Chandrasekhar lecture

Oct. 27, 2015

Random Matrix Theory and the Dynamics of Non-equilibrium Interfaces

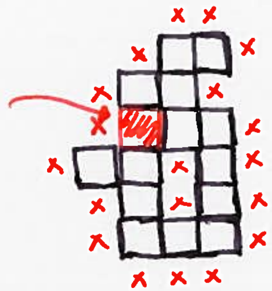
Herbert Spohn

TU München

non-equilibrium growth

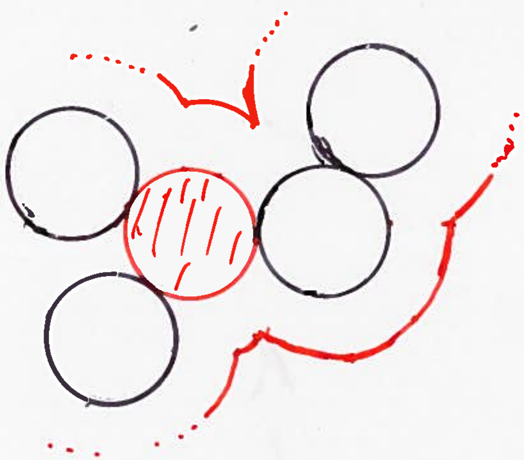
1. Eden growth

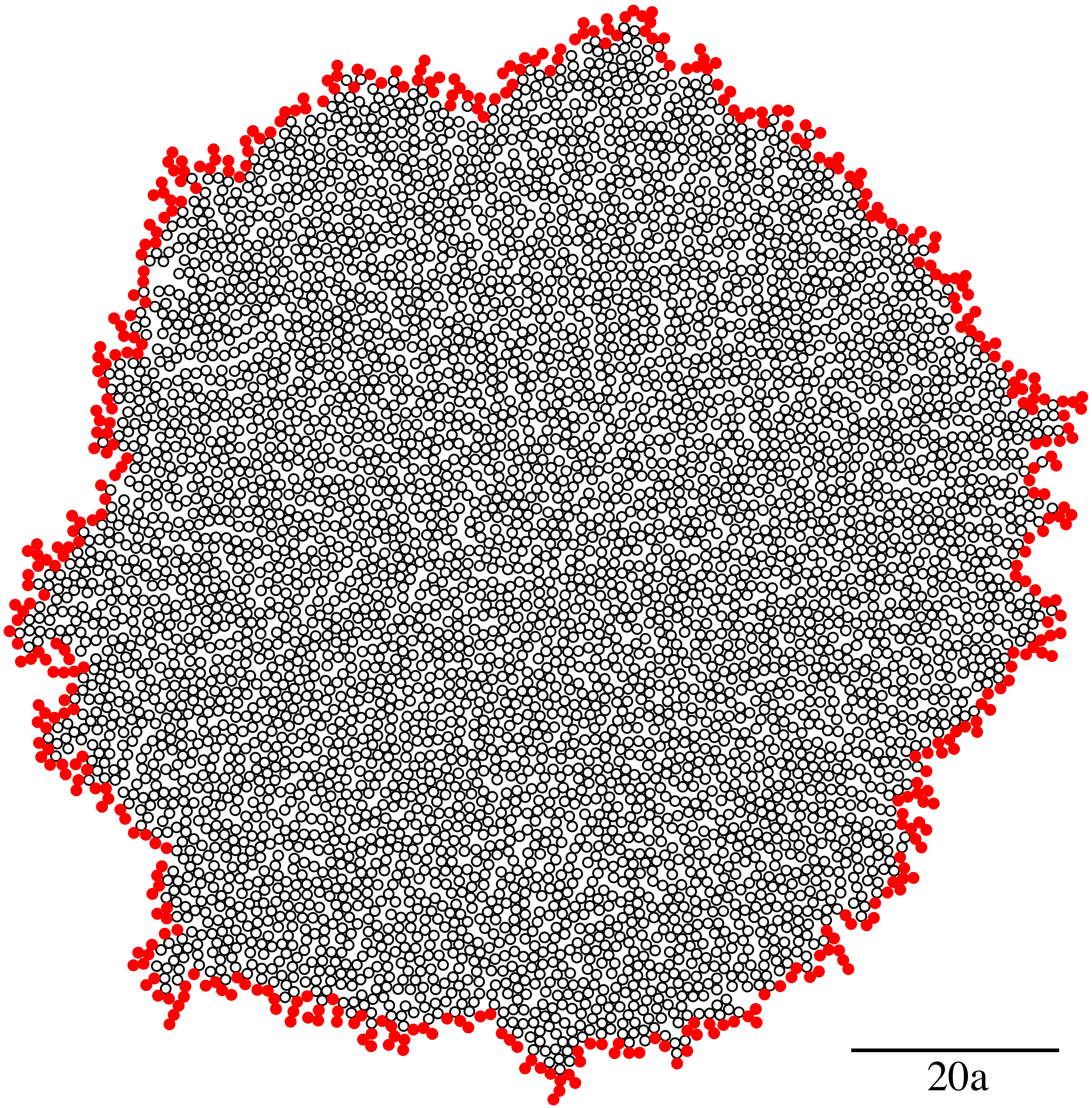
seed



- seed (also line seed)
- growth sites, filled with equal probability

⇒ isotropic version

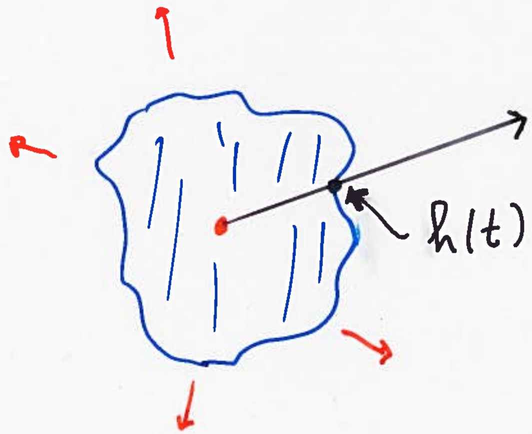



20a

- deterministic shape

→ shape fluctuations ⟸ {Statistical Mechanics}

non-equilibrium — reverse processes suppressed (detach)



