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Breakdown of the hydrodynamic limit for extreme current fluctuations

Yongjoo Baek[1], Yariv Kafri[1], Vivien Lecomte[2]

[1] Department of Physics, Technion
[2] LPMA, Université Paris VII

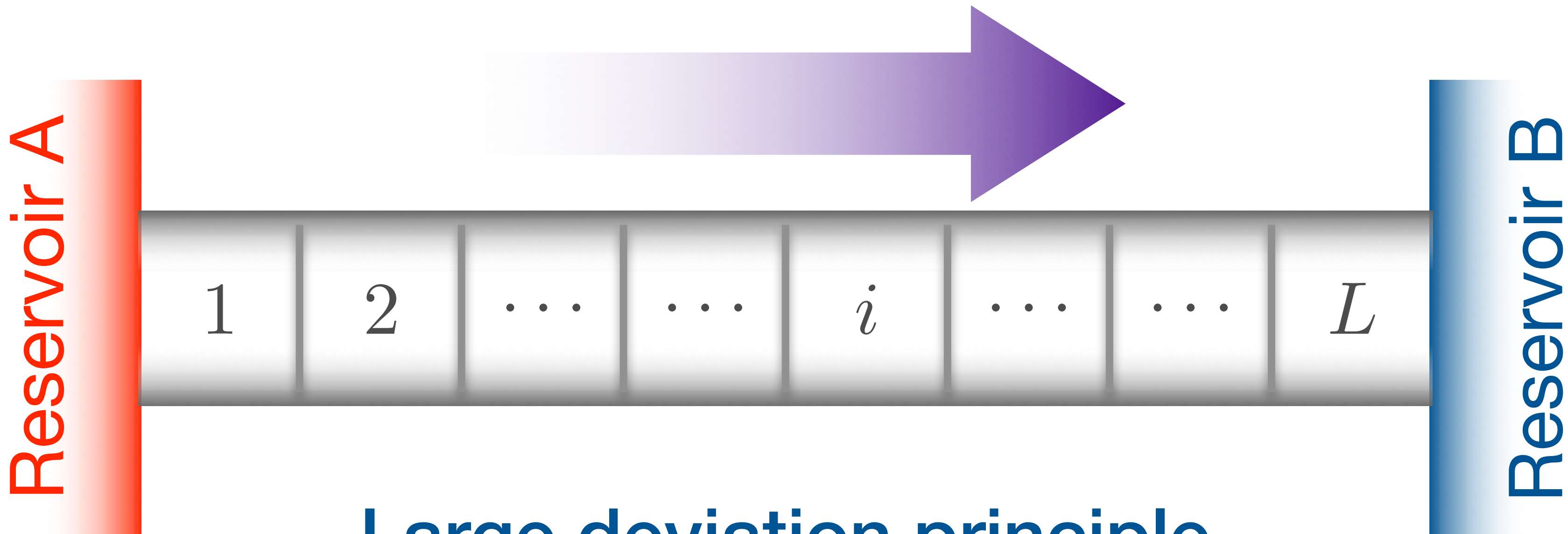
Introduction

Current fluctuations in the hydrodynamic limit

Boundary-driven systems

Time-averaged current

$$J = \frac{Q_B}{T}$$ (Total flux to B) (Observation time)



Large deviation principle

For $T \gg 1$, $P(J) \sim e^{-T\Phi(J)}$ Large deviation function (LDF)

Cumulant generating function

$$\langle e^{\lambda Q_B} \rangle = e^{T\mu(\lambda)}, \quad \Phi(J) = \sup_{\lambda}[\lambda J - \mu(\lambda)]$$

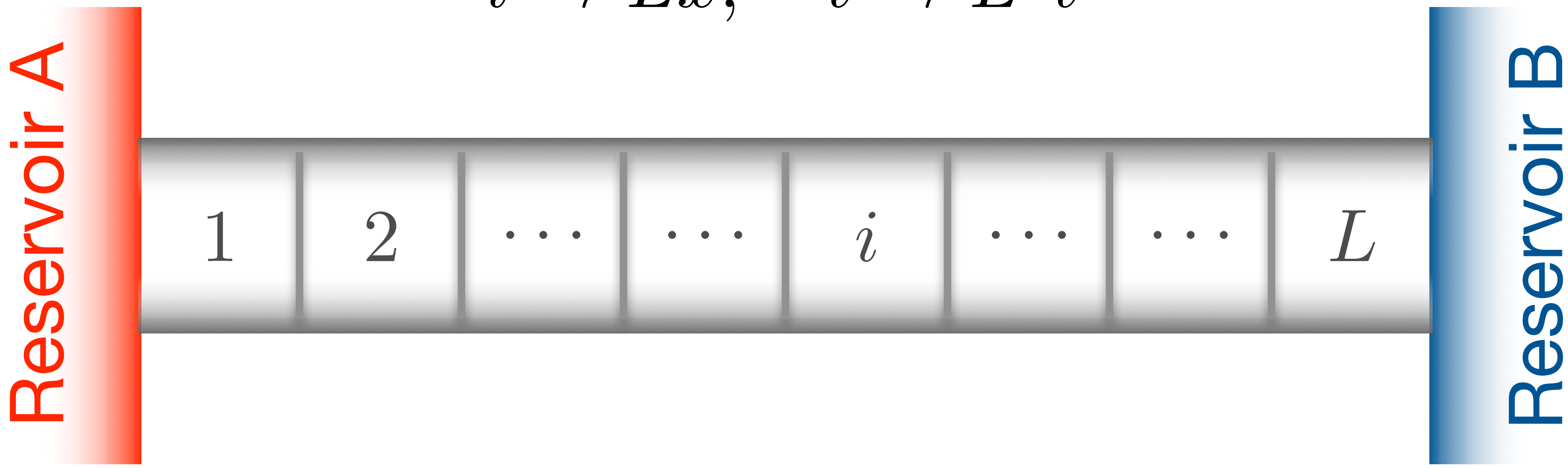
Hydrodynamic limit

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

$L \to \infty$, local equilibrium

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$



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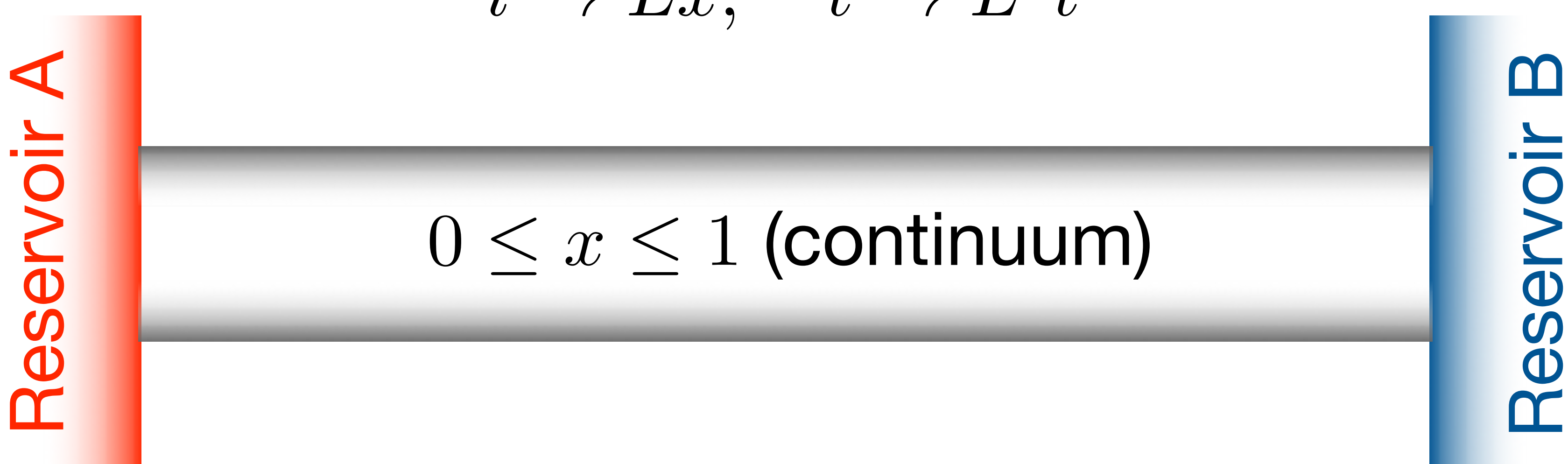

