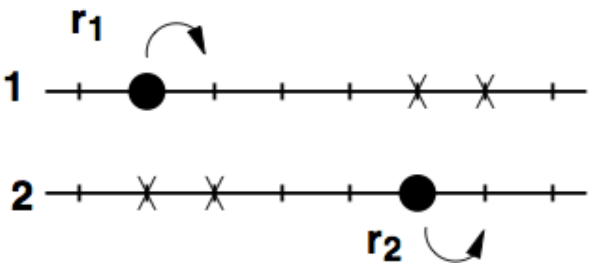
Relates stationarity property K with dynamical property A!

- Dynamical Fluctuations: Dynamical structure function $\check{S}_{\alpha\beta}(k,\tau)$

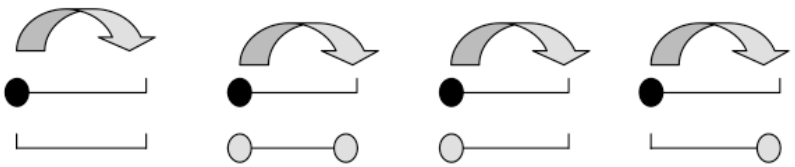
$\check{S}_{\alpha\beta}(k,\tau) = \langle(\xi_\alpha(k,\tau) - \rho_\alpha)(\xi_\beta(0,0) - \rho_\beta)\rangle = ??$

- Model: Interacting multi-lane TASEPs with densities ρ_l

For simplicity focus on two-lane model:



- Hopping rates: $r_1 = 1 + \gamma\, n^{(2)}/2$, $r_2 = b + \gamma\, n^{(1)}/2$ [Popkov, Salerno (2004)]



- Invariant measure: Canonical: Uniform ➔ Grandcanonical: Product

- Compressibility: $(K)_{\alpha\beta} = \rho_\alpha\,(1- \rho_\alpha)\,\delta_{\alpha,\beta}$

- Currents:

$$j_1(\rho_1,\rho_2) = \rho_1(1-\rho_1)(1+\gamma\rho_2)$$
$$j_2(\rho_1,\rho_2) = \rho_2(1-\rho_2)(b+\gamma\rho_1)$$

2. Nonlinear fluctuating hydrodynamics

Study large-scale dynamics under Eulerian scaling x = ka, t = τa,   a $\to$ 0:

==> Lattice continuity equation $\to$ system of hyperbolic conservation laws

$$\frac{\partial}{\partial t}\vec{\rho} + A \frac{\partial}{\partial x}\vec{\rho} = 0$$

- ◆ Current Jacobian A= A({ρ(x,t)}) nonlinear in densities ρ(x,t)
- ◆ Origin of hyperbolicity (real eigenvalues of A): Onsager-type symmetry
- ◆ General validity: Driven Diffusive Systems, Anharmonic chains, Hamiltonian dynamics, …

Density fluctuations: (1) Introduce fluctuation fields $u_\alpha(x,t) = \rho_\alpha(x,t) - \rho_\alpha$

(2) Expand PDE in u_α

(3) Add phenomenological diffusion and noise

A) <u>Noiseless linear theory:</u>

- Linearized hydrodynamic equation:

$$\partial_t u_\alpha = - \Sigma_\beta A_{\alpha\beta} \partial_x u_\beta$$

- Diagonalize A= A({ρ}): $RAR^{-1} = \mathrm{diag}(v_\alpha)$, Normalization $RKR^T = 1$

==> Eigenmode equation for normal modes $\phi = R\, \mathbf{u}$:

$$\partial_t \phi_\alpha = - v_\alpha \partial_x \phi_\alpha$$

Solution: Travelling waves $\phi_\alpha(x,t) = \phi_\alpha(x-v_\alpha t)$

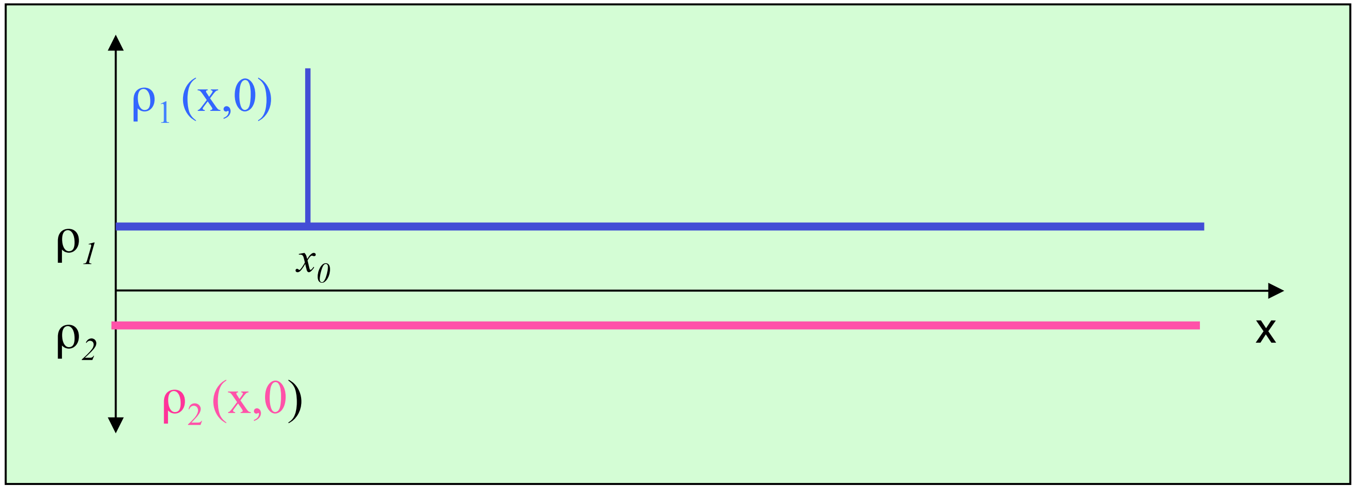
- Characteristic speeds $v_\alpha(\{\rho\})$ = eigenvalues of current Jacobian A

- Strict hyperbolicity: $v_\alpha \neq v_\beta$

Two-lane model:

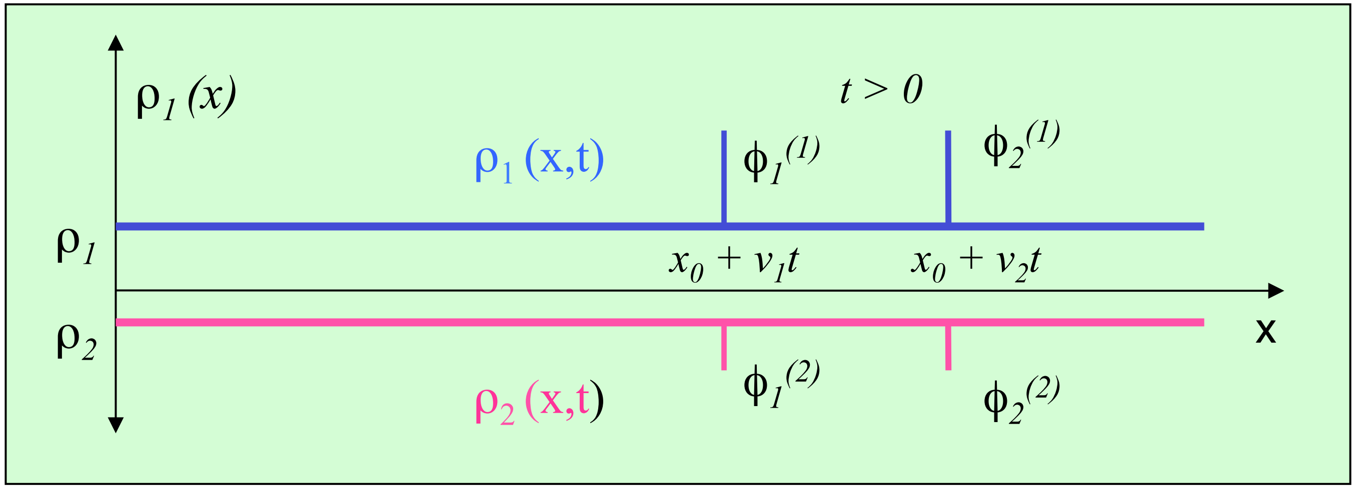
Initial density profile at t = 0

Blue: Lane 1
Magenta: Lane 2



Density profile at t > 0:

(a) Excitation due to interaction
(b) Splitting into eigenmodes



Normal modes well separated at time t!