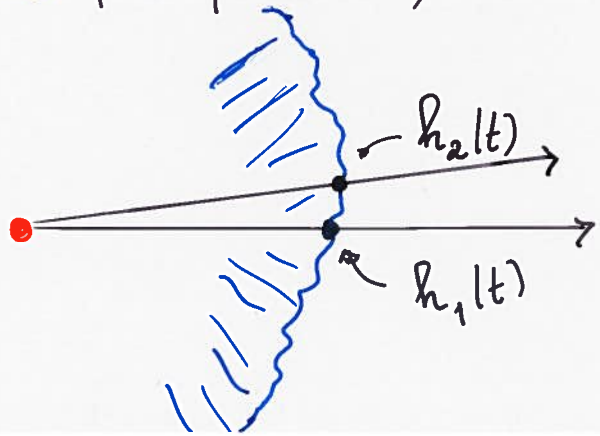
$$h(t) = \underbrace{v_0 t}_{\text{growth}} + (\lambda_e t)^{1/3} \underbrace{\xi}_{\text{random amplitude}}$$

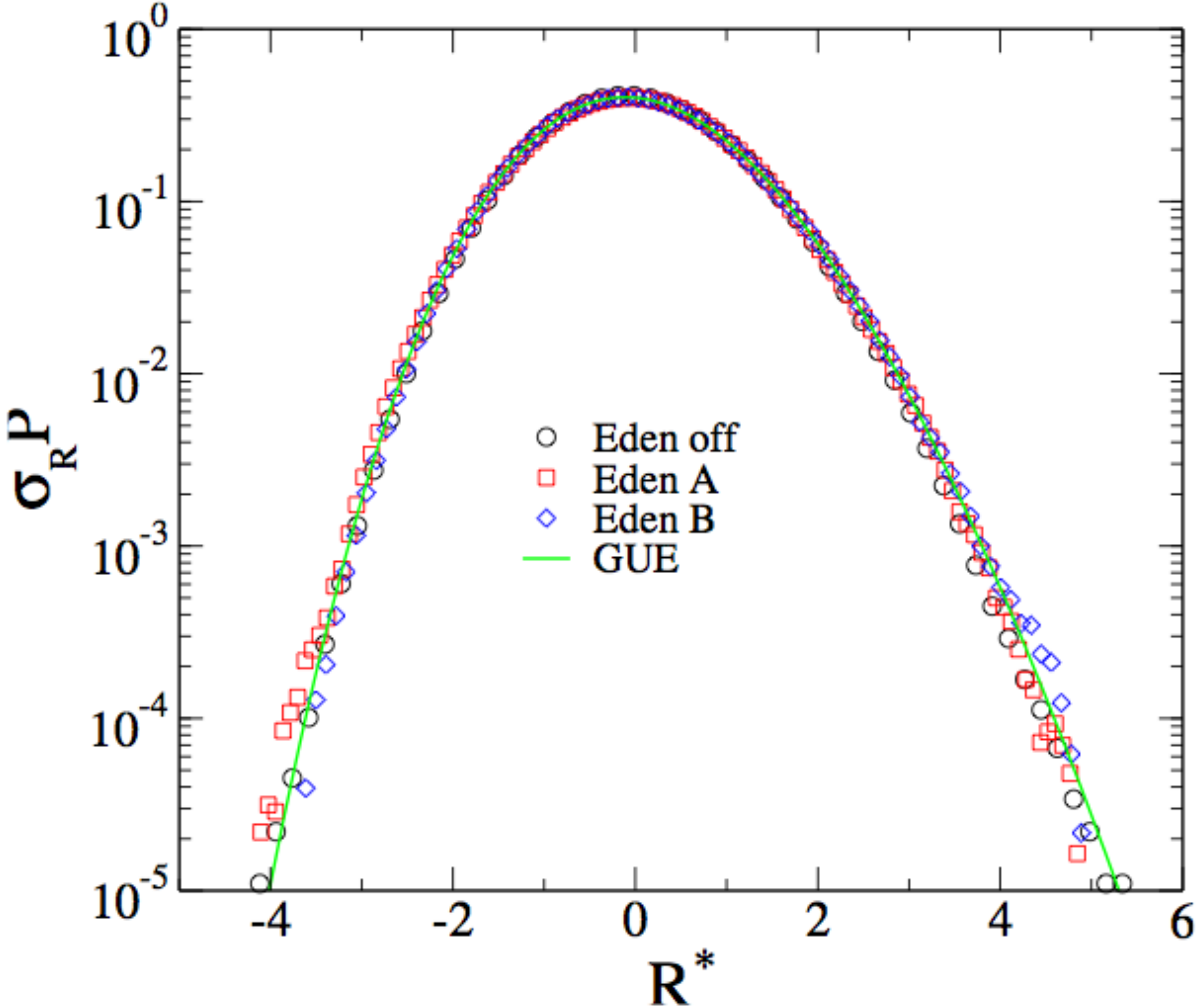
→ full probability density function of ξ

