Application of duality to stochastic non-equilibrium models

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References: arxiv:1407.3367, 1507.01478

Plan

1. (Self-)duality for SEP
2. KMP model
3. ASEP
4. A general construction and a few applications (in particular, an asymmetric version of the KMP model)

0. Introduction: Dualities

- Fourier transform
- Duality between electric and magnetic fields
- Self-dual if the dual object is the same as the original one.
- An important tool in statistical mechanics

  Ex: Kramers-Wannier duality for 2D Ising model

1. Stochastic self-duality

Ω: state space

$\eta(t), \xi(t), t \geq 0$: Two copies of a Markov process on Ω

$D : \Omega \times \Omega \to \mathbb{R}$: Duality function

Def The process is self-dual $\Leftrightarrow$

$$\mathbb{E}_{\eta} D(\eta(t), \xi) = \mathbb{E}_{\xi} D(\eta, \xi(t))$$

where $\eta = \eta(0), \xi = \xi(0)$.

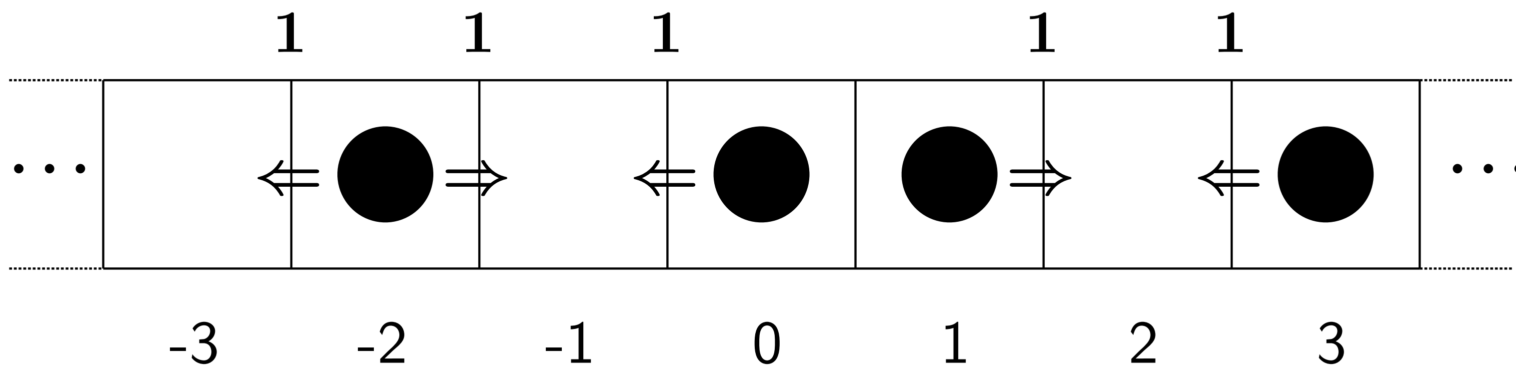
L: the generator of the Markov process

(For finite state space self-duality is equivalent to $LD = D^{\,t}L$.)

SEP

Symmetric simple exclusion process (SEP or SSEP)

1D case



$\eta_j = 1$ if site j is occupied, $\eta_j = 0$ if site j is empty.

Generator

$$Lf(\eta) = \sum_{j \in \mathbb{Z}} (\eta_j(1-\eta_{j+1}) + (1-\eta_j)\eta_{j+1})[f(\eta^{j,j+1}) - f(\eta)]$$

Self-duality for SEP

- In Liggett it is stated as

$$\mathbb{P}_\eta[\eta(t) = 1 \text{ on } A] = \mathbb{P}_A[\eta = 1 \text{ on } A_t]$$

where $A = \{x_1, \ldots, x_m\}, x_1 < \ldots < x_m, m \in \mathbb{N}$.