B) Nonlinear fluctuating theory

- Expand to second order, add phenomenological diffusion term and noise [Spohn (2014)]

$$\partial_t \phi_\alpha = - \partial_x [v_\alpha \phi_\alpha + ½ \Sigma_{\beta\gamma} \phi_\beta G_{\beta\gamma}{}^{\alpha} \phi_\gamma - \partial_x \Sigma_\beta D_{\alpha\beta}\phi_\beta + \Sigma_\beta B_{\alpha\beta}\eta_\beta$$

Noise strength B and diffusion matrix D related by FDT

- Mode coupling matrices G^α for eigenmodes

$$G^\alpha = \Sigma_\beta R_{\alpha\beta} \ (R^{-1})^T H^\beta R^{-1}$$

- Hessian H^γ with matrix elements $\partial^2 j_\gamma / (\partial\rho_\alpha \partial\rho_\beta)$

One component: $\partial_t \phi = - \partial_x [v \phi + g \phi^2 - D \partial_x \phi + B \xi]$

KPZ equation $\phi(x,t) = \partial_x h(x,t)$, $g = j''/2)$

Two components ==> Two coupled KPZ equations

==>

$$\begin{cases} \partial_t \phi_1 = - \partial_x [v_1 \phi_1 + G_{11}^1 (\phi_1)^2 + G_{22}^1 (\phi_2)^2 + \text{diff.} + \text{noise}] \\ \partial_t \phi_2 = - \partial_x [v_2 \phi_2 + G_{11}^2 (\phi_1)^2 + G_{22}^2 (\phi_2)^2 + \text{diff.} + \text{noise}] \end{cases}$$

Remarks:

1) Higher order terms irrelevant in RG sense (if second order non-zero)

2) Self-coupling terms $G_{\alpha\alpha}^{\alpha}$ leading, other diagonal terms $G_{\beta\beta}^{\alpha}$ subleading

3) Offdiagonal terms $G_{\alpha\beta}^{\gamma}$ neglible for strictly hyperbolic systems (no overlap between modes)

3. Mode coupling theory

Study fluctuations of eigenmodes using dynamical structure function:

Transform to eigenmodes Š → S using R

- Large scales: $S_{\alpha\beta}(x,t) = \langle\phi_\alpha(x,t)\ \phi_\beta(0,0)\rangle$

- Strict hyperbolicity: $S_{\alpha\beta}(x,t) = S_\alpha(x,t)\ \delta_{\alpha\beta}$

- Mode coupling equations: Represent effect of noise by memory

$$\partial_t S_\alpha(x,t) = \left(-v_\alpha \partial_x + D_\alpha \partial_x^2\right) S_\alpha(x,t) + \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y S_\alpha(x-y,t-s)\partial_y^2 M_{\alpha\alpha}(y,s)$$

$$M_{\alpha\alpha}(y,s) = \sum_{\beta,\gamma} \left(G^\alpha_{\beta\gamma}\right)^2 S_\beta(y,s) S_\gamma(y,s)$$

❖ Exact scaling solution with Fourier and Laplace transform $(x,t) \to (p,\omega)$! ❖

General solution of mode coupling equations (noise → memory kernel)

Scaling ansatz, with $\tilde{\omega}_\alpha := \omega + i v_\alpha p \;\; \zeta_\alpha = \tilde{\omega}_\alpha |p|^{-z_\alpha}$

$$\tilde{S}_\alpha(p, \tilde{\omega}_\alpha) = p^{-z_\alpha} g_\alpha(\zeta_\alpha)$$

Subballistic scaling z>1 (short range interactions) and strict hyperbolicity:

$$g_\alpha(\zeta_\alpha) = \lim_{p \to 0} \left[\zeta_\alpha + D_\alpha |p|^{2-z_\alpha} + Q_{\alpha\alpha} \zeta_\alpha^{2-z_\alpha-\frac{1}{z_\alpha}} |p|^{3-2z_\alpha} + \sum_{\beta \neq \alpha} Q_{\alpha\beta} \left(-i v_p^{\alpha\beta}\right)^{\frac{1}{z_\beta}-1} |p|^{1+\frac{1}{z_\beta}-z_\alpha} \right]^{-1}$$

with $\quad v_p^{\alpha\beta} := |v_\alpha - v_\beta| \mathrm{sgn}[p(v_\alpha - v_\beta)]$

$$Q_{\alpha\beta} = 2(G^\alpha_{\beta\beta})^2 \Gamma\left(1 - \frac{1}{z_\beta}\right) \Omega[\hat{S}_\beta] \geq 0$$

$$\Omega[\hat{f}] = \int_{-\infty}^{\infty} \mathrm{d}p \, \hat{f}(p) \hat{f}(-p)$$ (Integral over square of real-space scaling function)