finer details

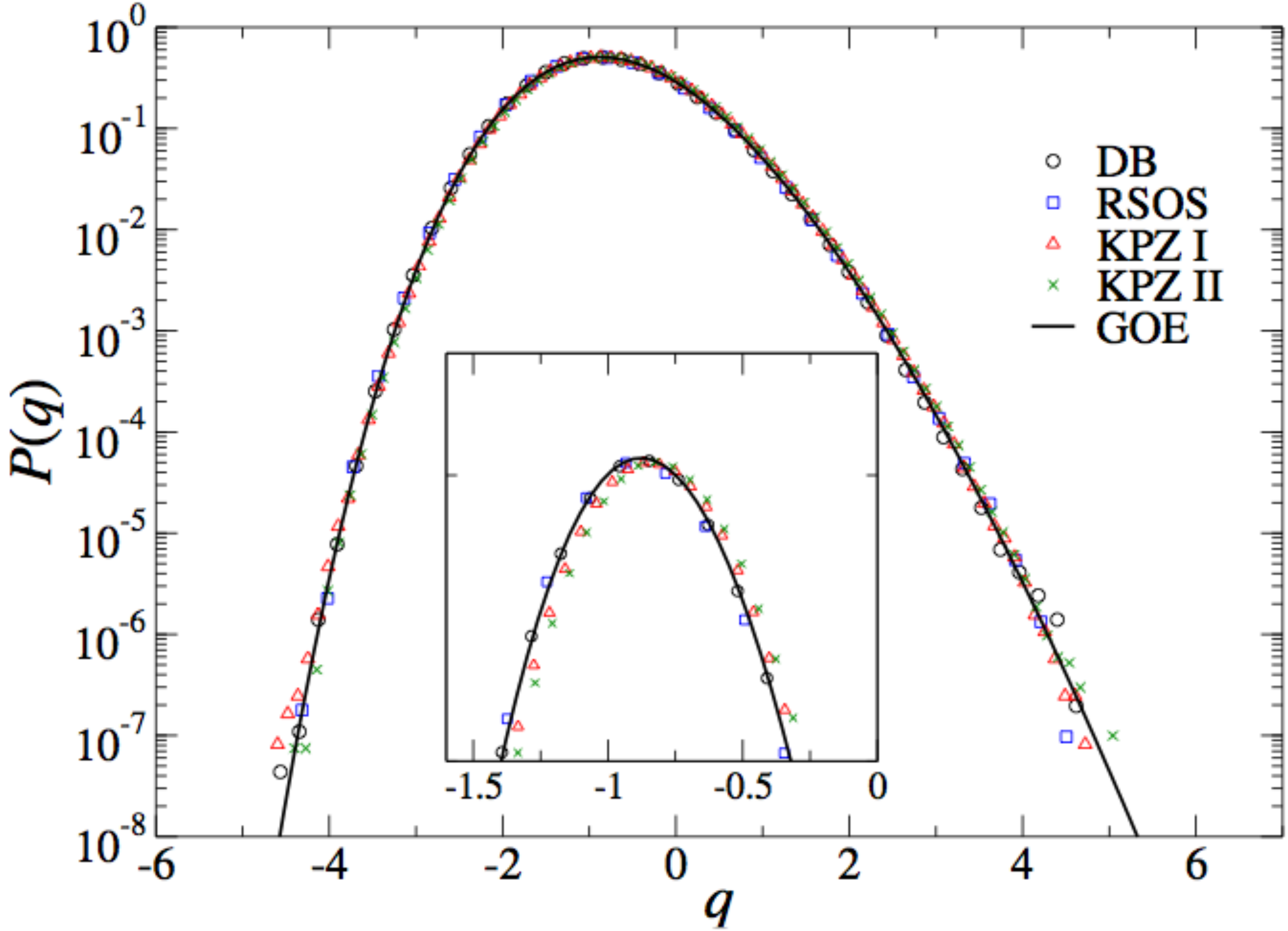
$h_1(t)$, $h_2(t)$

surface correlations





10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$\sigma_R P$
-4
-2
0
2
4
6
R^*
Eden off
Eden A
Eden B
GUE



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$P(q)$
-6
-4
-2
0
2
4
6
q
DB
RSOS
KPZ I
KPZ II
GOE
-1.5
-1
-0.5
0

universality

⇒ any realized system is more complex

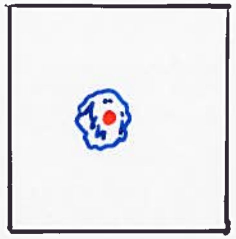
⇒ theory relies on a few models with "exact" solutions