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$

$$\langle \eta(x,t)\eta(x',t') \rangle = L^{-1}\delta(x-x')\delta(t-t')$$

Hydrodynamic limit

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

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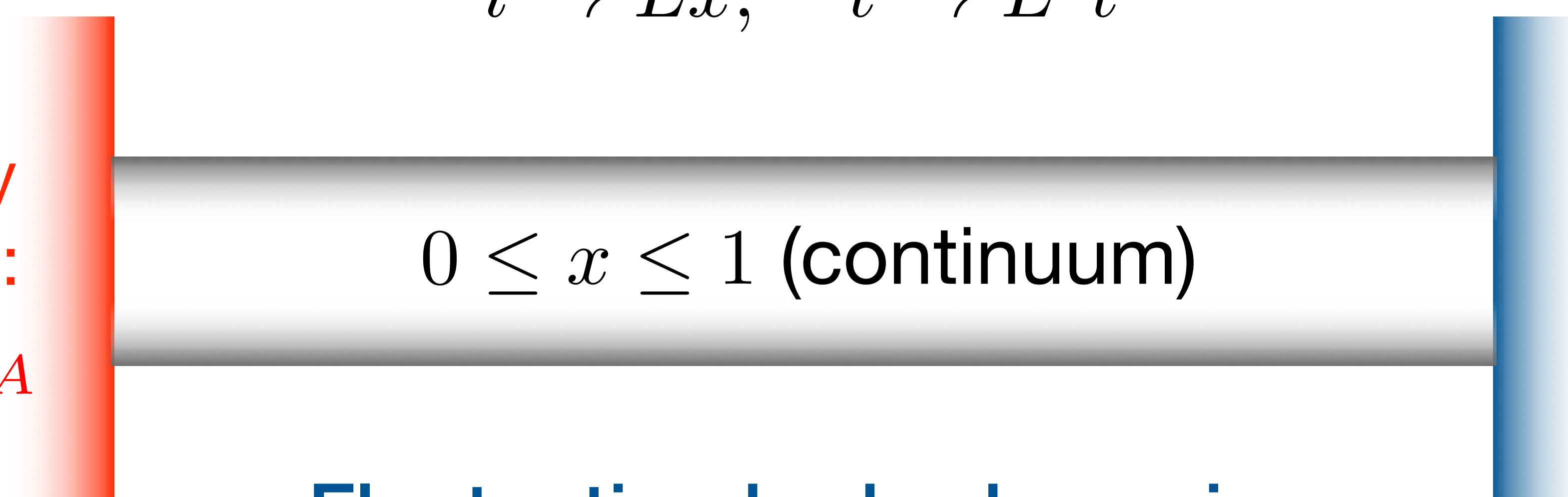
$$i \to Lx, \quad t \to L^2 t$$

Boundary condition: $\rho(0) = \bar{\rho}_A$



Boundary condition: $\rho(1) = \bar{\rho}_B$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-\underline{D(\rho)}\partial_x \rho + \sqrt{\underline{\sigma(\rho)}}\eta \right]$$

Diffusivity — Mobility

$$\langle \eta(x,t)\eta(x',t') \rangle = \underline{L^{-1}}\delta(x-x')\delta(t-t')$$

Weak noise
(microscopic origin)

Calculation of current LDFs

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$

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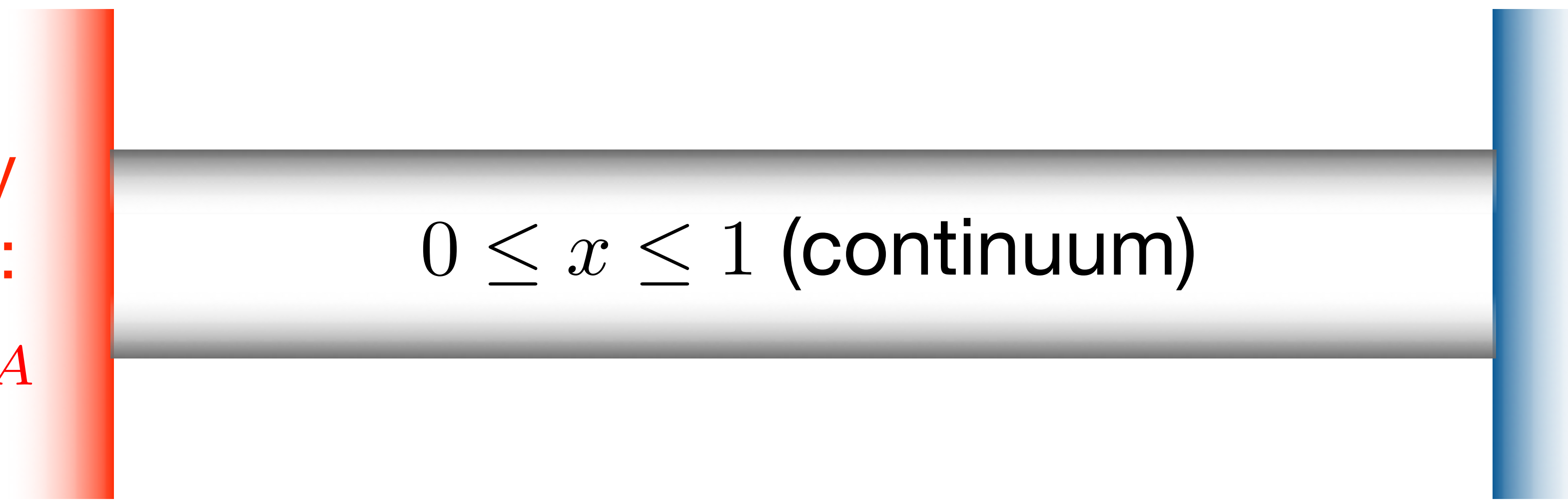


Boundary condition: $\rho(1) = \bar{\rho}_B$

Microscopic approaches (take the limit later)

Derrida, Douçot, Roche, J. Stat. Phys. **115**, 717 (2004)

Bodineau & Derrida, Phys. Rev. Lett. **92**, 180601 (2004)

Macroscopic fluctuation theory (take the limit first)

Bertini et al., Phys. Rev. Lett. **94**, 030601 (2005)

Bertini et al., J. Stat. Phys. **123**, 237 (2006)

Simplification by saddle-point approximation

Macroscopic fluctuation theory

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

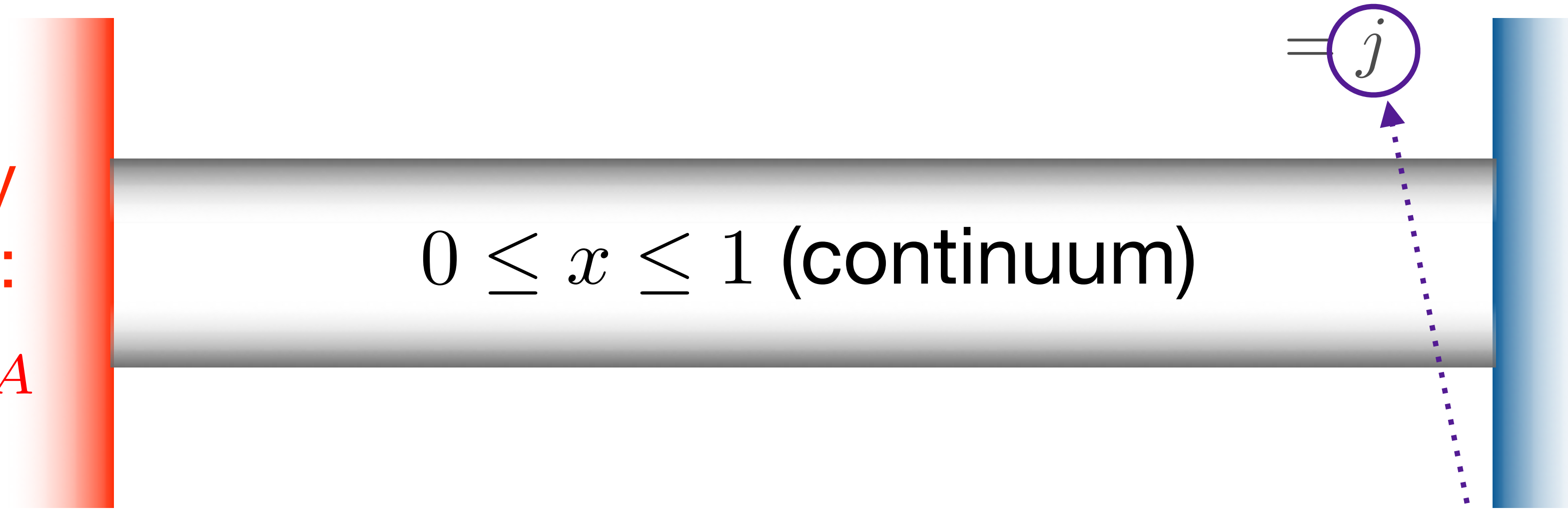
Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \Big[\underbrace{-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta}_{=j} \Big]$$

Boundary condition: $\rho(0) = \bar{\rho}_A$



Boundary condition: $\rho(1) = \bar{\rho}_B$

1. Express the CGF in a path integral form.

$$\langle e^{\lambda Q_B} \rangle = \int \mathcal{D}\rho \Big\langle \underbrace{\delta(\partial_t \rho + \partial_x j)}_{\text{Hydrodynamic equation}} \, e^{\underbrace{L \int_0^{T/L^2} \mathrm{d}t \int_0^1 \mathrm{d}x \, \lambda j}_{= \lambda Q_B}} \Big\rangle_j$$

