Crossover to the stochastic Burgers equation from the WASEP with a slow bond

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Outline

1. Slowed exclusion process: the dynamics.

2. Scaling limits for the density of particles:

   Hydrodynamic Limits.

   Equilibrium Fluctuations.

3. Equilibrium Fluctuations:

   Edwards-Wilkinson universality class.

   KPZ universality class.

Slowed exclusion processes: the dynamics

- η_t is an exclusion process with space state $\Omega = \{0,1\}^{\mathbb{Z}}$.
- For $x \in \mathbb{Z}$, $\eta(x) = 1$ if the site is occupied, otherwise $\eta(x) = 0$.

The rates are given by:

- $x \neq -1$,

$$\xi^n_{x,x+1}(\eta) = \left\{\frac{1}{2} + \frac{a}{2n^\gamma}\right\} \eta(x)(1-\eta(x+1)),$$

$$\xi^n_{-1,0}(\eta) = \left\{\frac{\alpha}{2n^\beta} + \frac{a}{2n^\gamma}\right\} \eta(-1)(1-\eta(0)),$$

- for $x \neq 0$,

$$\xi^n_{x,x-1}(\eta) = \left\{\frac{1}{2} - \frac{a}{2n^\gamma}\right\} \eta(x)(1-\eta(x-1)),$$

$$\xi^n_{0,-1}(\eta) = \left\{\frac{\alpha}{2n^\beta} - \frac{a}{2n^\gamma}\right\} \eta(0)(1-\eta(-1))\,,$$

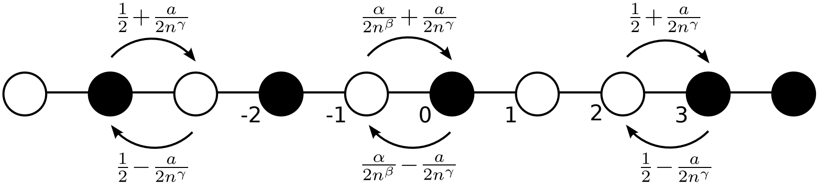
where $\alpha > 0$, $a \geq 0$, $\beta \geq 0$, $\gamma \geq \frac{1}{2}$.

Slowed exclusion processes

We assume $\gamma > \beta$ or $\beta = \gamma$ and $\alpha \geq a$.
We allow $\gamma = \beta$ and $a = \alpha$ for which $\{-1, 0\}$ is totally asymmetric with $\xi^n_{0,-1} = 0$ and $\xi^n_{-1,0} = \alpha / n^\beta$.

- For $a = 0$, we obtain the SSEP with a slow bond.

- For $\alpha = 1$ and $\beta = 0$ we obtain the WASEP - weak asymmetry.



- ν_ρ the Bernoulli product measure of parameter ρ is invariant.

THE HYDRODYNAMIC LIMIT $a = 0$

Hydrodynamic limit: initial state

- For η let $\pi^n(\eta; du)$ be the empirical measure:

$$\pi^n(\eta; du) = \frac{1}{n} \sum_{x \in \mathbb{Z}} \eta(x) \delta_{x/n}(du)$$

and $\pi_t^n(\eta, du) := \pi^n(\eta_{tn^2}, du)$.

- Fix $\rho_0 : \mathbb{R} \to [0, 1]$ and μ_n such that for every $\delta > 0$ and every continuous function $H : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n \to \infty} \mu_n \Big\{ \eta : \Big| \frac{1}{n} \sum_{x \in \mathbb{Z}} H(\tfrac{x}{n})\, \eta(x) - \int_{\mathbb{R}} H(u)\, \rho_0(u)\, du \Big| > \delta \Big\} \; = \; 0 \, .$$

- Then for any $t > 0$, $\pi_t^n \to \rho(t, u) du$, where $\rho(t, u)$ evolves according to the hydrodynamic equation.