- This means that m-point correlation functions of SEP satisfy the m-particle SEP dynamics. For example for $m = 1$

$$\frac{d}{dt}\mathbb{E}\eta_x(t) = \mathbb{E}\eta_{x-1}(t) + \mathbb{E}\eta_{x+1}(t) - 2\mathbb{E}\eta_x(t)$$

Matrix representation for finite SEP

- For finite SEP with L sites, $\Omega = \{0,1\}^L$ (finite state space).
- Duality function
$$D(\eta,\xi) = \prod_{i=1,\xi_i=1}^{L} \eta_i$$
- The adjoint generator ${}^t L_{\mathsf{SEP}}$ of SEP
$${}^t L_{\mathsf{SEP}} = \frac{1}{2}\sum_{j=1}^{L-1}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z - 1)$$
where $\sigma^{x,y,z}$ are Pauri matrices
$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
- For these one can check $LD = D\,{}^t L$.

SU(2) symmetry

- The matrix ${}^t\boldsymbol{L}_{\mathsf{SEP}}$ is also known as the Hamiltonian of the Heisenberg chain ($= \boldsymbol{H}_{\mathsf{Hei}}$).

- $\boldsymbol{SU(2)}$ algebra

$$[S^z, S^\pm] = \pm S^\pm$$
$$[S^+, S^-] = 2S^z$$

The spin-$\frac{1}{2}$ representation is written in terms of Pauri matrices. One can consider the tensor product representation for $\boldsymbol{L}$ spin-$\frac{1}{2}$ spins.

- Set

$$S^+ = \frac{1}{2}\sum_{j=1}^{L}(\sigma_j^x + i\sigma_j^y)$$

$$S^- = \frac{1}{2}\sum_{j=1}^{L}(\sigma_j^x - i\sigma_j^y)$$

$$S^z = \frac{1}{2}\sum_{j=1}^{L}\sigma_j^z$$

They satisfy the $SU(2)$ algebra.

- Prop. H_{Hei} commutes with these generators:

$$[H_{\mathsf{Hei}}, S^{\pm}] = [H_{\mathsf{Hei}}, S^z] = 0$$

- The self-duality of SEP is a consequence of this symmetry. (1993 Sandow-Schütz)

Derivation of the self-duality relation

With $\langle N| = \langle 0|(S^+)^N/N!$ and $|I_N\rangle$: the initial state

$$
\begin{aligned}
&\langle \eta_{x_1} \cdots \eta_{x_m} \rangle \\
&= \langle N|\eta_{x_1} \cdots \eta_{x_m} e^{Ht}|I_N\rangle \\
&= \langle x_1, \cdots x_m| \frac{(S^+)^{N-m}}{(N-m)!} e^{Ht}|I_N\rangle \\
&\quad [\text{Comute } \ S^+ \ \text{ with } \ H] \\
&= \sum_{1 \le z_1 < \cdots < z_m \le L} \langle x_1, \cdots x_m|e^{Ht}|z_1, \cdots z_m\rangle \langle N|\eta_{z_1} \cdots \eta_{z_m}|I_N\rangle
\end{aligned}
$$

In the last equality, we use

$\mathbf{1} = \sum_{1 \le z_1 < \cdots < z_m \le L} |z_1, \cdots z_m\rangle\langle z_1, \cdots z_m|$

2. SU(1,1)

$\boldsymbol{SU(1,1)}$ algebra

$$[\boldsymbol{K^0}, \boldsymbol{K^\pm}] = \pm \boldsymbol{K^\pm}$$

$$[\boldsymbol{K^-}, \boldsymbol{K^+}] = \boldsymbol{2K^0}$$

A representation

$$\boldsymbol{K^+} = \frac{1}{2}\boldsymbol{x^2}$$

$$\boldsymbol{K^-} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$\boldsymbol{K^0} = \frac{1}{4}\left(\frac{\partial}{\partial x}x + x\frac{\partial}{\partial x}\right)$$

Brownian energy process

We consider the tensor product representation of $SU(1,1)$. The corresponding generator is given by

$$L = -4 \sum_j L_{j,j+1}$$

with

$$\begin{aligned} L_{j,j+1} &= K_j^+ K_{j+1}^+ + K_j^- K_{j+1}^- - 2K_j^0 K_{j+1}^0 + 1/2 \\ &= \left(x_j \frac{\partial}{\partial x_{j+1}} - x_{j+1} \frac{\partial}{\partial x_j} \right)^2 \end{aligned}$$

$L_{j,j+1}$ conserves the energy $x_j^2 + x_{j+1}^2$ and generates a Brownian rotation of the angle $\arctan(x_{j+1}/x_j)$.