Fluctuating curent

Macroscopic fluctuation theory

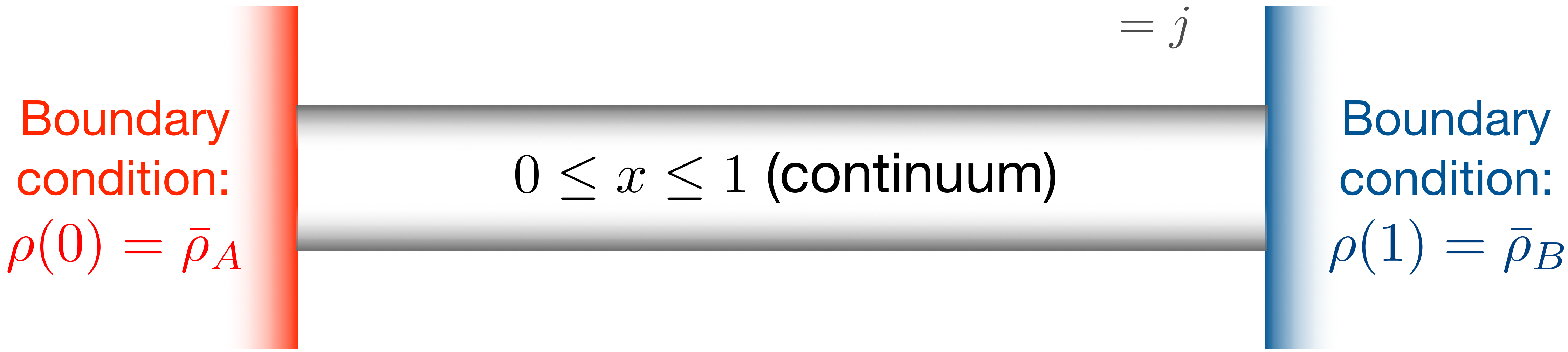
Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \Big[\underbrace{-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta}_{=j} \Big]$$



2. Introduce the conjugate field $\hat{\rho}$.

$$\left\langle e^{\lambda Q_B} \right\rangle = \int \mathcal{D}\rho \left\langle \underline{\delta(\partial_t \rho + \partial_x j)}\, e^{L \int_0^{T/L^2} \mathrm{d}t \int_0^1 \mathrm{d}x\, \lambda j} \right\rangle_j$$

$$= \int \mathcal{D}\hat{\rho}\, e^{-L \int_0^{T/L^2} \mathrm{d}t\, \hat{\rho}(\partial_t \rho + \partial_x j)}$$

Martin, Siggia, Rose, Phys. Rev. A **8**, 423 (1973)

Macroscopic fluctuation theory

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

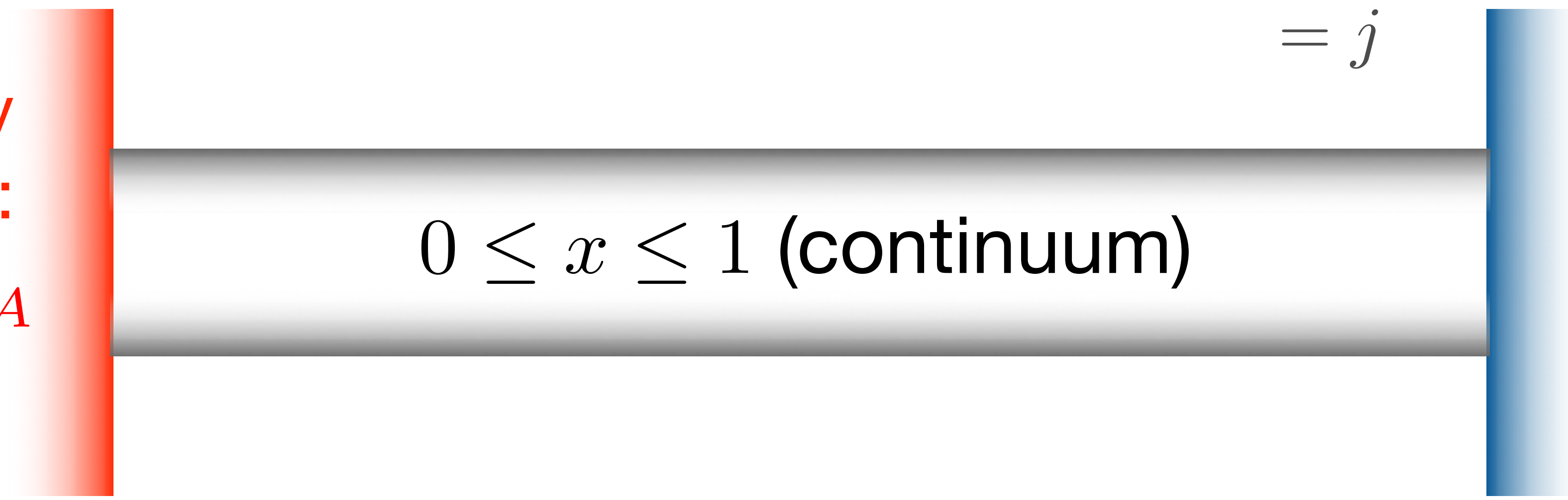
Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \Big[\underbrace{-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta}_{=j} \Big]$$

Boundary condition:
$\rho(0) = \bar{\rho}_A$
$\hat{\rho}(0) = 0$



Boundary condition:
$\rho(1) = \bar{\rho}_B$
$\hat{\rho}(1) = \lambda$

3. Average over the fluctuating current.

$$\left\langle e^{\lambda Q_B} \right\rangle = \int \mathcal{D}\rho \mathcal{D}\hat{\rho} \left\langle e^{-L \int_0^{T/L^2} \mathrm{d}t \int_0^1 \mathrm{d}x \, (\hat{\rho}\partial_t \rho - j \partial_x \hat{\rho})} \right\rangle_{j}$$

Macroscopic fluctuation theory

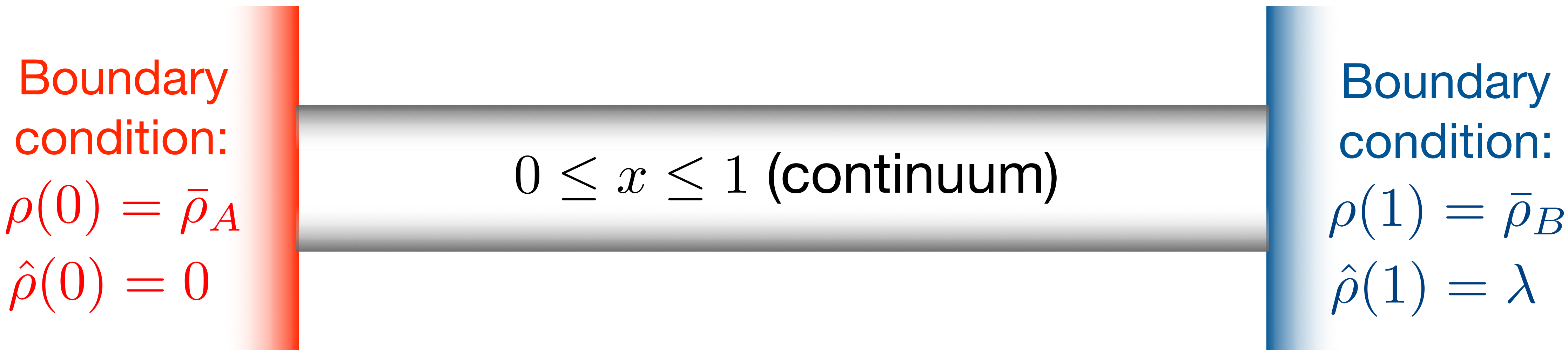
Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$



4. Apply the saddle-point approximation.

$$e^{T\mu(\lambda)} = \int \mathcal{D}\rho \mathcal{D}\hat{\rho}\, e^{-\underline{L} \int_0^{T/L^2} \mathrm{d}t \int_0^1 \mathrm{d}x\, [\hat{\rho}\partial_t \rho - \underline{H(\rho,\hat{\rho})}]}$$

$L \gg 1$

$$H(\rho, \hat{\rho}) = -D(\rho)(\partial_x \rho)(\partial_x \hat{\rho}) + \frac{1}{2}\sigma(\rho)(\partial_x \hat{\rho})^2$$

