Equilibrium fluctuations for one-dimensional conservative systems

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in collaboration with P. Gonçalves

Non-equilibrium statistical physics
ICTS, Bangalore

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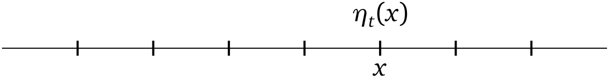
I. Particle systems in 1D



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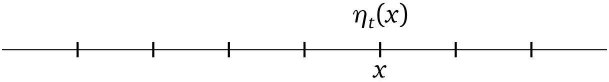

- **Time-dependent system** $\{\eta_t(x) \; ; \; x \in \mathbb{Z}\}$, state space $\mathcal{X}^{\mathbb{Z}}$
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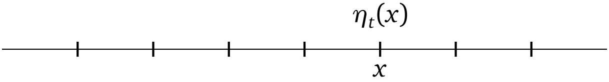

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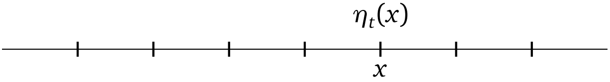

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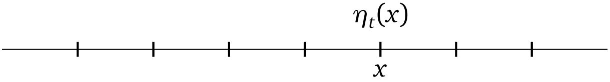

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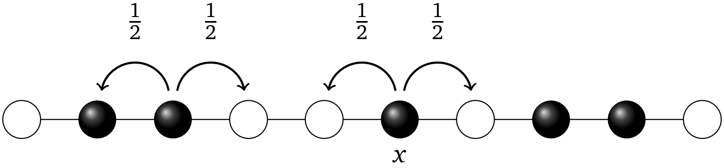
- **Generator of the Markov process:**

$$\boxed{\frac{d\mu}{dt} = \mu\mathcal{L}} \qquad \mu(t) = \text{ law of } \{\eta_t(x)\,;\, x \in \mathbb{Z}\}.$$

Example: porous media



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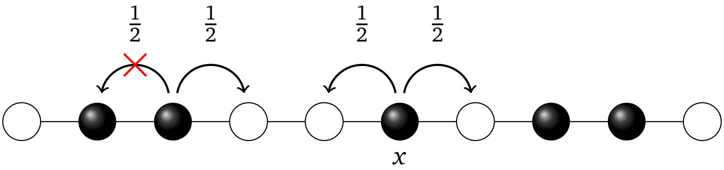

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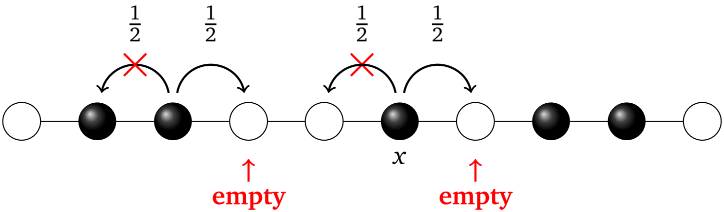

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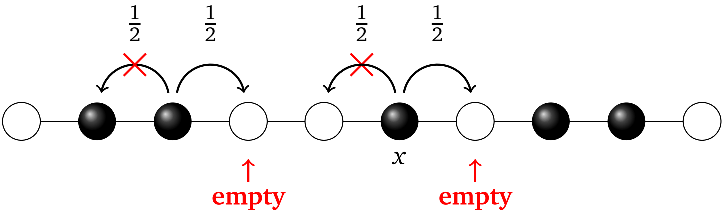

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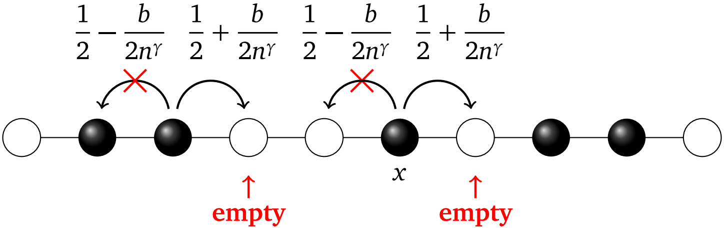

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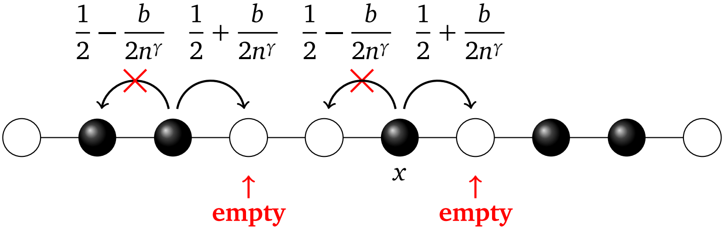

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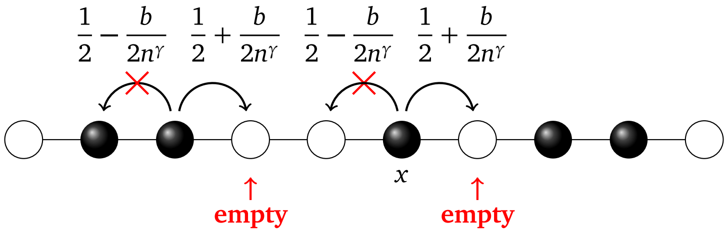

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$$\mathbb{P}[\eta(x)=1]=\rho, \qquad \mathbb{P}[\eta(x)=0]=1-\rho$$

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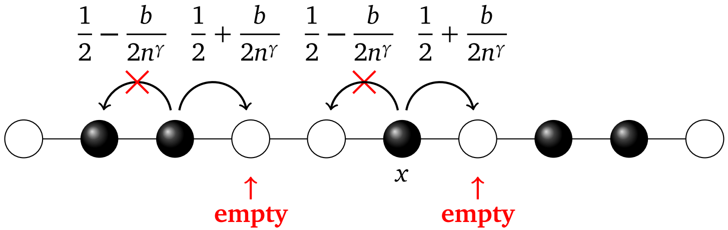

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Fluctuations around equilibrium??