Mode coupling scenarios [van Beijeren (2012), Spohn (2013), Popkov, Schmidt, GMS (2014)]

Some scenarios for two-component systems:

A) Both self-coupling coefficients nonzero: $G_{11}^1 \neq 0$, $G_{22}^2 \neq 0$

==> two KPZ modes ($z_1 = 3/2$, $z_2=3/2$)

B) One self-coupling coefficient nonzero, all other diagonal terms of mode-coupling matrices 0, e.g., $G_{11}^1 \neq 0$, $G_{22}^1 = G_{22}^2 = G_{11}^2 = 0$

==> one KPZ mode, one diffusive mode ($z_1 = 3/2$, $z_2=2$)

C) One self-coupling coefficient nonzero, subleading diagonal of other mode-coupling matrix 0, e.g., $G_{11}^1 \neq 0$, $G_{11}^2 \neq 0$, $G_{22}^2 = 0$

==> one KPZ mode, second non-KPZ superdiffusive mode ($z_1 = 3/2$, $z_2=5/3$)

Remark: Heat mode with $z=5/3$, two KPZ sound modes in Hamiltonian dynamics with three conservation laws [van Beijeren (2012)]

General solution of mode coupling equations for any number of conservation laws

Power counting:

$$\boxed{z_\alpha = \begin{cases} 2 & \text{if} \quad \mathbb{I}_\alpha = \emptyset \\ 3/2 & \text{if} \quad \alpha \in \mathbb{I}_\alpha \\ \min_{\beta \in \mathbb{I}_\alpha} \left[\left(1 + \frac{1}{z_\beta} \right) \right] & \text{else} \end{cases}}$$

where

$$\boxed{\mathbb{I}_\alpha := \{\beta : G^\alpha_{\beta\beta} \neq 0\}}$$

Structure of diagonal elements of mode coupling matrix ➔ universality classes

1) All diagonal elements of mode α vanish: Diffusive mode (z=2)

2) Self coupling $G_{\alpha\alpha}^{\alpha} \neq 0$: KPZ or modified KPZ (for non-zero coupling to diffusive mode [Stoltz, Spohn (2015)]

3) Otherwise: ?

General solution of mode coupling equations (Case 3)

A) Consider sequential coupling with KPZ mode 1:

➔ Recursion for dynamical exponents $$z_\alpha = 1 + \frac{1}{z_{\alpha-1}}$$

Solution: Kepler ratios $$\boxed{z_\alpha = \frac{F_{\alpha+3}}{F_{\alpha+2}}}$$

of neighbouring Fibonacci numbers 1, 1, 2, 3, 5, 8,…

➔ z = 3/2, 5/3, 8/5, … → φ = (1+√5)/2 ≈1.618 (golden mean)

Scaling functions: z-stable Levy distributions

$$\hat{S}_\alpha(p,t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-iv_\alpha pt - E_\alpha |p|^{z_\alpha} t \left(1 - i\sigma_p^{\alpha\beta} \tan\left(\frac{\pi z_\alpha}{2}\right)\right)}$$

Time scales and asymmetries given recursively in terms of mode coupling coefficients and (numerically known) integral over square of KPZ scaling function

General solution of mode coupling equations (Case 3 cont')

B) Consider sequential coupling to *diffusive* mode 1:

➔ Same recursion for dynamical exponents, but shifted initial value

Solution: Shifted Kepler ratios z= 2, 3/2, 5/3, 8/5, …

Scaling functions: Levy for z<2, time scales contain diffusion coefficient instead of KPZ-integral