Hydrodynamic limit: the result

- $\beta < 1$: Heat equation

$$\begin{cases} \partial_t \rho(t,u) = \Delta \rho(t,u)\,, \\ \rho(0,u) = \rho_0(u). \end{cases}$$

- $\beta = 1$: Heat equation with a type of Robin's boundary conditions

$$\begin{cases} \partial_t \rho(t,u) = \Delta \rho(t,u)\,, \\ \partial_u \rho(t,0^-) = \partial_u \rho(t,0^+) = \alpha(\rho(t,0^+) - \rho(t,0^-))\,, \\ \rho(0,u) = \rho_0(u). \end{cases}$$

- $\beta > 1$: Heat equation with Neumann's boundary conditions

$$\begin{cases} \partial_t \rho(t,u) = \Delta \rho(t,u)\,, \\ \partial_u \rho(t,0^-) = \partial_u \rho(t,0^+) = 0\,, \\ \rho(0,u) = \rho_0(u). \end{cases}$$

EQUILIBRIUM FLUCTUATIONS

Density fluctuation field: from the invariant state

- Fix a density $\rho \in (0,1)$.
- Consider the process starting from ν_ρ.
- The *density fluctuation field* $\{\mathcal{Y}_t^{\beta,\gamma,n}\ ;\ t \in [0,T]\}$ is given on $H \in \mathcal{S}_\beta(\mathbb{R})$ by

$$\begin{aligned}\mathcal{Y}_t^{\beta,\gamma,n}(H) :&= \sqrt{n}\Big\{ \int H(u)\pi_t^n(\eta, du) - \mathbb{E}_\rho\Big[\int H(u)\pi_t^n(\eta, du)\Big]\Big\} \\ &= \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} H\Big(\frac{x}{n}\Big)(\eta_{tn^2}(x) - \rho).\end{aligned}$$

Density fluctuation field for $a = 0$

Theorem (Franco, G., Neumann - 2013)

If $a = 0$, the sequence of processes $\{\mathcal{Y}_t^{\beta,\gamma,n}\,;\ t \in [0,T]\}_{n\in\mathbb{N}}$ converges to the Ornstein-Uhlenbeck process given by

$$d\mathcal{Y}_t^{\beta} = \frac{1}{2}\Delta_\beta \mathcal{Y}_t^{\beta}\,dt + \sqrt{\chi(\rho)}\nabla_\beta\, d\mathcal{W}_t^{\beta},$$

where $\{\mathcal{W}_t^{\beta}\,;\ t \in [0,T]\}$ is an $\mathcal{S}'_\beta(\mathbb{R})$-valued Brownian motion and $\chi(\rho) = \rho(1-\rho)$.

Density fluctuation field for $a \neq 0$: removing the drift

We redefine for any $H \in \mathcal{S}_\beta(\mathbb{R})$

$$\mathcal{Y}_t^{\beta,\gamma,n}(H) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} (\eta_{tn^2}(x) - \rho) H\left(\frac{x - n^{2-\gamma} a(1-2\rho)t}{n}\right).$$

Theorem (Ornstein-Uhlenbeck process)

If one of these two conditions are satisfied:

- $\beta \leq 1/2$ *and* $\gamma > 1/2$,
- $\beta > 1/2$ *and* $\begin{cases} \gamma \geq \beta & \text{if} \quad \beta \neq 1, \\ \gamma > 1 & \text{if} \quad \beta = 1, \end{cases}$

then $\{\mathcal{Y}_t^{\beta,\gamma,n}\ ;\ t \in [0,T]\}_{n \in \mathbb{N}}$ *converges to the same OU process as in* $a = 0$.

- The influence of the asymmetry is NOT SEEN in the limit.