- **Initial distribution:** ν_ρ (equilibrium)
  - ▷ **Stationarity**: for any $t \geqslant 0$, the law of $\{\eta_t(\cdot)\}$ is ν_ρ
  - ▷ **Acceleration of time**:

$$\mathbb{P}_\rho = \text{ law of the process } \{\eta_{tn^a}(\cdot)\ ;\ t \in [0,T]\}$$

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- **Density fluctuation field:**

$$\boxed{\mathcal{Y}_t^n(\varphi) := \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \varphi\Big(\frac{x}{n}\Big)\big(\eta_{tn^a}(x) - \rho\big)} \qquad \varphi \in \mathcal{S}(\mathbb{R})$$

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  - ▷ For each **fixed time**

$$\mathcal{Y}_t^n(\cdot) \xrightarrow[n\to\infty]{\text{distr.}} \chi(\rho)\mathcal{W}(\cdot) \qquad \mathcal{W} = \text{ white noise} \qquad \textbf{(CLT)}$$

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  - ▷ **Limiting process for $t \in [0,T]$ and $n \to \infty$?**

$$\frac{d}{dt}\{\mathcal{Y}_t^n(\varphi)\} \quad ?? \qquad \Rightarrow \qquad \boxed{n^a \mathcal{L}\big(\mathcal{Y}^n(\varphi)\big)}$$

II. Boltzmann-Gibbs Principles (BG)

- **Density current:**
  - **Conservation law**:

$$\frac{d\eta_t(x)}{dt} = \mathcal{L}\left(\eta(x)\right) = -\nabla\left(j_{x,x+1}\right) = j_{x-1,x}(\eta) - j_{x,x+1}(\eta)$$

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  - ▷ **Integral part**:

$$\int_0^t n^a \mathcal{L}\left(\mathcal{Y}_s^n(\varphi)\right) ds = \frac{n^{a-1}}{\sqrt{n}} \int_0^t \sum_{x\in\mathbb{Z}} \varphi'\left(\frac{x}{n}\right) \left\{ j_{x,x+1}(\eta_{sn^a}) - \mathbb{E}_\rho\left[j_{x,x+1}(\eta)\right]\right\} ds$$

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- **Decomposition of the current:**

$$j_{x,x+1}(\eta) = \underbrace{\nabla(h_x)}_{\text{gradient}} + \frac{b}{2n^\gamma}\left\{ \underbrace{\eta(x)\eta(x+1)}_{\text{polynomial}} + \underbrace{\eta(x)\eta(x+1)\eta(x-1) + \dots}_{\text{degree 3}} \right.$$

1. Gradient part $\nabla(h_x)$: First-order BG

- **Second integration by part:**

$$\frac{n^{a-2}}{\sqrt{n}}\int_0^t \sum_{x\in\mathbb{Z}} \varphi''\Big(\frac{x}{n}\Big)\Big\{h_x(\eta_{sn^a}) - \underbrace{\mathbb{E}_\rho\big[h_x(\eta)\big]}_{:=H(\rho)}\Big\}\,ds$$

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- **Limiting process:** for $a=2$ and $\gamma > \frac{1}{2}$

Ornstein-Uhlenbeck process (OU)

$\{\mathcal{Y}_t^n(\cdot)\}$ converges in distribution to the stationary solution of

$$d\mathcal{Y}_t = D(\rho)\,\Delta\mathcal{Y}_t\,dt + \sqrt{2\chi(\rho)D(\rho)}\,\nabla(d\mathcal{B}_t), \qquad \mathcal{B}_t = \text{ Brownian}$$

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- **Second-order Boltzmann-Gibbs principle** [2014]

$$\mathbb{E}_\rho\left[\left(\int_0^t \sum_{x\in\mathbb{Z}} v(x)\left\{\overline{\eta}_{sn^a}(x)\overline{\eta}_{sn^a}(x+1)-\left[\left(\overline{\eta}^{\ell}_{sn^a}(x)\right)^2-\frac{\chi(\rho)}{\ell}\right]\right\}ds\right)^2\right]$$

$$\leq \underbrace{C(v,\rho,t)}_{\text{constant}} \times \underbrace{\varepsilon(a,\ell,n)}_{\text{error}}$$

where

$$\overline{\eta}(x)=\eta(x)-\rho, \qquad \overline{\eta}^{\ell}(x)=\frac{1}{\ell}\sum_{y=x+1}^{x+\ell}\left(\eta(y)-\rho\right).$$

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  - ▷ Multiscale analysis using the **spectral gap**
  - ▷ Proof for every **local function**

3. Consequences of the second-order BG

- **Limiting process:** for $a = 2$ and $\gamma = \frac{1}{2}$

Stochastic Burgers Equation

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  - ▷ **Hamiltonian oscillators** with harmonic or exponential interactions
  - ▷ **Exclusion processes** with slow bonds (⟶ *Talk of Patricia Gonçalves, next week*)