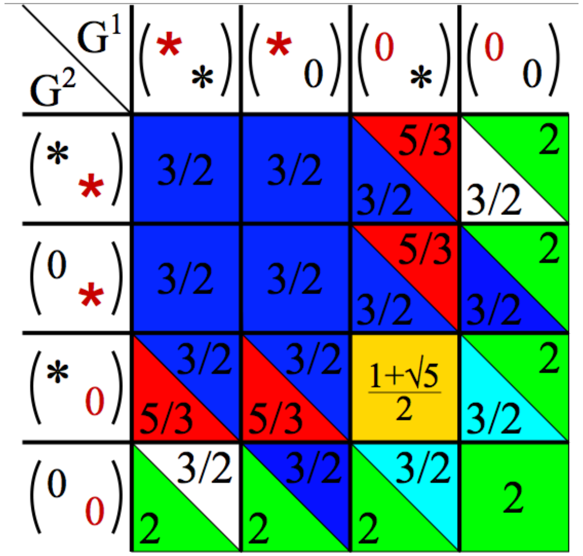
Non-sequential coupling: Still valid for large general class of mode-coupling matrices (work in progress)

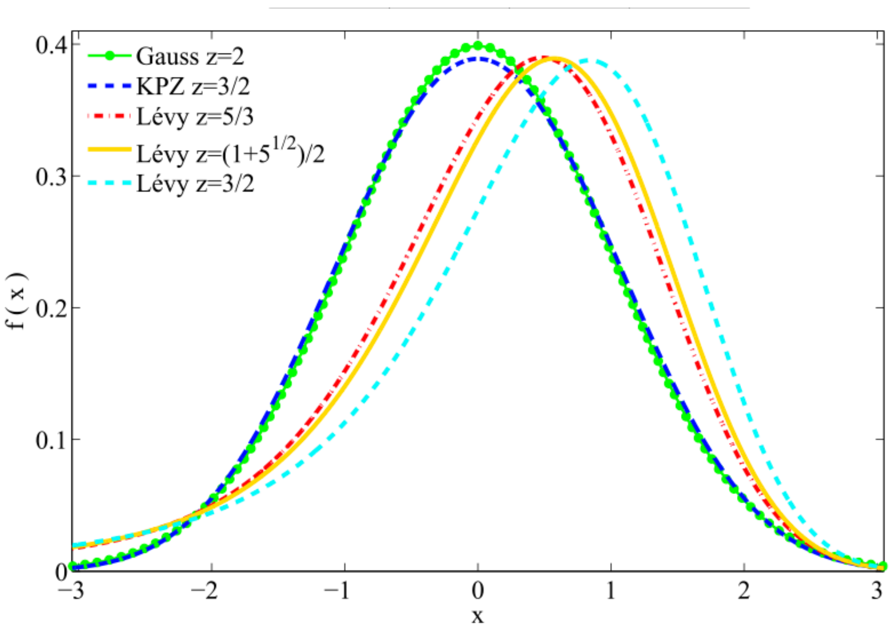
C) No coupling to either diffusive or KPZ-type mode

➔ Golden mean universality class $z_\alpha = \varphi$

Universality classes for two-component systems

[Popkov, Schmidt, GMS (2015); Spohn, Stoltz (2015)]





Universality classes with dynamical exponents z_α

Universal scaling functions with dynamical exponents z_α

- Difference between Gauss and KPZ verified experimentally (unambiguous)
  [Takeuchi & Sano (2010)]

4. Simulation results

Measure dynamical exponents z_α

1) Monte Carlo random sequential update

2) Excite modes at site k=L/2 at t=0 and measure dynamical structure function of each mode

3) Compute center of mass motion of excitation :

$$\langle X_\alpha(t)\rangle = v_\alpha t \quad \Rightarrow \quad v_\alpha$$

4) Measure amplitudes $A_\alpha(t)$ at maximum:

$$A_\alpha(t) \sim t^{-1/z\varphi} \quad \Rightarrow z_\alpha$$

5) Fit predicted scaling functions

A) (D, 3/2L) universality class (Two-lane model)

Choose equal densities $\rho_1 = \rho_2 = \rho$, and symmetric lanes b=1

$$G^1 = -4A_0(1+\gamma\rho)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G^2 = -4A_0\begin{pmatrix} 1+\gamma(1-\rho) & 0 \\ 0 & 1-\gamma(1-3\rho) \end{pmatrix}$$

$\gamma = 1/(1-3\rho)$: Diffusive and 3/2-Fibonacci mode

$$S_1(x,t) = \frac{1}{\sqrt{4\pi D_1 t}} \mathrm{e}^{-\frac{(x-v_1 t)^2}{4D_1 t}}$$

$$\hat{S}_2(p,t) = \frac{1}{\sqrt{2\pi}} \exp\left(-iv_2 pt - C_0|p|^{3/2} t\left[1 - i\,\mathrm{sgn}(p(v_1-v_2))\right]\right)$$