2. turbulent liquid crystal film

Takeuchi, Sano 2012

cell

top

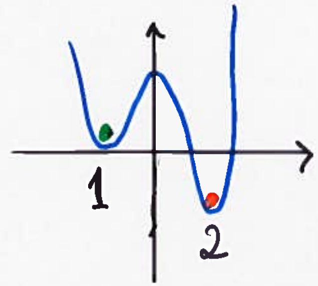


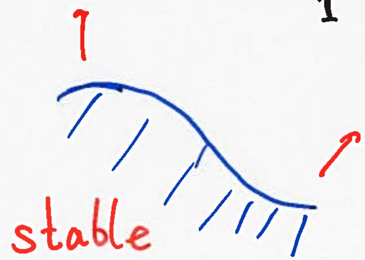
← 16mm →

side

12 μm

oscillating electric field



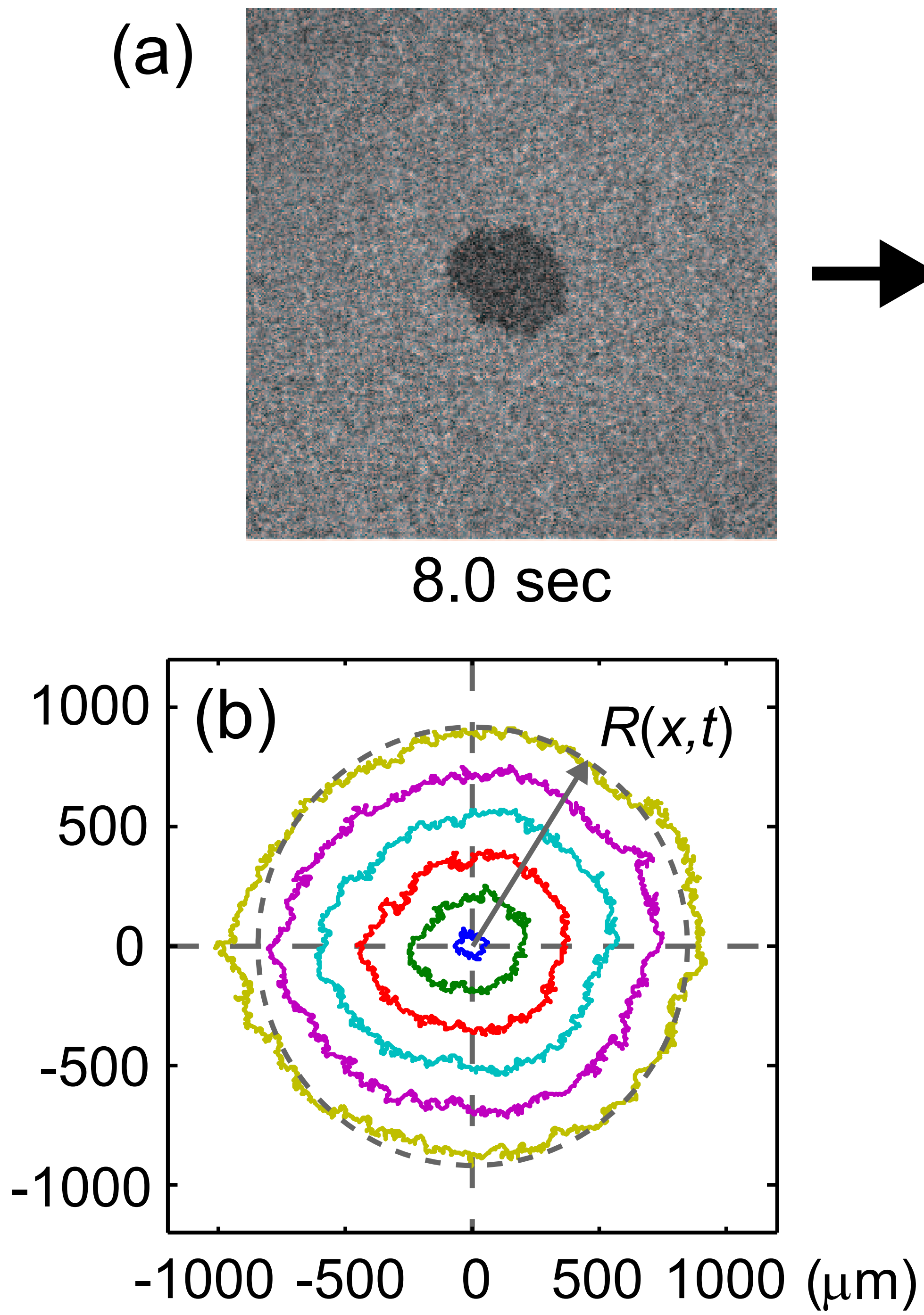


	light transmission
• metastable DLS 1 phase	YES
• stable DLS 2 phase	NO

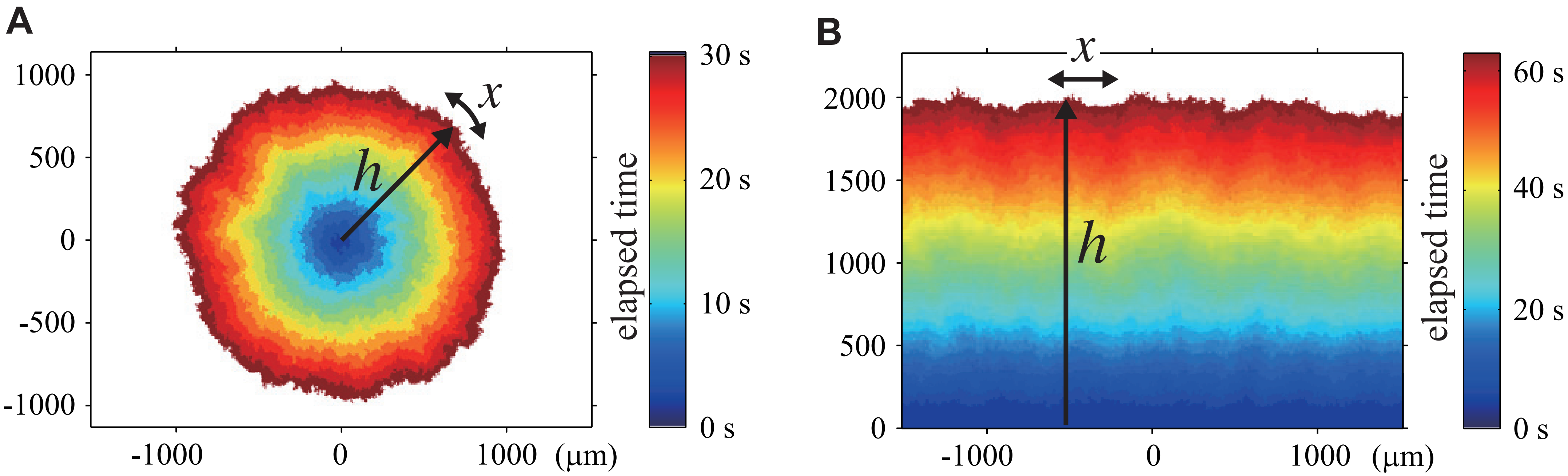
transitions: metastable ⇝ stable

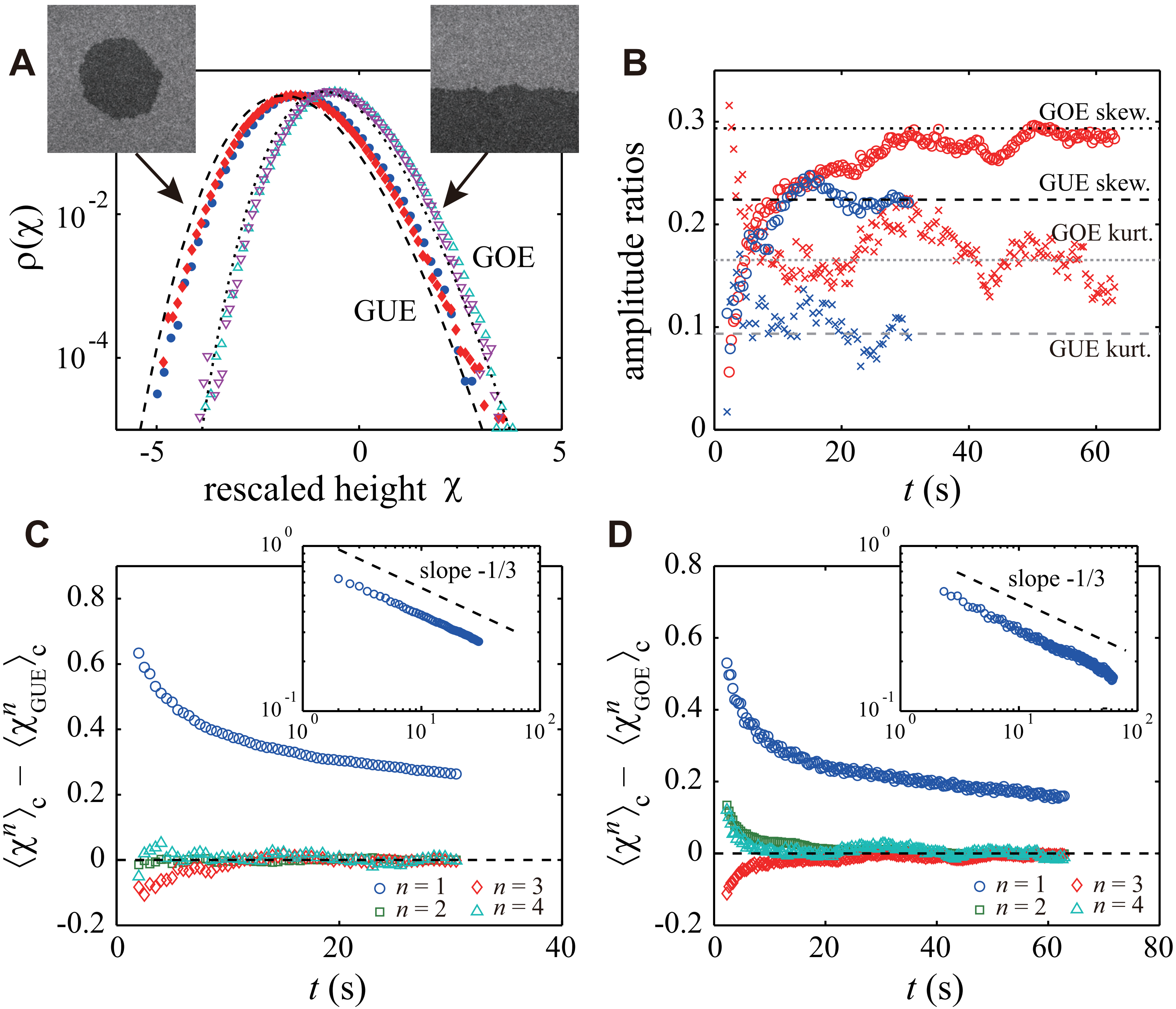
only at interface

// NO mass transport //



(a)
8.0 sec
18.0 sec
500μm
28.0 sec
(b)
R(x,t)
1000
500
0
-500
-1000
-1000 -500 0 500 1000 (μm)


A
1000
500
0
-500
-1000
-1000
0
1000
(μm)
h
x
elapsed time
30 s
20 s
10 s
0 s
B
2000
1500
1000
500
0
-1000
0
1000
(μm)
x
h
elapsed time
60 s
40 s
20 s
0 s


A
B
C
D
ρ(χ)
rescaled height χ
GUE
GOE
amplitude ratios
GOE skew.
GUE skew.
GOE kurt.
GUE kurt.
t (s)
⟨χ^n⟩_c − ⟨χ^n_GUE⟩_c
⟨χ^n⟩_c − ⟨χ^n_GOE⟩_c
slope -1/3
n = 1
n = 2
n = 3
n = 4