Macroscopic fluctuation theory

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$

Boundary condition:
$\rho(0) = \bar{\rho}_A$
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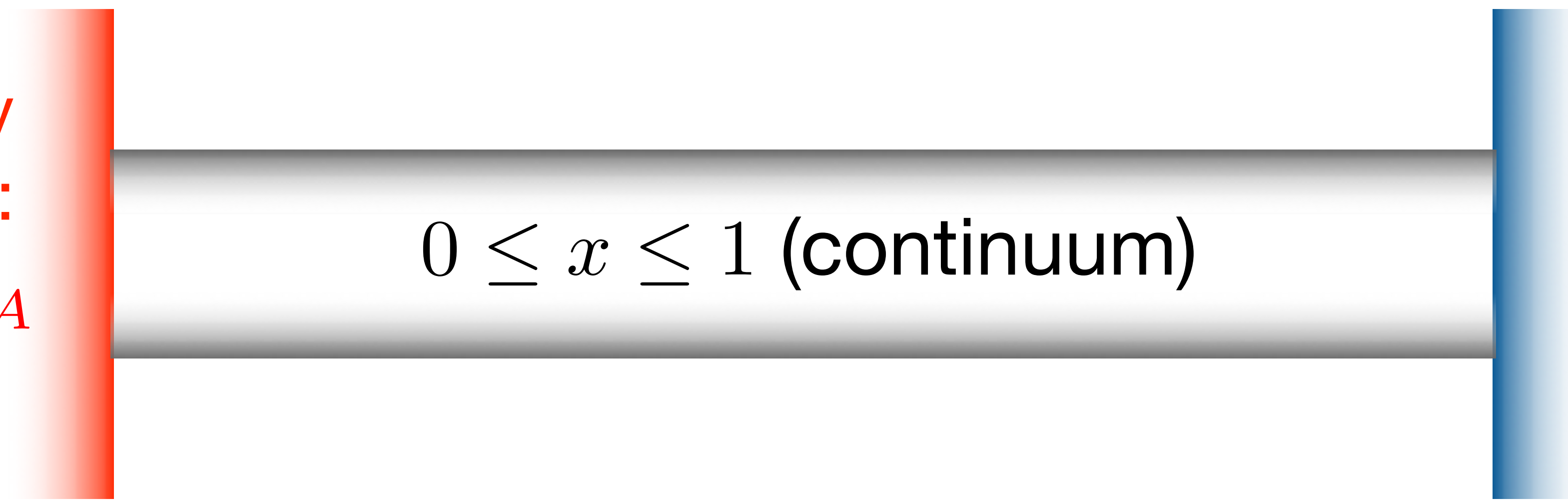

Boundary condition:
$\rho(1) = \bar{\rho}_B$
$\hat{\rho}(1) = \lambda$

5. Derive Hamilton's equations.

Position Momentum

$$T\mu(\lambda) = -L \min_{\rho,\hat{\rho}} \int_0^{T/L^2} \mathrm{d}t \int_0^1 \mathrm{d}x \left[\hat{\rho}\partial_t \rho - \underline{H(\rho,\hat{\rho})} \right]$$

$$H(\rho,\hat{\rho}) = -D(\rho)(\partial_x \rho)(\partial_x \hat{\rho}) + \frac{1}{2}\sigma(\rho)(\partial_x \hat{\rho})^2$$

Macroscopic fluctuation theory

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$

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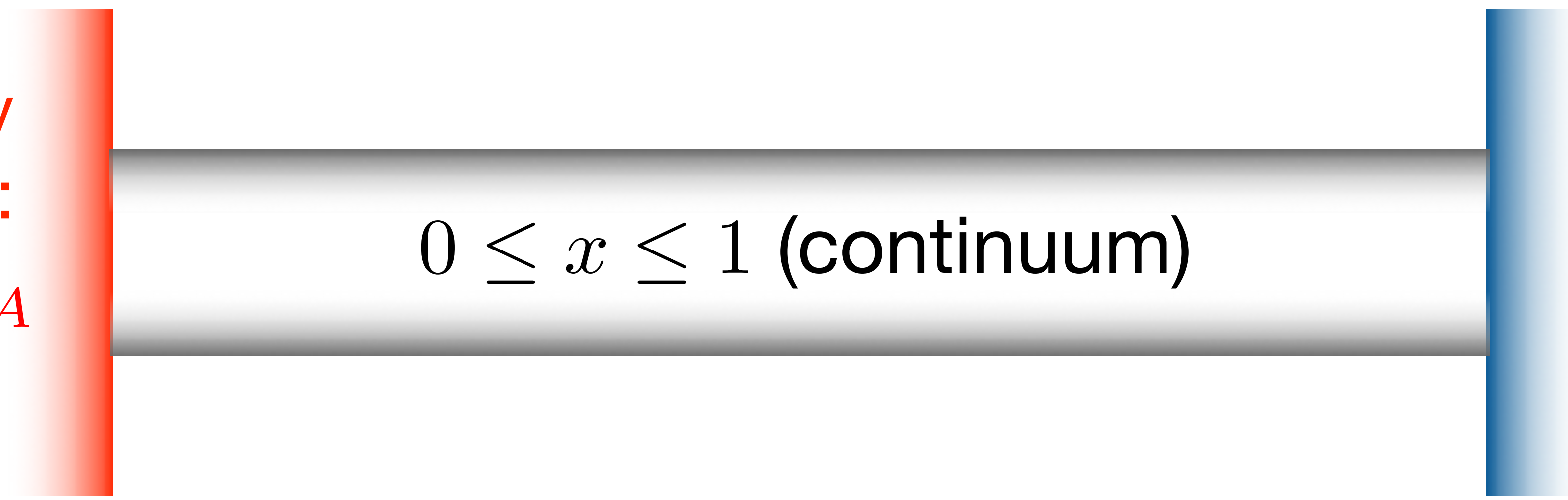


Boundary condition:
$\rho(1) = \bar{\rho}_B$
$\hat{\rho}(1) = \lambda$

5. Derive Hamilton's equations. $H(\rho, \hat{\rho}) = -D(\rho)(\partial_x \rho)(\partial_x \hat{\rho}) + \frac{1}{2}\sigma(\rho)(\partial_x \hat{\rho})^2$

$$T\mu(\lambda) = -L \min_{\rho, \hat{\rho}} \int_0^{T/L^2} \mathrm{d}t \int_0^1 \mathrm{d}x \left[\hat{\rho}\partial_t \rho - H(\rho, \hat{\rho}) \right]$$

$$\partial_t \rho = \frac{\partial H}{\partial \hat{\rho}} = -\partial_x [-D(\rho)\partial_x \rho + \sigma(\rho)\partial_x \hat{\rho}] \qquad \partial_t \hat{\rho} = -\frac{\partial H}{\partial \rho}$$

$\partial_x \hat{\rho} =$ noise realization

Macroscopic fluctuation theory

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

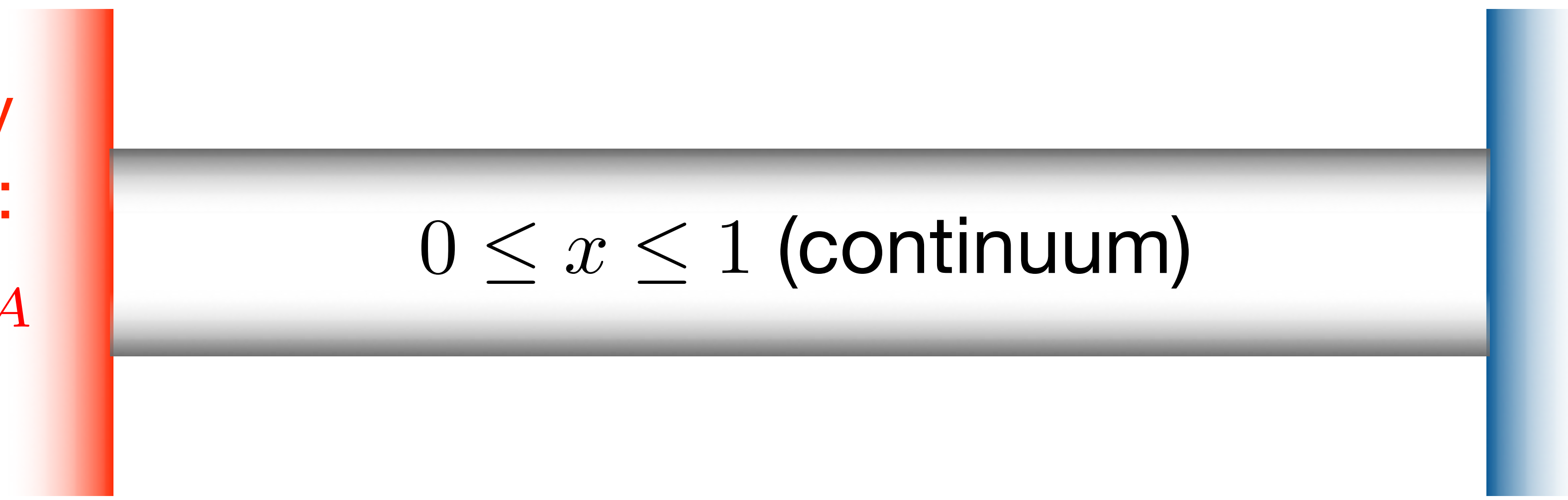
Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$

Boundary condition:
$\rho(0) = \bar{\rho}_A$
$\hat{\rho}(0) = 0$



Boundary condition:
$\rho(1) = \bar{\rho}_B$
$\hat{\rho}(1) = \lambda$

6. Assume that the minimizing solution is stationary.

$$T\mu(\lambda) = -L \min_{\rho,\hat{\rho}} \int_0^{T/L^2} \mathrm{d}t \int_0^1 \mathrm{d}x \left[\hat{\rho}\partial_t \rho - H(\rho, \hat{\rho}) \right]$$

$$\partial_t \rho = \frac{\partial H}{\partial \hat{\rho}} = 0, \;\; \partial_t \hat{\rho} = -\frac{\partial H}{\partial \rho} = 0 \quad \Rightarrow \quad \rho(x,t) = \rho^*_\lambda(x), \;\; \hat{\rho}(x,t) = \hat{\rho}^*_\lambda(x)$$