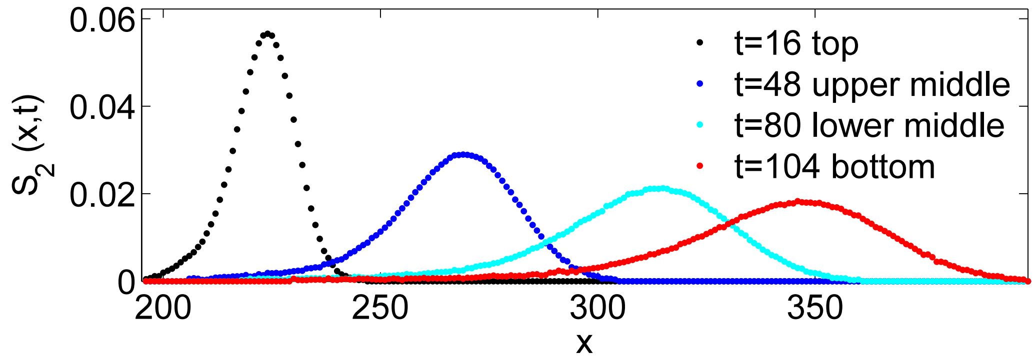
$$C_0 = \frac{(G^2_{11})^2}{2\sqrt{D_1|v_2-v_1|}}$$

Monte-Carlo data:

3/2L mode: $\rho_1 = \rho_2 = 0.2$, $\gamma = 2.5$, b = 1 (symmetry between lanes)

Theory: $v_2 = 1.3$, $2/z=1.333$, $\beta = -1$

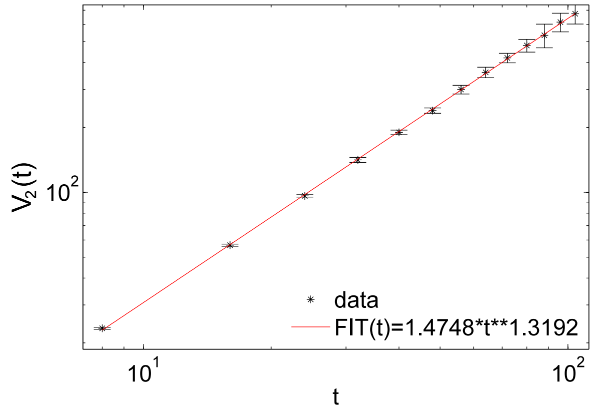
$V_2 = 1.30(1)$ ✔



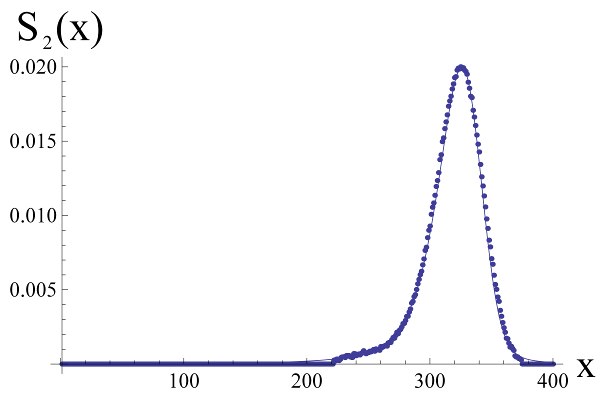
Variance: $t^{2/z}$

z = 1.52 ✓

β = - 0.692 ✖



$1+\beta \sim t^{-1/6}$



t = 88

B) Golden mean universality class (Two-lane model)

$G^{(1)}_{11} = G^{(2)}_{22} = 0, \quad G^{(1)}_{22} \neq 0; G^{(2)}_{11} \neq 0;$

Dynamical structure function:

$$\hat{S}_{\pm}(p,t) = \frac{1}{\sqrt{2\pi}} \exp\left(-iv_{\pm}pt - C_{\pm}|p|^{\varphi}t\left[1 \pm i\,\mathrm{sgn}(p(v_{-} - v_{+}))\tan\left(\frac{\pi\varphi}{2}\right)\right]\right)$$

$\varphi = (1+\sqrt{5})/2 \approx 1.618$ (golden mean)

$$C_{\pm} = \frac{1}{2}|v_{+} - v_{-}|^{1-\frac{2}{\varphi}} \left(\frac{2G^{1}_{22}G^{2}_{11}}{\varphi \sin\left(\frac{\pi\varphi}{2}\right)}\right)^{\varphi-1} \left(\frac{G^{1}_{22}}{G^{2}_{11}}\right)^{\pm(1+\varphi)}$$

➔ All parameters given by J and K!