THE EFFECT OF A STRONG ASYMMETRY

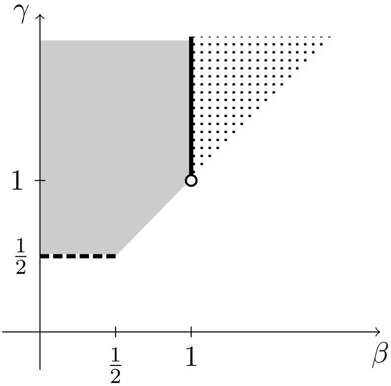
Density fluctuation field for $a \neq 0$: the KPZ scaling

Theorem (Stochastic Burgers equation)

For $\beta \leq 1/2$ and $\gamma = 1/2$, $\{\mathcal{Y}_t^{\beta,\gamma,n}\,;\ t \in [0,T]\}_{n\in\mathbb{N}}$ is tight and any limit point is a stationary energy solution of the stochastic Burgers equation

$$d\mathcal{Y}_t^{\beta} = \frac{1}{2}\Delta_\beta \mathcal{Y}_t^{\beta}\,dt + a\nabla(\mathcal{Y}_t^{\beta})^2\,dt + \sqrt{\chi(\rho)}\nabla\,d\mathcal{W}_t^{\beta},$$

where $\{\mathcal{W}_t^{\beta}\,;\ t \in [0,T]\}$ is an $\mathcal{S}'(\mathbb{R})$-valued Brownian motion.



Stochastic Burgers equation (KPZ regime)

OU process with no boundary conditions

OU process with Robin's boundary conditions

OU process with Neumann's boundary conditions

On the universality of KPZ: exclusion processes

- Let $r : \Omega \to \mathbb{R}$ be a local function that satisfies:

[i] There exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < r(\eta) < \varepsilon_0^{-1}$ for any $\eta \in \Omega$.

[ii] For any η, ξ such that $\eta(x) = \xi(x)$ for $x \neq 0, 1$, then $r(\eta) = r(\xi)$.

[iii] *Gradient condition.* There exists $\omega : \Omega \to \mathbb{R}$ such that

$$r(\eta)\big(\eta(1) - \eta(0)\big) = \tau_1 \omega(\eta) - \omega(\eta), \text{for any } \eta \in \Omega.$$

On the universality of KPZ: zero-range processes

- η_t a Markov process with space state $\Omega := \mathbb{N}^{\mathbb{Z}}$.
- for $x \in \mathbb{Z}$, $\eta(x)$ counts the number of particles at the site x.
- the jump rate from x only depends on the number of particles at x.
- and is given by a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ such that $g(0) = 0$, $g(k) > 0$ for $k \geq 1$ and g is Lipschitz:

(LIP) $\sup_{k \geq 0} |g(k+1) - g(k)| < \infty$.

Examples of zero-range models:

- If g satisfies (LIP) and

(U) There exists x_0 and $\varepsilon_0 > 0$ such that $g(x + x_0) - g(x) \geq \varepsilon_0$ for all $x \geq 0$.

- If g is sublinear, that is $C^{-1}x^{\gamma} \leq g(x+1) - g(x) \leq Cx^{\gamma}$ for $0 < \gamma < 1$ and $C > 0$.

- If $g(x) = 1(x \geq 1)$.

On the universality of KPZ: kinetically constrained exclusion processes

• η_t is a Markov process with space state $\Omega = \{0,1\}^{\mathbb{Z}}$.

• here particles more likely hop to unoccupied nearest-neighbor sites when at least $m-1 \geq 1$ other neighboring sites are full.