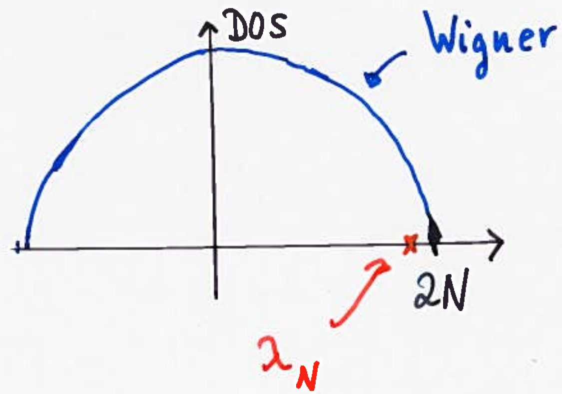
Random Matrix Theory

3. Tracy-Widom (1993)

Gaussian Unitary Ensemble — random A $N \times N$ hermitian

$$\frac{1}{Z} e^{-\frac{1}{2N} \operatorname{tr} A^2}$$

eigenvalues $\lambda_1 < \dots < \lambda_N$, large N



$$\lambda_N \cong 2N + N^{1/3} \xi_{GUE} \Leftarrow$$

$$\mathbb{P}(\xi_{GUE} \leq s) = \det(1 - K_{2,s})_{L^2(\mathbb{R}_+)}$$

Airy kernel

$$K_{2,s}(x,y) = \int_0^\infty d\lambda \, Ai(x+s+\lambda)\, Ai(y+s+\lambda)$$

Gaussian Orthogonal Ensemble random A real symmetric

$$\lambda_N \cong 2N + N^{1/3} \xi_{GOE}$$

$$\mathbb{P}(\xi_{GOE} \leq s) = \det(1 - K_{1,s})_{L^2(\mathbb{R}_+)}$$

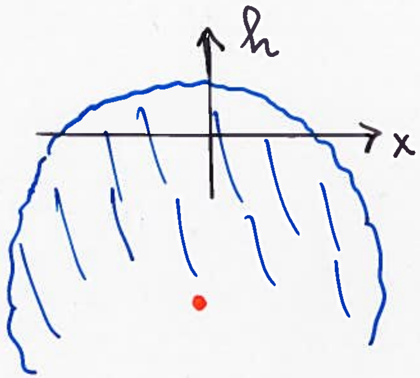
$$K_{1,s}(x,y) = \mathrm{Ai}(x+y+s$$

- droplet / curved $\xi = \xi_{GUE}$
- flat interface $\xi = \xi_{GOE}$

Why random matrix theory ?

4. KPZ equation

Kardar, Parisi, Zhang (1986)



height function $h(x,t)$ $\quad t \geq 0$

$$\| \; \partial_t h = \underbrace{(\partial_x h)^2}_{\text{nonlinearity}} + \tfrac{1}{2}\partial_x^2 h + W \; \|$$

space-time white noise

‖ universality ‖

- isotropic growth $\quad \partial_t h = \sqrt{1+(\partial_x h)^2} \cong \frac{1}{2}(\partial_x h)^2$

- Cole-Hopf transformation

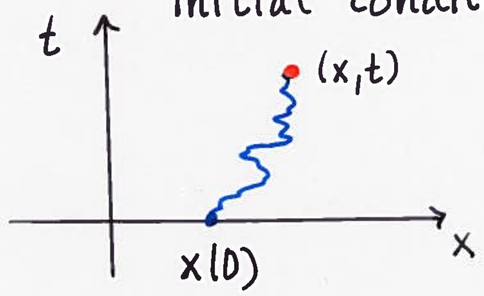
$$Z = e^h \quad \rightsquigarrow \quad \partial_t Z = \tfrac{1}{2}\partial_x^2 Z + W Z$$

random potential

"solved" by Feynman path integral

$$Z(x,t) = \int \mathcal{D}x \; \exp\left[- \int_0^t ds \left\{ \frac{1}{2} \dot{x}(s)^2 - W(x(s),s) \right\} \right] Z_0(x(0))$$

initial condition



- directed polymer in random medium
- Z partition function disordered
- $\log Z = h$ free energy $\Leftarrow$ fluctuations (shape fluctuations)

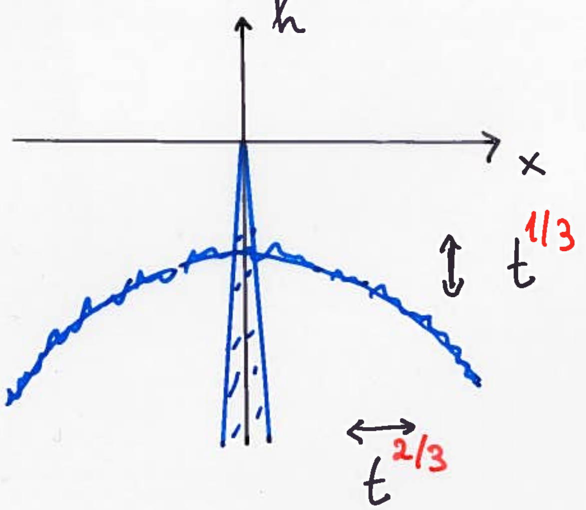
$\rightarrow$ sharp wedge $Z_0(x) = \delta(x)$ point - point