(No free fitting parameters)

Simulation results:

Choose manifold: $\rho_1 = \dfrac{1-b}{3\gamma}, \quad \rho_2 = \dfrac{\gamma - 1}{3\gamma}$

$$v_\pm = (1+\gamma\rho_2)(1-2\rho_1) \pm \gamma\sqrt{\rho_1(1-\rho_1)\rho_2(1-\rho_2)}$$

$\rho_1 = 0.25$, $\rho_2 = 0.20$, $\gamma = 2.5$, $b = 0.625$ ➔ $v_- = 0.317$, $v_+ = 1.183$

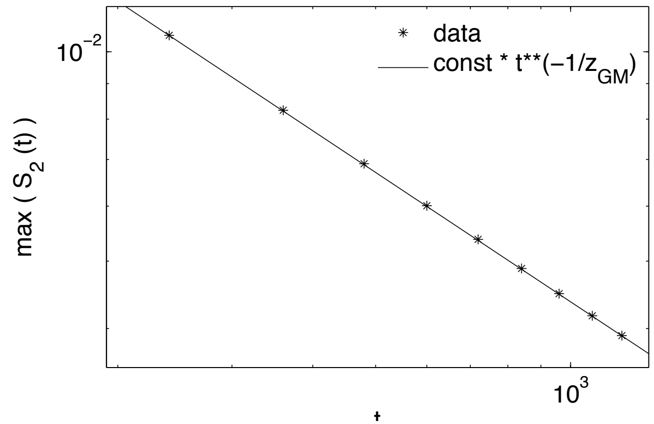
$$G^1 = \begin{pmatrix} 0 & -0.406416 \\ -0.406416 & -0.105726 \end{pmatrix}, \quad G^2 = \begin{pmatrix} -0.812833 & -0.052863 \\ -0.052863 & 0 \end{pmatrix}$$

Measurement of center of mass: Error << 1%

Asymmetry: + mode: $\beta \approx -1$ for t=600;
- mode: $|\beta| < 1$ for all measured t (small coupling constant)

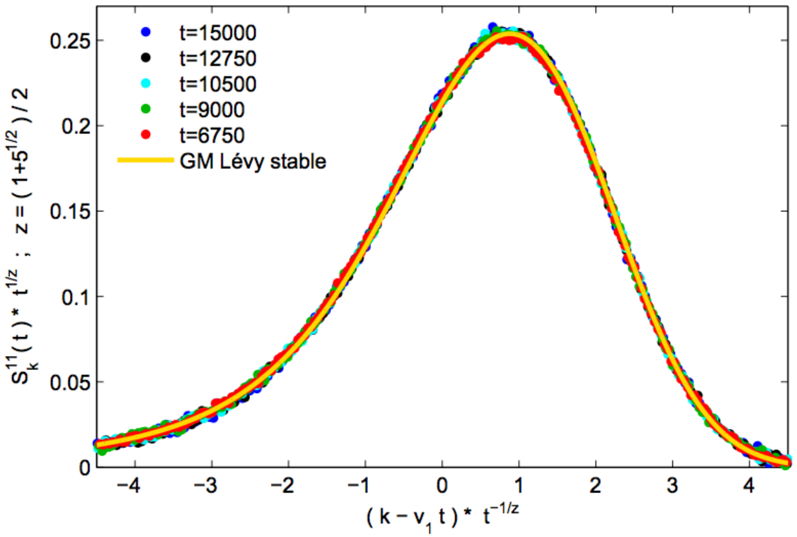
Simulation results (cont')

Amplitude at maximum:



➔ z = 1.618(5)