The dynamics of x_i^2 is called the Brownian energy process(BEP).

k-BEP

Another representation of $SU(1,1)$ with parameter k

$$K^+ = \frac{1}{2} z$$
$$K^- = 2z\partial^2 + k\partial$$
$$K^0 = z\partial + k/4$$

For this

$$\begin{aligned} L_{j,j+1} &= K_j^+ K_{j+1}^+ + K_j^- K_{j+1}^- - 2K_j^0 K_{j+1}^0 + k^2/8 \\ &= (\partial_j - \partial_{j+1})^2 - 2k(z_j - z_{j+1})(\partial_j - \partial_{j+1}) \end{aligned}$$

$k = 2$ case is the usual BEP.

BEP can also be obtained as a limiting case of a particle system.

Symmetric Inclusion Process(SIP)

2010 Giardina Redig Vafayi

By considering the tensor product of another discrete representation of $SU(1,1)$ with parameter k, one can construct a process, SIP(k), with generator

$$(L^{SIP(k)}f)(\eta) := \sum_{i=1}^{L-1} (L_{i,i+1}^{SIP(k)}f)(\eta) \qquad \text{with}$$

$$(L_{i,i+1}^{SIP(k)}f)(\eta) :=$$
$$= (\eta_i(2k+\eta_{i+1}) + (2k+\eta_i)\eta_{i+1})(f(\eta^{i,i+1}) - f(\eta))$$

Prop. This process has a self-duality related to $SU(1,1)$.

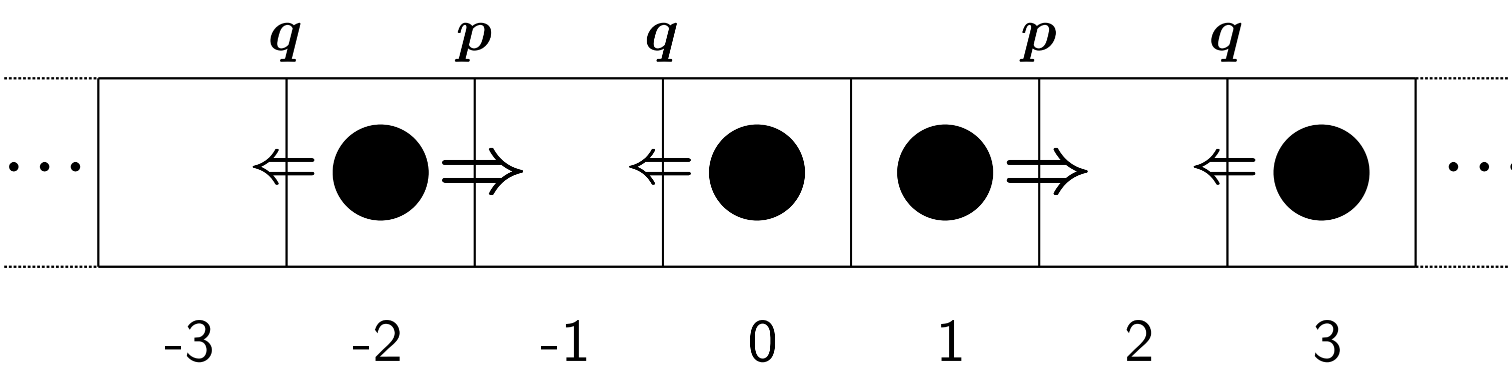
Prop. In a diffusion scaling limit, this tends to k-BEP.

KMP model

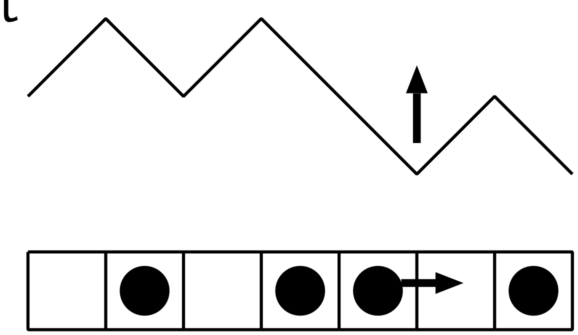
- KMP(Kipnis-Marchioro-Pressutti) model
- A bond $(i, i+1)$ is randomly selected and the energies of the two sites $i, i+1$ are uniformly redistributed under the constraint of conservation of $E_i + E_j$.
- KMP is the "instantaneous thermalization" limit of BEP.
- This is one of the few models for which one can do concrete analysis about fluctuations.