• for $m = 2$, the jump rate to the right is given by:

$$\eta(x)(1-\eta(x+1))\Big[\eta(x-1)+\eta(x+2)+\frac{\theta}{2n}\Big]$$

and the jump rate to the left is given by

$$\eta(x+1)(1-\eta(x))\Big[\eta(x-1)+\eta(x+2)+\frac{\theta}{2n}\Big].$$

Crossover to the Stochastic Burgers equation

♣ KPZ equation: $dh_t = A\Delta h_t dt + B(\nabla h_t)^2 dt + \sqrt{C} d\mathcal{W}_t,$

♣ Cole-Hopf transformation: $u(t,x) = \exp Bh(t,x)/A$

$$du_t = A\Delta u_t dt + \frac{BC}{A} u_t d\mathcal{W}_t.$$

♣ Stochastic Burgers equation: $\mathcal{Y}_t = \nabla h_t$

$$d\mathcal{Y}_t = A\Delta \mathcal{Y}_t dt + B\nabla \mathcal{Y}_t^2 dt + \sqrt{C}\nabla d\mathcal{W}_t.$$

♣ use the microscopic Cole-Hopf:
Dittrich and Gartner, Bertini and Giacomin, Dembo and Tsai...

♣ use the original process - need a Boltzmann-Gibbs Principle!

The KPZ scaling: stationary energy solution

To show that $\mathcal{Y}_t$ is a stationary energy solution of

$$d\mathcal{Y}_t^\beta = \frac{1}{2}\Delta_\beta \mathcal{Y}_t^\beta dt + a\nabla(\mathcal{Y}_t^\beta)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t^\beta,$$

we need to prove that $\{\mathcal{M}_t : t \in [0, T]\}$ given by

$$\mathcal{M}_t(H) \ := \ \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \frac{1}{2}\int_0^t \mathcal{Y}_s(\Delta H)\, ds + a\mathcal{A}_t(H)$$

is a continuous martingale with quadratic variation

$$\langle \mathcal{M}(H)\rangle_t = \rho(1-\rho)\|\nabla H\|_2^2,$$

where

$$\mathcal{A}_t(H) = \lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}} \nabla H(x)\Big[\mathcal{Y}_u(\iota_\varepsilon(x))\Big]^2 dx du$$

in $L^2(\mathbb{R})$, where $\iota_\varepsilon(x,y) = \frac{1}{\varepsilon}\mathbf{1}_{x\le y<x+\varepsilon}$, for $y \in \mathbb{R}$.

The instantaneous current

Note that

$$j^n_{x,x+1}(\eta) = j^{n,S}_{x,x+1}(\eta) + j^{n,A}_{x,x+1}(\eta)$$

with

$$\begin{aligned} j^{n,A}_{x,x+1}(\eta) &= \frac{an^2}{2n^\gamma}(\eta(x+1)-\eta(x))^2, && x \in \mathbb{Z}, \\ j^{n,S}_{x,x+1}(\eta) &= \frac{n^2}{2}(\eta(x)-\eta(x+1)), && x \neq -1, \\ j^{n,S}_{-1,0}(\eta) &= \frac{\alpha n^2}{2n^\beta}(\eta(-1)-\eta(0)). \end{aligned}$$

The martingale problem

Simple computations show that

$$\mathcal{M}_t^n(H) := \mathcal{Y}_t^n(H) - \mathcal{Y}_0^n(H) - \mathcal{I}_t^n(H) - \mathcal{B}_t^n(H),$$

plus some negligible term, where

$$\mathcal{I}_t^n(H) := \frac{1}{2}\int_0^t \mathcal{Y}_s^n(\Delta_\beta H)\ ds = \frac{1}{2}\int_0^t \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} (\eta_{sn^2}(x) - \rho)\Delta_\beta H\Big(\frac{x}{n}\Big)\ ds,$$

and

$$\mathcal{B}_t^n(H) = -a\frac{\sqrt{n}}{n^\gamma}\int_0^t \sum_{x\in\mathbb{Z}} \bar{\eta}_{sn^2}(x+1)\bar{\eta}_{sn^2}(x)\nabla H\Big(\frac{x}{n}\Big)\ ds.$$

Last term is the hard one!