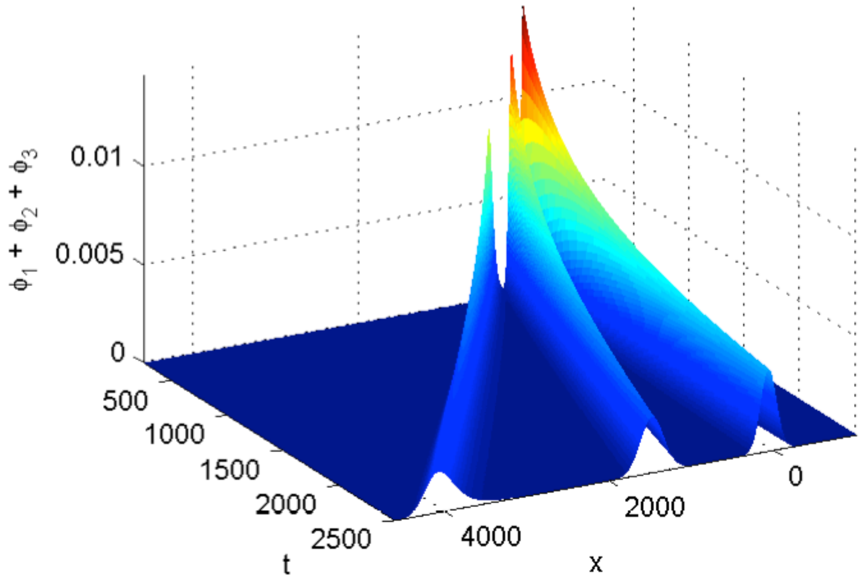
Three-lane model:



Scaling plot and fit with max. asym. φ−Levy

Three lane model

New Fibonacci universality class: z= 2, 3/2, 5/3, 8/5



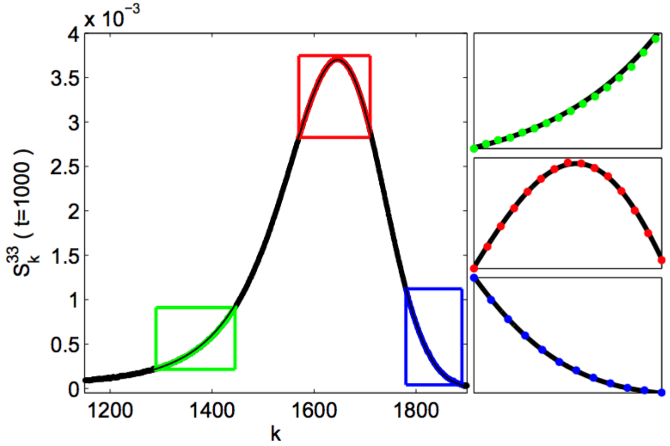
Mode 1: 8/5-Fibonacci, Mode 2: 5/3-Fibonacci, Mode 3: 3/2-KPZ.

Simulation results for 8/5 Fibonacci and GM

Choose large coupling constants ➔ maximal asymmetry at finite t

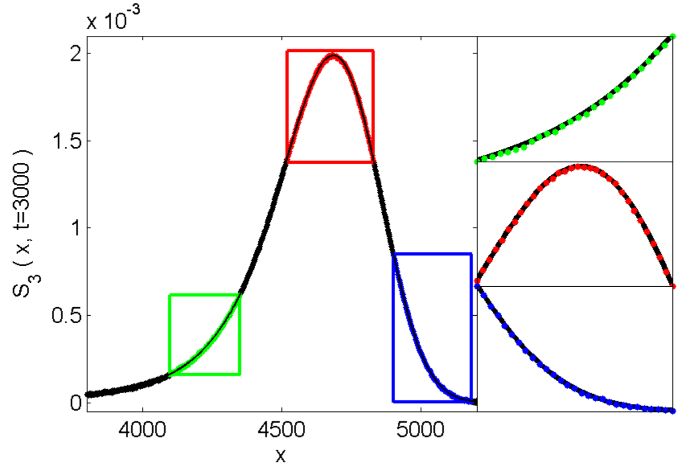
8/5 Fibonacci mode at t=1000:

Fit with max. asym. 8/5−Levy



Golden mean mode at t=3000:

Fit with max. asym. φ−Levy



5. Conclusions

- Mode coupling theory predicts infinite discrete family of non-equilibrium universality classes for fluctuating hydrodynamics for hyperbolic systems

- Universality classes completely fixed by macroscopic current-density relation

- Dynamical exponents are Kepler ratios of consecutive Fibonacci numbers or the golden mean limit

- Scaling functions completely fixed by current-density relation and macroscopic stationary compressibility matrix

- Agreement of scaling functions with simulation data: MCT exact?