3. ASEP

ASEP = asymmetric simple exclusion process



- SEP($p = q$), TASEP(Totally ASEP, $p = 0$ or $q = 0$)
- $N(x, t)$: Integrated current at $(x, x + 1)$ upto time t
- In a certain weakly asymmetric limit
  ASEP $\Rightarrow$ KPZ equation



Self-duality

- 1997 Schütz

  The n-point function of the form $\mathbb{E}[\prod_{i=1}^{n} q^{N(x_i,t)}]$ satisfies the n particle dynamics of the same process (self-duality).

- The adjoint generator of ASEP is equivalent to the Hamiltonian of XXZ spin chain by a similarity transformation.

  The self-dality is related to $U_q(sl_2)$ symmetry of XXZ and ASEP.

- 2012-2015 Borodin-Corwin-TS

  The self-duality of ASEP can be used to study the fluctuations of current $N(x,t)$.

Deformed algebra $U_q(sl_2)$

$$[J^+, J^-] = [2J^0]_q, \qquad [J^0, J^\pm] = \pm J^\pm$$

and

$$[2J^0]_q := \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}$$

Casimir element

$$C = J^- J^+ + [J^0]_q [J^0 + 1]_q$$

XXZ spin chain

By considering the tensor product representation of L spin-$\frac{1}{2}$ spins, we see that the XXZ spin chain Hamiltonian with boundary magnetic fields

$$H_{\mathrm{XXZ}} = h\sigma_1^z + \frac{1}{2}\sum_{j=1}^{L-1}[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)] - h\sigma_L^z$$

with $h = (Q - Q^{-1})/4, \Delta = (Q + Q^{-1})/2$ has the $U_Q(sl_2)$ symmetry.

ASEP and XXZ

Adjoint generator of ASEP (with reflective bounaries)

$$ {}^{t}L_{\mathrm{ASEP}} = \sum_j \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{j,j+1} $$

With $Q = \sqrt{q/p}, \Delta = (Q + Q^{-1})/2$ and $V = \prod_j Q^{jn_j}$ where $n_j = \frac{1}{2}(1 - \sigma_j^z)$ this is related to XXZ hamiltonian by

$$ V\,{}^{t}L_{\mathrm{ASEP}} V^{-1}/\sqrt{pq} = H_{\mathrm{XXZ}} $$

4. A general construction

- $\boldsymbol{H}$: $\boldsymbol{n \times n}$ symmetric matrix with non-negative off diagonal elements ($\boldsymbol{n = |\Omega|}$). The lowest eigenvalue is taken to be 0.

- By Perron-Frobenius theorem, there exist $\boldsymbol{g} \in \mathbb{R}^{|\boldsymbol{\Omega}|}$ with strictly positive entries such that $\boldsymbol{Hg = 0}$.

- Let us denote by $\boldsymbol{G}$ the diagonal matrix with entries $\boldsymbol{G(x,x) = g(x)}$ for $\boldsymbol{x \in \Omega}$.

- The matrix
$$\boldsymbol{L = G^{-1}HG}$$
is a generator of a Markov process.

- If $\boldsymbol{[H,S] = 0}$, then $\boldsymbol{[L, G^{-1}SG] = 0}$ and $\boldsymbol{D = G^{-1}SG^{-1}}$ is a self-duality function for the process with generator $\boldsymbol{L}$.

Main results

By applying the general scheme in the previous slide to a deformed algebra, one can systematically try to construct Markov processes with asymmetry which has self-duality.

- By applying the scheme to $U_q(sl_2)$, one can construct a generalization of ASEP in which there could be more than one particles on each site.
- By applying the scheme to $U_q(su(1,1))$, one can construct a generalization of BEP and as a limiting case an asymmetric version of the KMP model.
- The scheme was applied to $U_q(sl_3)$ and $U_q(sp_4)$ by Kuan ($U_q(sl_3)$ also by Belitsky-Schütz).