Bodineau & Derrida, Phys. Rev. Lett. **92**, 180601 (2004)

Macroscopic fluctuation theory

Goal: obtain $\mu(\lambda)$ and $\Phi(J)$

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Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$

Boundary condition:
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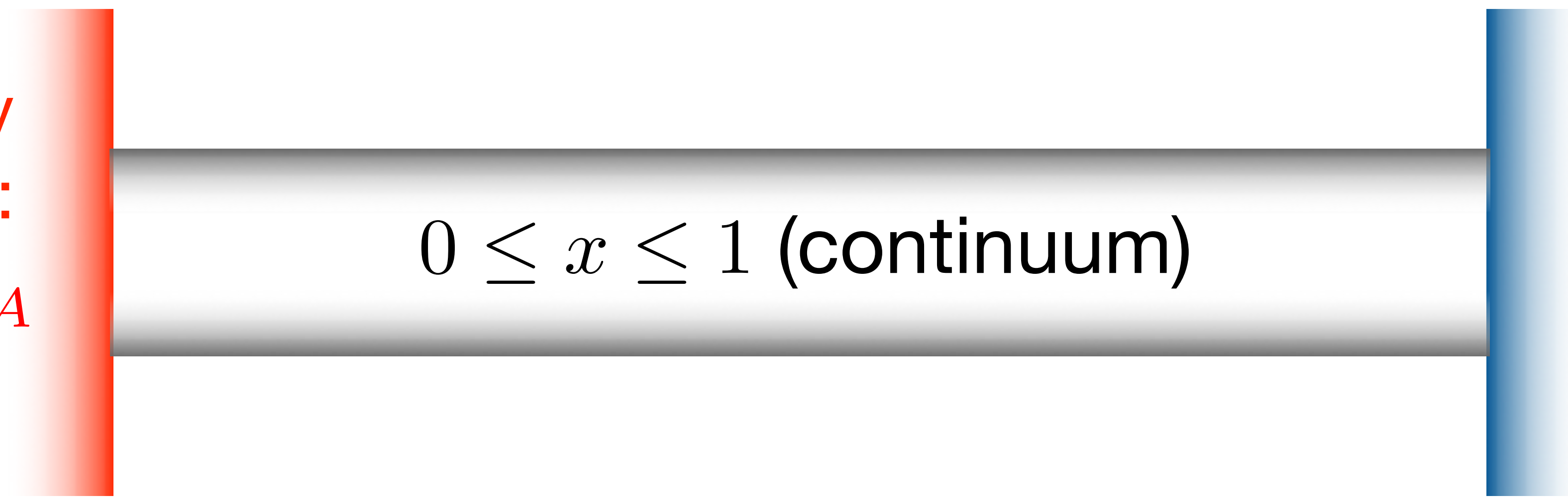


Boundary condition:
$\rho(1) = \bar{\rho}_B$
$\hat{\rho}(1) = \lambda$

7. Evaluate the Hamiltonian at the “optimal profile”.

Current CGF

$$\mu(\lambda) = \frac{1}{L}\int_0^1 \mathrm{d}x \, H(\rho^*_\lambda, \hat{\rho}^*_\lambda)$$

Optimal profile

$$\rho(x,t) = \rho^*_\lambda(x), \;\; \hat{\rho}(x,t) = \hat{\rho}^*_\lambda(x)$$

Bodineau & Derrida, Phys. Rev. Lett. **92**, 180601 (2004)

Bertini et al., Rev. Mod. Phys. **87**, 593 (2015)

Macroscopic fluctuation theory

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Continuum limit

Fluctuating hydrodynamics

$$\partial_t \rho = -\partial_x \left[-D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$

Saddle-point approximation

Effective Hamiltonian

$$H(\rho, \hat{\rho}) = -D(\rho)(\partial_x \rho)(\partial_x \hat{\rho}) + \frac{1}{2}\sigma(\rho)(\partial_x \hat{\rho})^2$$

Stationary solution

Optimal profile

$$\rho(x,t) = \rho^*_\lambda(x), \;\; \hat{\rho}(x,t) = \hat{\rho}^*_\lambda(x)$$

Current CGF

$$\mu(\lambda) = \frac{1}{L}\int_0^1 \mathrm{d}x \, H(\rho^*_\lambda, \hat{\rho}^*_\lambda)$$

Current fluctuations in the hydrodynamic limit

Current CGF

For $D(\rho) = 1$ and $\sigma(\rho) = c_2\rho^2 + c_1\rho$,

$$\mu(\lambda) = \frac{1}{L}\int_0^1 \mathrm{d}x\, H(\rho^*_\lambda, \hat{\rho}^*_\lambda) = \begin{cases} -\frac{2}{Lc_2}\left(\mathrm{arcsinh}\sqrt{\omega}\right)^2 & \text{for } \omega > 0 \\ +\frac{2}{Lc_2}\left(\mathrm{arcsin}\sqrt{-\omega}\right)^2 & \text{for } \omega < 0 \end{cases}$$

$$\omega(\lambda, \bar{\rho}_A, \bar{\rho}_B) = \frac{c_2}{c_1^2}(1 - e^{-c_1\lambda/2})[c_1(\bar{\rho}_B - e^{c_1\lambda/2}\bar{\rho}_A) - c_2(e^{c_1\lambda/2} - 1)\bar{\rho}_A\bar{\rho}_B]$$

Imparato, Lecomte, van Wijland, Phys. Rev. E **80**, 011131 (2009)

Current LDF $\Phi(J) = \sup_\lambda[\lambda J - \mu(\lambda)]$

Symmetric exclusion process

(SEP: $c_1 = 2$, $c_2 = -2$)

$\Phi(J) \sim J^2$ (Gaussian)

Kipnis–Marchioro–Presutti model

(KMP: $c_1 = 0$, $c_2 = 2$)

$\Phi(J) \sim J$ (Exponential)

Current fluctuations in the hydrodynamic limit

Current CGF

$$\mu(\lambda) = \frac{1}{L}\int_0^1 \mathrm{d}x\, H(\rho_\lambda^*, \hat{\rho}_\lambda^*)$$

Current LDF

$$\Phi(J) = \sup_\lambda[\lambda J - \mu(\lambda)]$$

In any case, $J(\lambda) = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = O(L^{-1})$

Interpretation

In the hydrodynamic limit, current fluctuations are always comparable to the average density gradient.

Questions

- How can we study even larger current fluctuations?
- Can extreme current fluctuations induce non-hydrodynamic tail behaviors?

Current fluctuations in the hydrodynamic limit

Questions

- How can we study even larger current fluctuations?
- Can extreme current fluctuations induce non-hydrodynamic tail behaviors?

Objectives

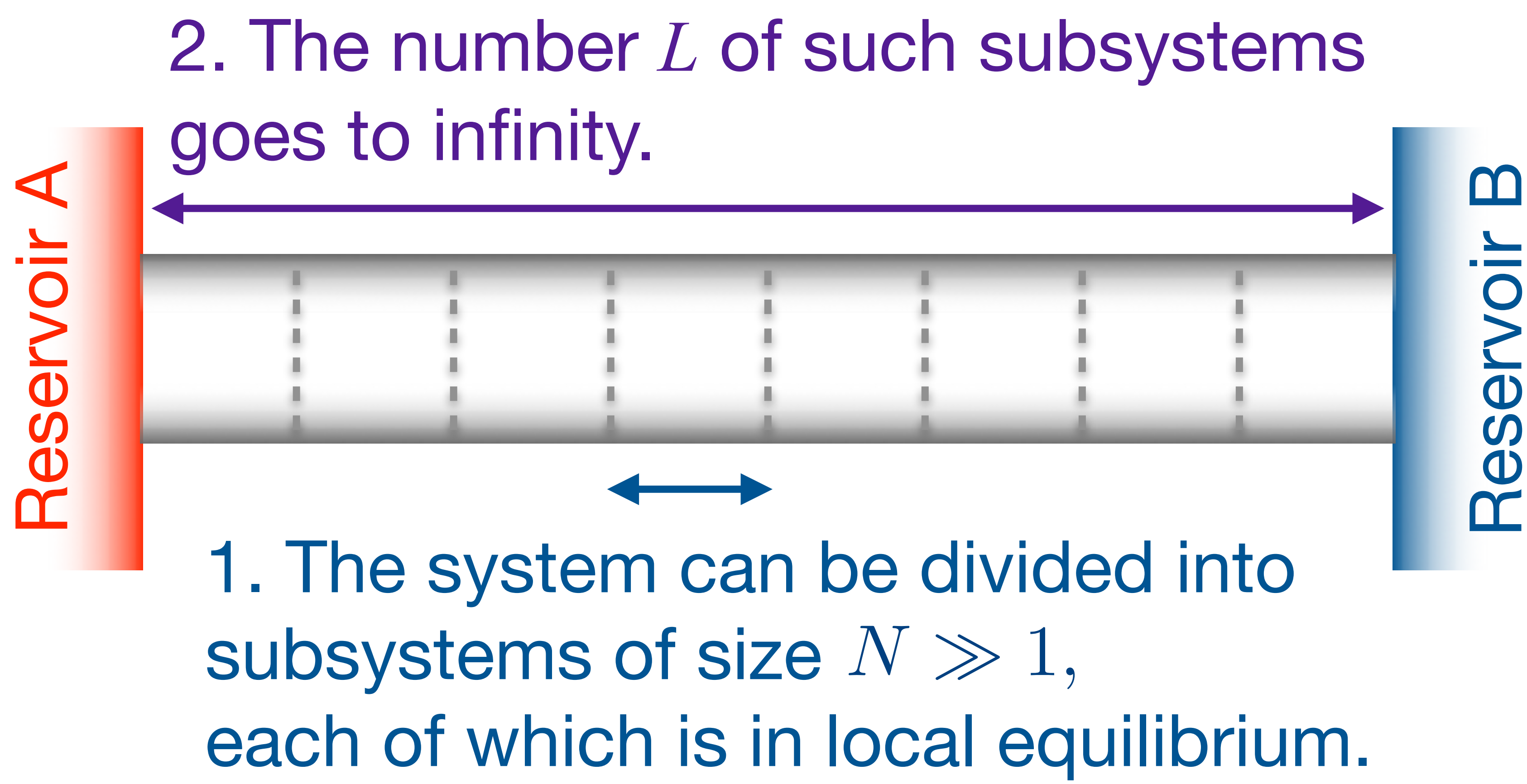
- Develop a rescaling scheme which enables saddle-point techniques but does not directly lead to the hydrodynamic limit.
- Check if the rescaling scheme can recover the hydrodynamic limit under appropriate conditions.
- Obtain current CGFs assuming stationary saddle-point solutions.