$h(x,0) = -\frac{1}{\delta}|x|$, $\delta \to 0$

$(0,0) \rightsquigarrow (x,t)$

5. exact solution

Amir, Corwin, Quastel 2011
Sasamoto, H.S. 2011



$$\eta(x,t) = h(x,t) + \frac{1}{2t}x^2 + \frac{1}{24}t$$

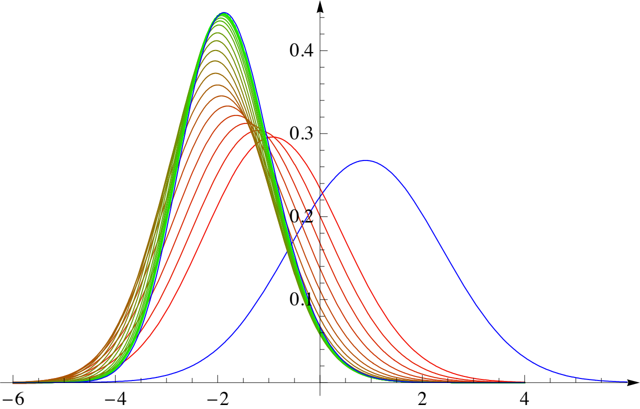
stationary in x

generating function

$$\left\langle e^{-e^{(\eta(0,t) - t^{1/3}s)}} \right\rangle = \det(1 - K_{t,s})_{L^2(\mathbb{R}_+)}$$

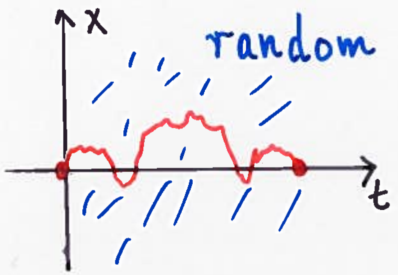
$$K_{t,s}(x,y) = \int d\lambda \, \frac{1}{1 + e^{-t^{1/3}\lambda}} \, \mathrm{Ai}(x+s+\lambda)\,\mathrm{Ai}(y+s+\lambda)$$

limit $t \to \infty$ $\qquad \Vert \qquad \xi_{GUE} \qquad \Vert$



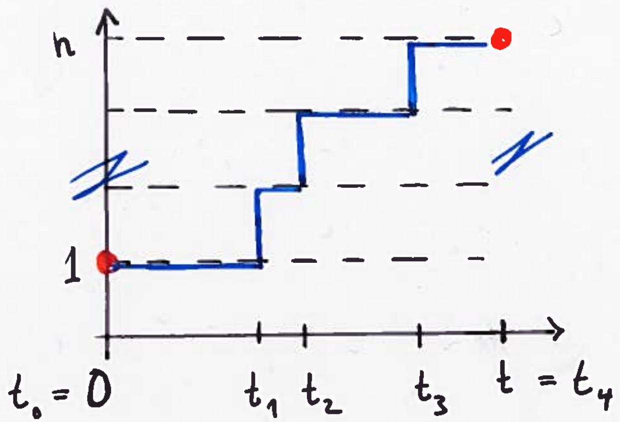
0.4
0.3
0.2
0.1
−6
−4
−2
2
4

6. discretize



$\Rightarrow$

universality



up-right path $x(\cdot)$

$x(0) = 1, \quad x(t) = n$

energy

$$E(x(s), 0 \le s \le t) = \int_0^t ds\, W(x(s), s)$$

$$= \sum_{j=1}^{n} \{ b_j(t_j) - b_j(t_{j-1}) \}$$

$$b_j(t) = \int_0^t ds\, W(j, s)$$

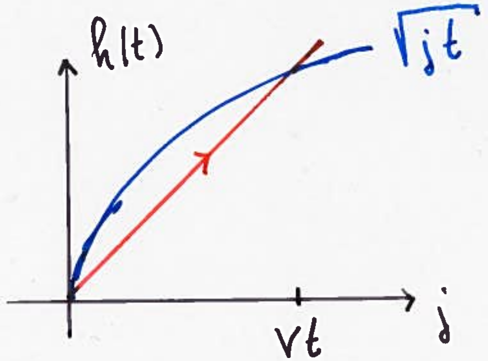
$$Z_d(n,t) = \sum_{\substack{\text{path} \\ (0,1) \to (t,n)}} e^{-\beta E(x(s), 0 \le s \le t)}$$

$\beta \to \infty$

→ minimization of ground state energy

$$h_n(t) = \min_{\text{path } (0,1) \to (t,n)} E(x(s), 0 \le s \le t)$$

Baryshnikov, 2002



$h_n(t)$ = largest eigenvalue of $A(t)$ GUE

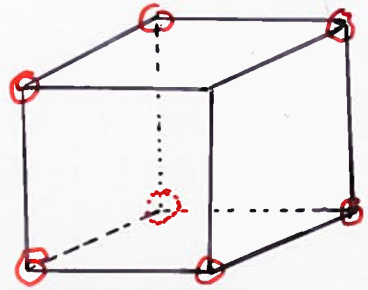
hermitian, $n \times n$, $\frac{1}{Z} e^{-\frac{1}{2Nt} \operatorname{tr} A^2}$

$$h_{\lfloor vt \rfloor}(t) \cong t + t^{1/3} \xi_{GUE}$$

$\beta < \infty$

Borodin, Corwin, Ferrari 2013

7. equilibrium crystal shape



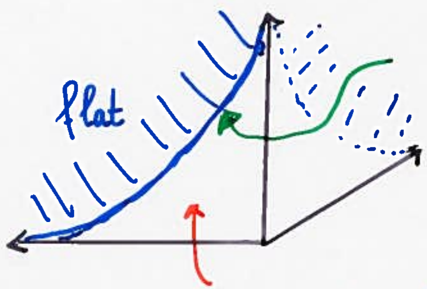
particles (lattice gas)

$N^3 - M$

$N \to \infty$, large M

Ising cube, T = 0, attractive

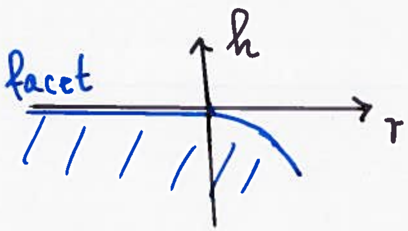
⇝ Ising corner microcanonical



facet edge (random)

rounded (massless Gaussian field)