The Boltzmann-Gibbs Principle

Theorem

Let $v : \mathbb{Z} \to \mathbb{R}$ *be a function such that* $\|v\|_{2,n}^2 := \frac{1}{n} \sum_{x \in \mathbb{Z}} v^2(x) < \infty$. *Then, there exists* $C > 0$ *such that for any* $t > 0$ *and* $\ell = \varepsilon n$:

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \sum_{x \in \mathbb{Z}} v(x) \Big\{ \bar{\eta}_{sn^2}(x) \bar{\eta}_{sn^2}(x+1) - \Big(\big(\bar{\eta}^{\ell}_{sn^2}(x) \big)^2 - \frac{\chi(\rho)}{\ell} \Big) \Big\} ds \Big)^2 \Big]$$

$$\leq Ct \Big\{ \frac{\ell}{n} + \frac{n^\beta}{\alpha n} + \frac{tn}{\ell^2} \Big\} \|v\|_{2,n}^2 + Ct \Big\{ \frac{n^\beta (\log_2(\ell))^2}{\alpha n} \Big\} \frac{1}{n} \sum_{x \neq -1} v^2(x),$$

where

$$\bar{\eta}^{\ell}(x) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} \bar{\eta}(y).$$

The idea of the proof

The idea consists in using the following decomposition of the local function

$$
\begin{aligned}
\bar\eta(x)\bar\eta(x+1) - \Big((\bar\eta^\ell(x))^2 - \frac{\chi(\rho)}{\ell}\Big)
&= \bar\eta(x)\Big\{\bar\eta(x+1) - \bar\eta^\ell(x+1)\Big\} \quad \text{(I)}\\
&+ \bar\eta(x)\Big\{\bar\eta^\ell(x+1) - \bar\eta^\ell(x)\Big\} \quad \text{(II)}\\
&+ \Big\{\bar\eta^\ell(x)\bar\eta(x) - (\bar\eta^\ell(x))^2 + \frac{1}{2\ell}(\bar\eta(x) - \bar\eta(x+1))^2\Big\} \quad \text{(III)}\\
&- \Big\{\frac{1}{2\ell}(\bar\eta(x) - \bar\eta(x+1))^2 - \frac{\chi(\rho)}{\ell}\Big\} \quad \text{(IV)}.
\end{aligned}
$$

(II) and (IV) can be bounded by Cauchy-Schwarz.
Let us see (I) and (III)!

Averaging over a box:

Proposition (One-block estimate)

For any $t > 0$:

$$\mathbb{E}_\rho\left[\left(\int_0^t ds \sum_{x\in\mathbb{Z}} v(x)\bar{\eta}_{sn^2}(x)\left\{\bar{\eta}_{sn^2}(x+1) - \bar{\eta}^{\ell_0}_{sn^2}(x+1)\right\}\right)^2\right]$$

$$\leq C(\rho)t\left(\frac{\ell_0^2}{n}\|v\|^2_{2,n} + \frac{\ell_0 n^\beta}{n^2\alpha}\sum_{x\in\Lambda_0^{\ell_0}} v^2(x)\right),$$

where for $y \in \mathbb{Z}$, $\Lambda_y^\ell := \{-\ell-1-y, \ldots, -2-y\}$.

Doubling the size of the box:

Let $\ell_k \in \mathbb{N}$ and $\ell_{k+1} = 2\ell_k$.

Proposition (Doubling the box)

For any $t > 0$*:*

$$\mathbb{E}_\rho\Big[\Big(\int_0^t ds \sum_{x\in\mathbb{Z}} v(x)\bar{\eta}_{sn^2}(x)\Big\{\bar{\eta}^{\ell_k}_{sn^2}(x+1) - \bar{\eta}^{\ell_{k+1}}_{sn^2}(x+1)\Big\}\Big)^2\Big] \leq C(\rho)t\Big(\frac{\ell_k}{n}\|v\|^2_{2,n} + \frac{n^\beta}{\alpha n^2}\sum_{x\neq -1} v^2(x)\Big).$$

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THANK YOU!