Rest of the talk

- Large N limit
- Current fluctuations of the SEP-like model
- Current fluctuations of the KMP-like model
- Properties of optimal profiles
- Conclusions and future works

Revisiting the hydrodynamic limit

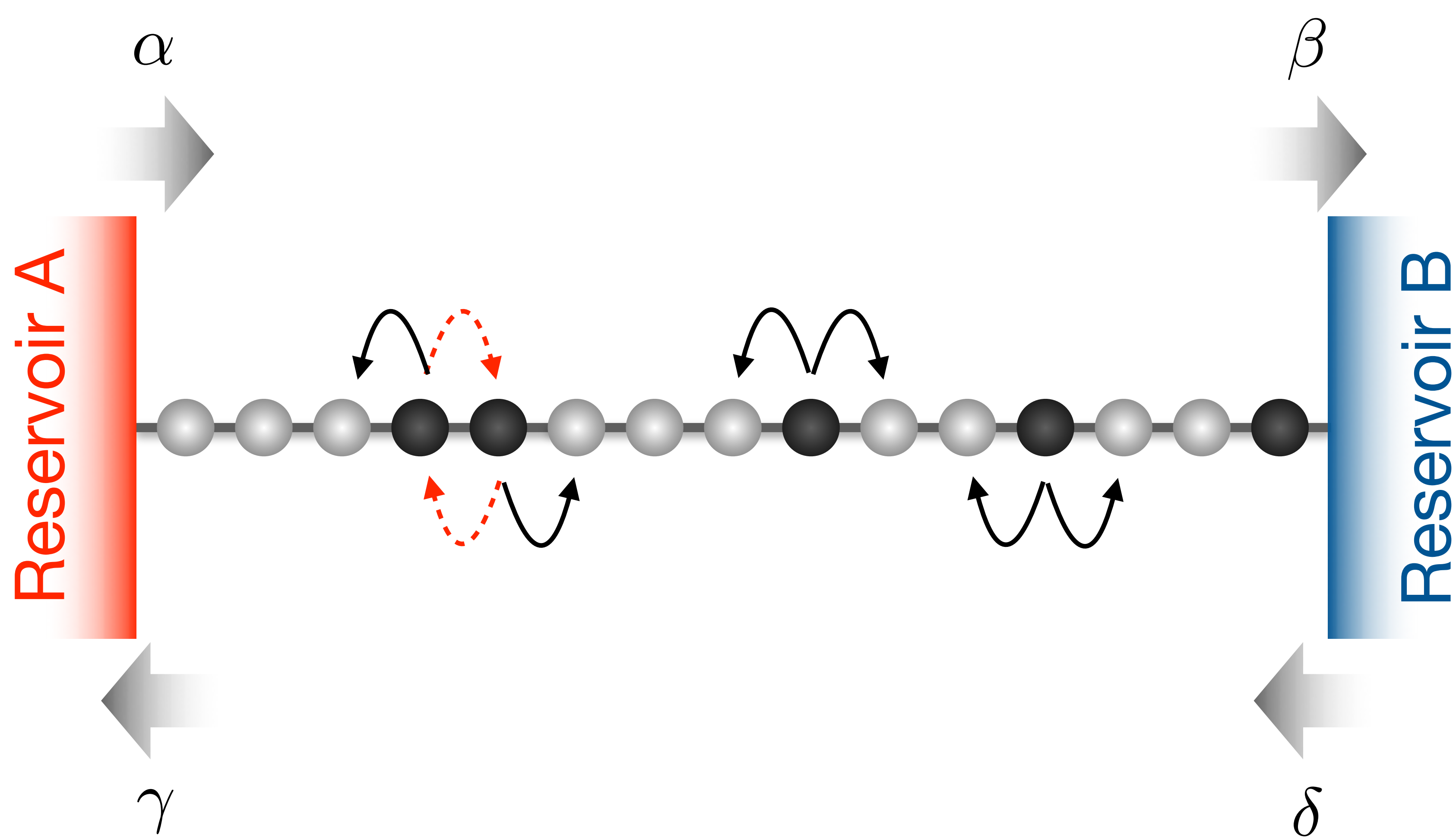
Hydrodynamic limit involves two limiting processes.



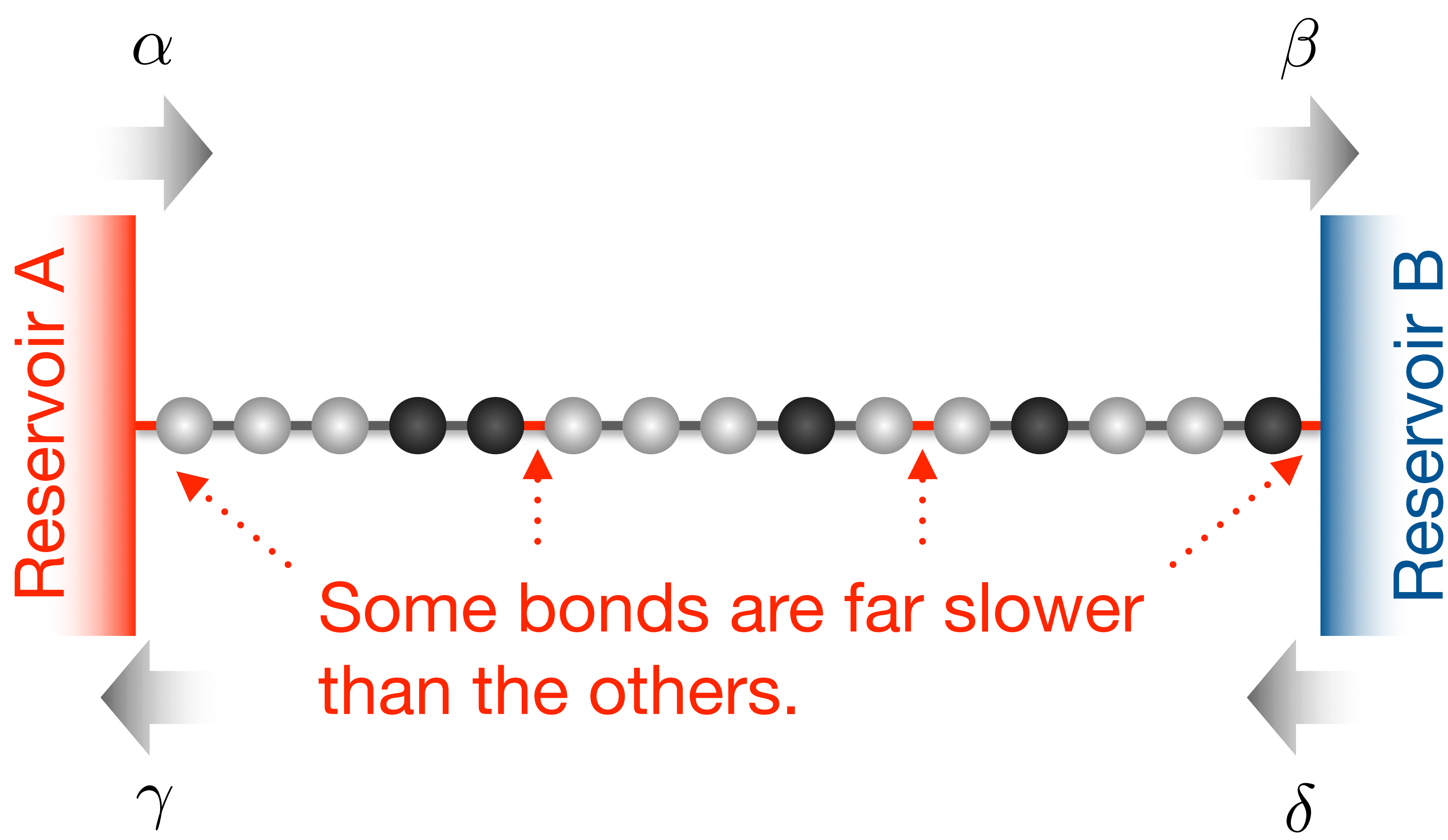
Idea: take only the first limit, leave the second limit untaken.