Prokovsky - Talapov



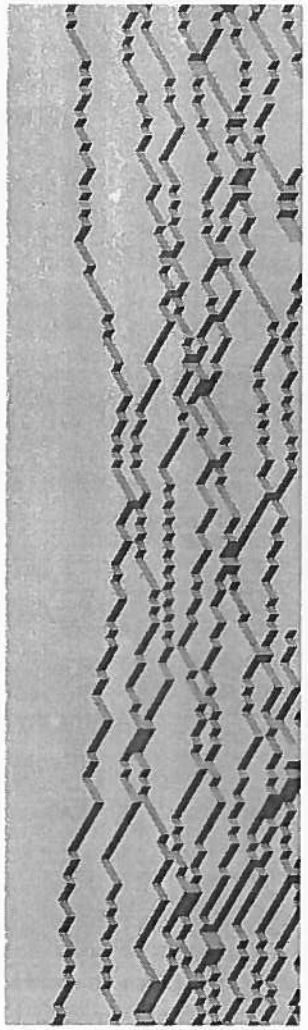
$h(r) = r^{3/2}$

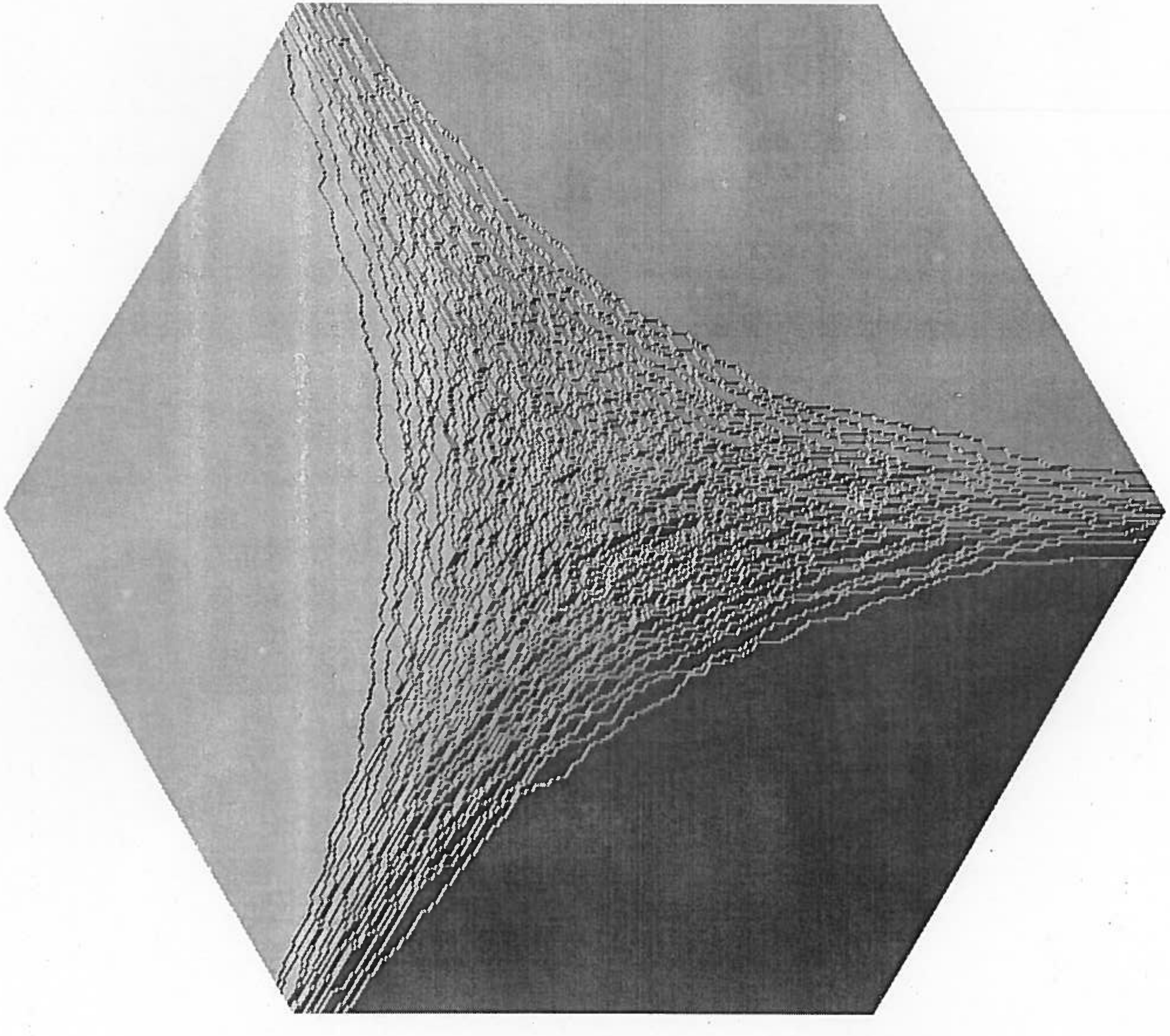
Wigner semicircle

Ferrari, Prähofer, H.S. 2005

equilibrium fluctuations of facet edge

= KPZ fluctuations, curved GUE





Outlook

- metastable - stable interface $\Rightarrow$ fluctuations as RMTheory

- higher dimensions KPZ $\partial_t h = (\nabla h)^2 + \Delta h + W$ $h(\underset{\sim}{\vec{x}}, t)$

- one dimension, several components $\underset{\sim}{\vec{h}}(x,t)$

please, come to my next lectures!