Application 1: Spin j representation of $U_q(sl_2)$

The Markov process ASEP(q,j) on $[1,L] \cap \mathbb{Z}$ with closed boundary conditions is defined by the generator

$$(Lf)(\eta) = \sum_{i=1}^{L-1} (L_{i,i+1}f)(\eta) \qquad \text{with}$$

$$(L_{i,i+1}f)(\eta) = q^{\eta_i - \eta_{i+1} - (2j+1)} [\eta_i]_q [2j - \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta))$$
$$+ q^{\eta_i - \eta_{i+1} + (2j+1)} [2j - \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))$$

$j = 1/2$ is the usual ASEP.

Thm. This process has a duality related to $U_q(sl_2)$.

Application 2: $U_q(su(1,1))$

For $q \in (0,1)$ we consider the algebra with generators K^+, K^-, K^0 satisfying the commutation relations

$$[K^0, K^\pm] = \pm K^\pm, \qquad [K^-, K^+] = [2K^0]_q$$

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

Casimir element

$$C = [K^0]_q [K^0 - 1]_q - K^+ K^-$$

Asymmetric process with self-duality

By considering the tensor product of a representation with parameter k, we can construct a process, ASIP(q,k), with closed boundary conditions with generator

$$(L^{ASIP(q,k)}f)(\eta) := \sum_{i=1}^{L-1} (L_{i,i+1}^{ASIP(q,k)}f)(\eta) \qquad \text{with}$$

$$\begin{aligned}
&(L_{i,i+1}^{ASIP(q,k)}f)(\eta) \\
&:= q^{\eta_i-\eta_{i+1}+(2k-1)}[\eta_i]_q[2k+\eta_{i+1}]_q(f(\eta^{i,i+1})-f(\eta)) \\
&\quad + q^{\eta_i-\eta_{i+1}-(2k-1)}[2k+\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i})-f(\eta))
\end{aligned}$$

Thm. This process has a duality related to $U_q(su(1,1))$.

Asymetric Brownian Energy Process ABEP

Consider the limit of weak asymmetry $q = 1 - \epsilon\sigma \to 1$ $(\epsilon \to 0)$ combined with the number of particles proportional to ϵ^{-1}, going to infinity, and work with rescaled particle numbers $x_i = \lfloor \epsilon \eta_i \rfloor$.

Generator

Let $\sigma > 0$ and $k \geq 0$. The generator of ABEP(σ, k) is

$$L^{ABEP^{(\sigma,k)}} f(x) = \sum_{i=1}^{L-1} [L_{i,i+1}^{ABEP^{(\sigma,k)}} f](x)$$

with

$$L_{i,i+1}^{ABEP^{(\sigma,k)}} f(x) = \frac{1}{4\sigma^2} (1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2$$

$$- \frac{1}{2\sigma} \left\{ (1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) + 2k(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}}) \right\}$$

$$\times \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) f(x)$$

$\sigma \to 0$ correspondes to k-BEP.

Asymmetric version of the KMP model

By considering an "instantaneous thermalization" limit of the ABEP, we can define am asymmetric KMP with asymmetry parameter $\sigma \in \mathbb{R}_+$ as the process with generator given by:

$$L^{AKMP(\sigma)} f(x) = \sum_{i=1}^{L-1} \left\{ \frac{2\sigma(x_i + x_{i+1})}{e^{2\sigma(x_i+x_{i+1})} - 1} \right.$$

$$\cdot \int_0^1 [f(x_1, \ldots, w(x_i + x_{i+1}), (1-w)(x_i + x_{i+1}), \ldots, x_L) - f(x)$$

$$\left. \times \; e^{2\sigma w(x_i+x_{i+1})} \, dw \right\}$$

- This is an example with duality but without integrability.
- Properties of the process are yet to be studied.

Summary

- (Self-)duality: The m-point correlation function can be reduced to m-particle problem
- Self-dualities for asymmetric processes. Current fluctuations for ASEP
- A general scheme to construct Markov processes with (deformed) symmetry
- Examples of spin $U_q(sl_2)$ and $U_q(su(1,1))$.
- Properties of the asymmetric KMP model?