Derrida, J. Stat. Mech., P07023 (2007)

Example: coarse-grained SEP



Example: coarse-grained SEP



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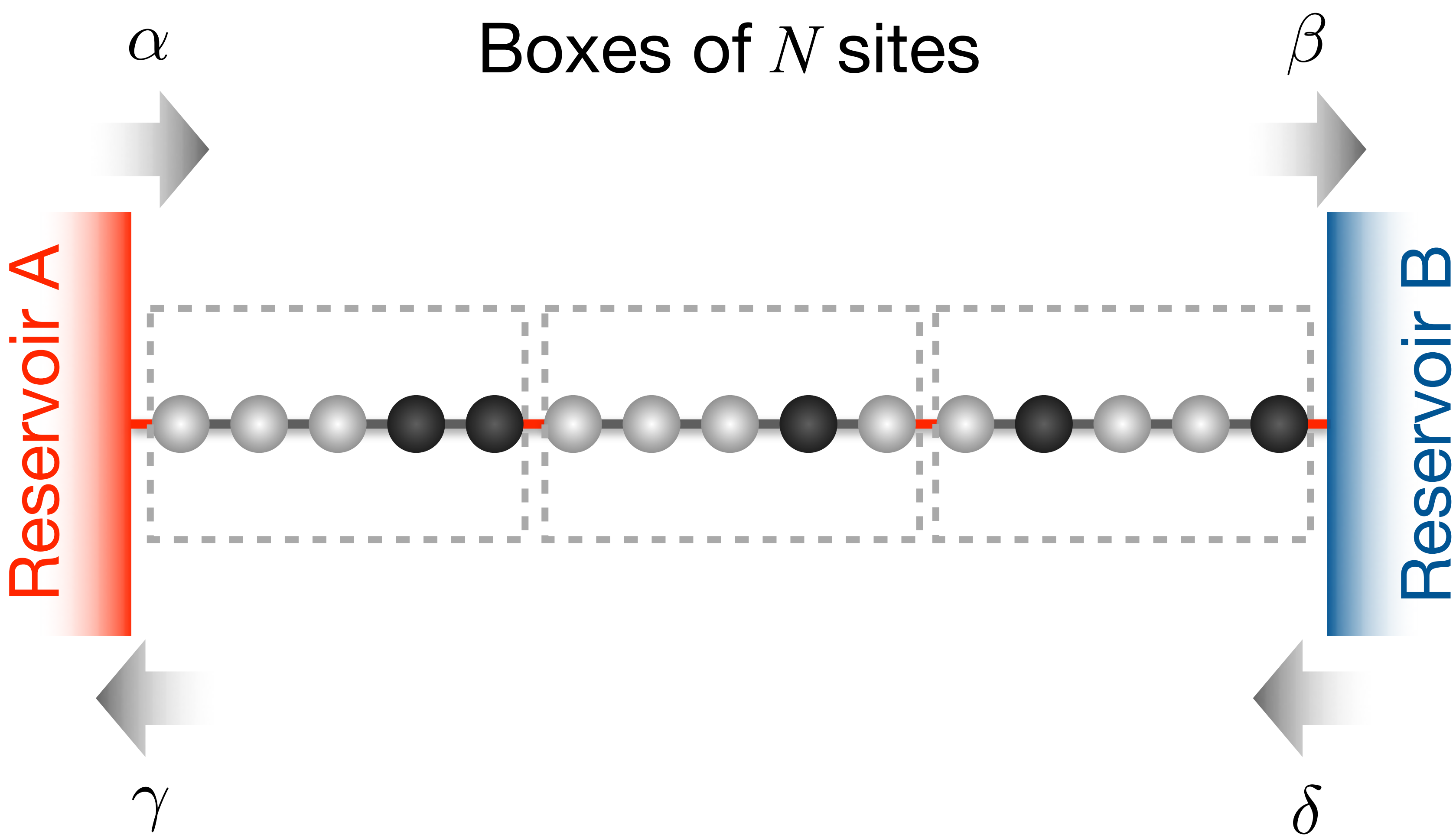
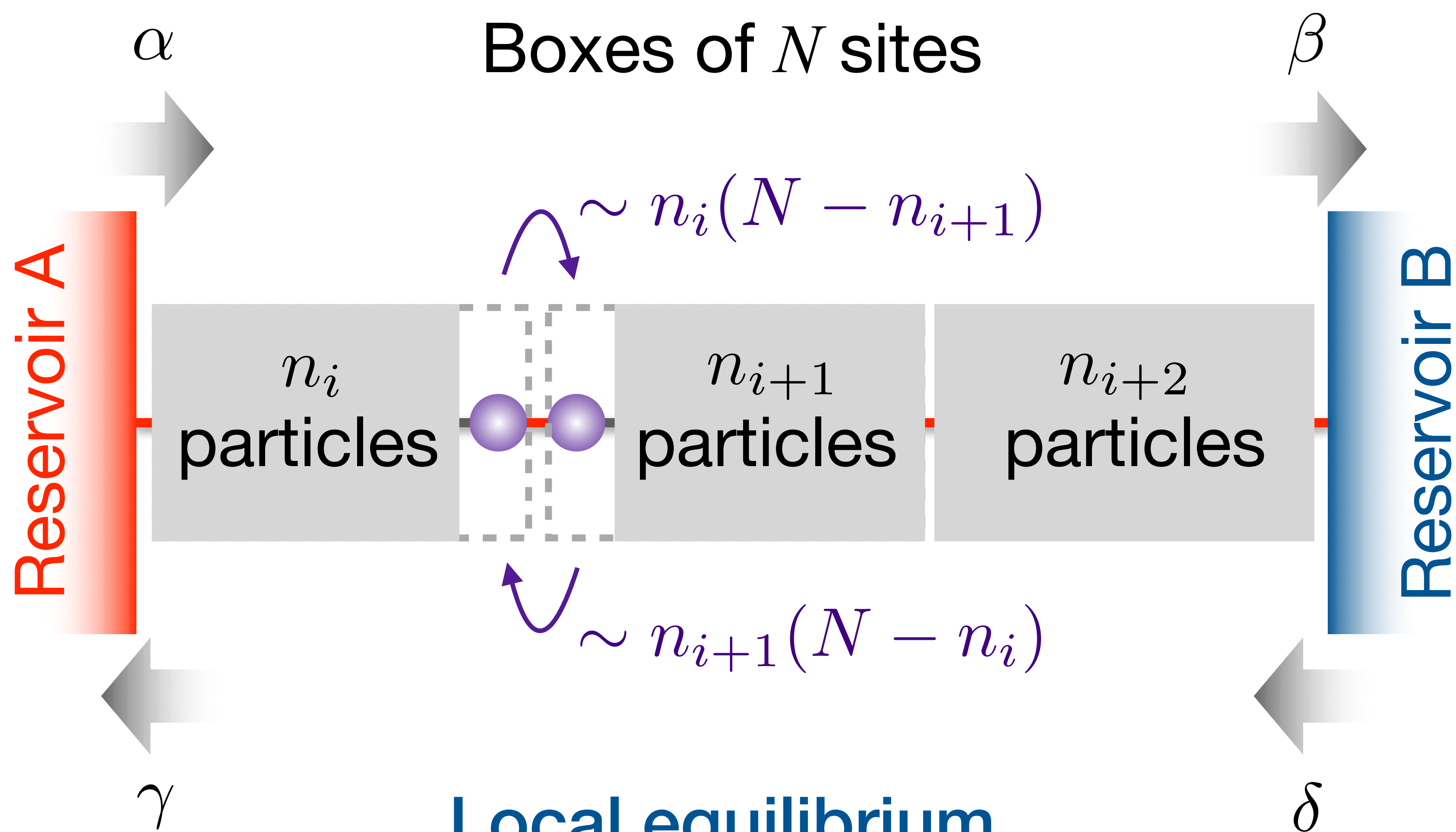


Example: coarse-grained SEP

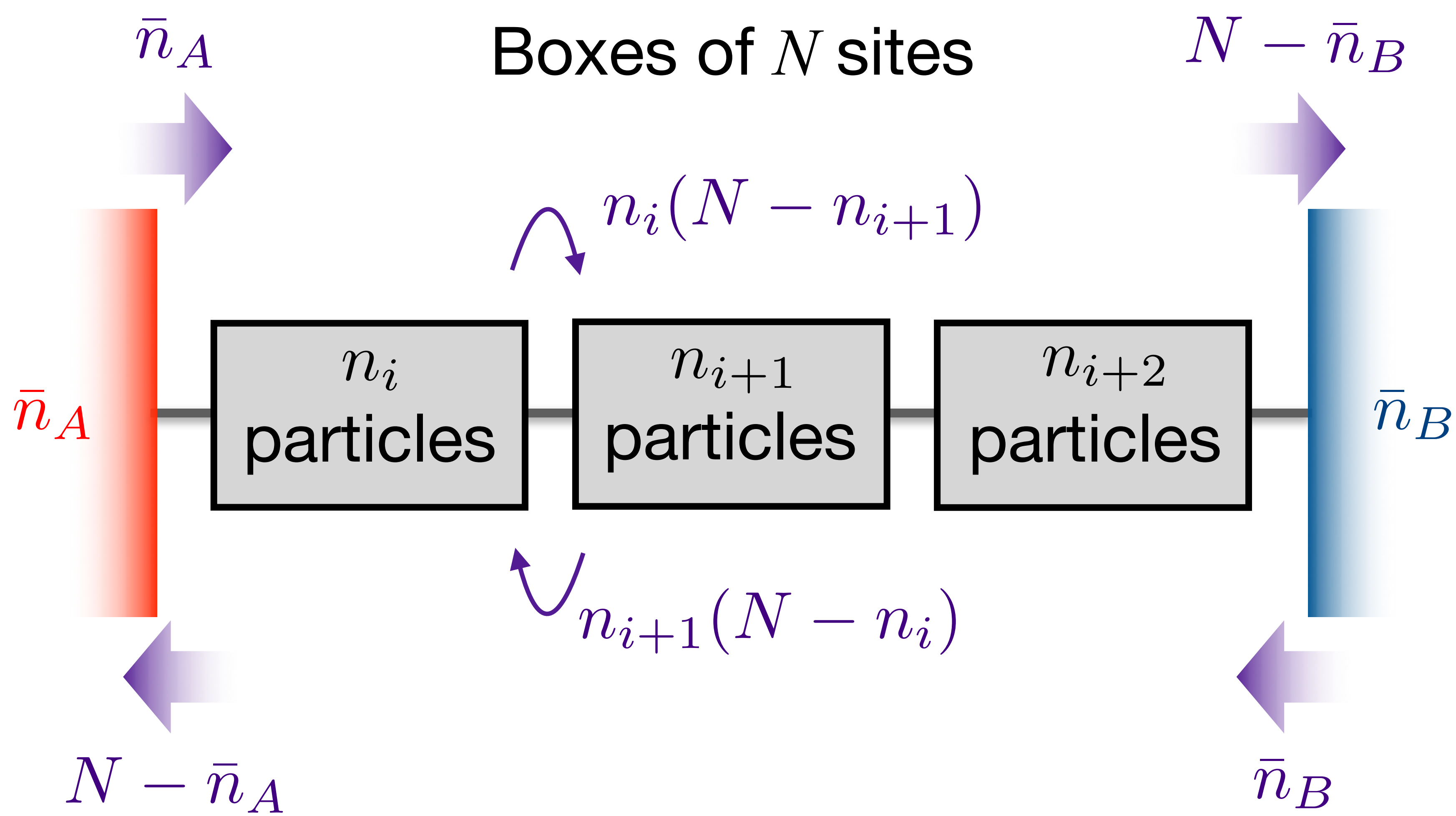


Local equilibrium

For each box, any configuration with a fixed number of particles is equally likely.

Large N limit

Example: coarse-grained SEP

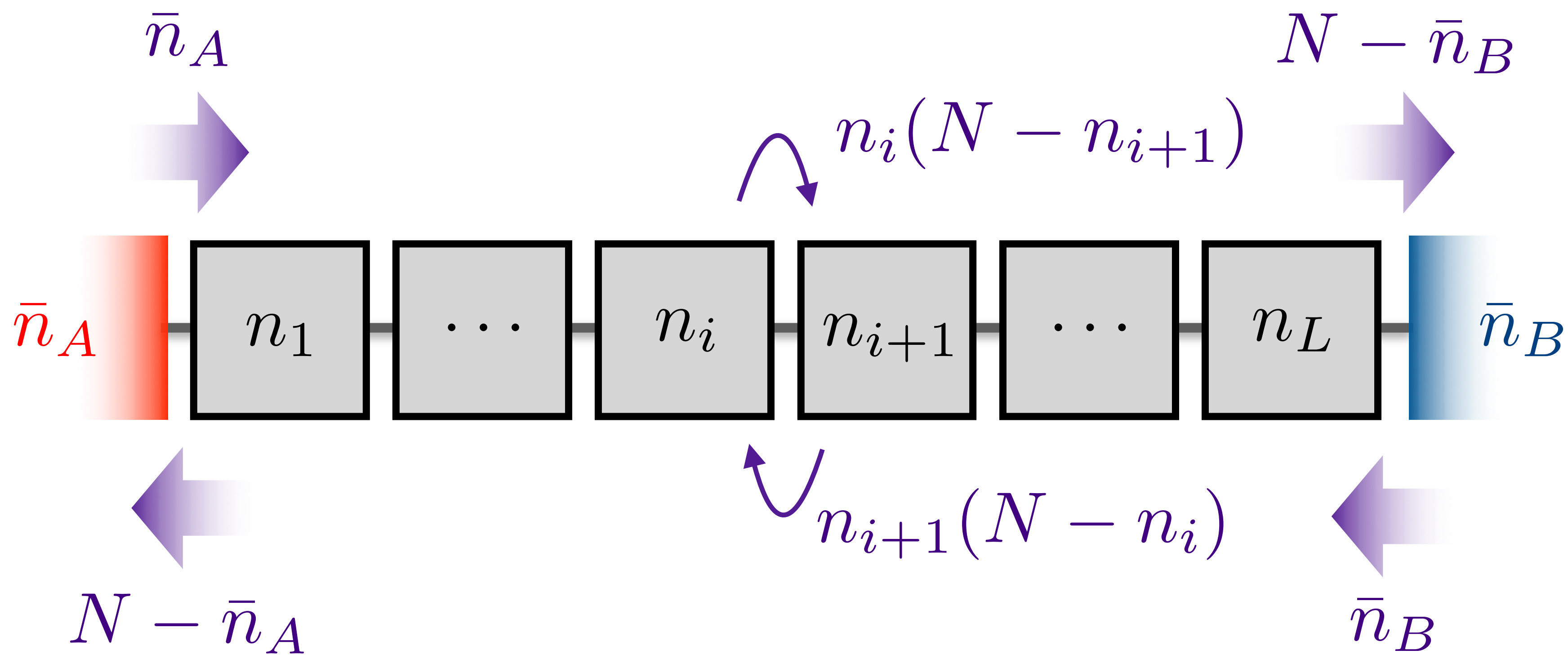


Coarse-grained dynamics

(with appropriate boundary rates and rescaling of time)

Large N limit

Example: coarse-grained SEP



Propagator for the number configuration

$$P[\mathbf{n}_\mathsf{f}, t_\mathsf{f} | \mathbf{n}_\mathsf{i}, t_\mathsf{i}] = \int D\mathbf{n} D\hat{\mathbf{n}} \exp \left\{ - \int_{t_\mathsf{i}}^{t_\mathsf{f}} \mathrm{d}t \, [\hat{\mathbf{n}} \cdot \dot{\mathbf{n}} - H(\mathbf{n}, \hat{\mathbf{n}})] \right\}$$

Lefèvre & Biroli, J. Stat. Mech., P07024 (2007)

Propagator for the number configuration

$$P[\mathbf{n}_\mathsf{f}, t_\mathsf{f} | \mathbf{n}_\mathsf{i}, t_\mathsf{i}] = \int D\mathbf{n} D\hat{\mathbf{n}} \, \exp\left\{ - \int_{t_\mathsf{i}}^{t_\mathsf{f}} \mathrm{d}t \, [\hat{\mathbf{n}} \cdot \dot{\mathbf{n}} - H(\mathbf{n}, \hat{\mathbf{n}})] \right\}$$

Rescaling to density

$$\bar{n}_{A,B} \to N \bar{\rho}_{A,B}$$
$$n_i \to N \rho_i$$
$$\hat{n}_i \to \hat{\rho}_i$$
$$t \to N^{-1} t$$

Propagator for the density configuration

$$P[\boldsymbol{\rho}_\mathsf{f}, t_\mathsf{f} | \boldsymbol{\rho}_\mathsf{i}, t_\mathsf{i}] = \int \mathcal{D}\boldsymbol{\rho} \mathcal{D}\hat{\boldsymbol{\rho}} \, \exp\left\{ - \underline{N} \int_{t_\mathsf{i}}^{t_\mathsf{f}} \mathrm{d}t \, [\hat{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}} - H(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}})] \right\}$$

$N \gg 1$ enables the saddle-point approximation

Pre-Hamiltonian

Large N limit

$$H(\mathbf{n},\hat{\mathbf{n}}) \equiv \sum_{i=1}^{L-1} \left[n_i(N-n_{i+1})\left(e^{\hat{n}_{i+1}-\hat{n}_i}-1\right)+n_{i+1}(N-n_i)\left(e^{\hat{n}_i-\hat{n}_{i+1}}-1\right)\right]$$
$$+n_1(N-\bar{n}_A)\left(e^{-\hat{n}_1}-1\right)+\bar{n}_A(N-n_1)\left(e^{\hat{n}_1}-1\right)$$
$$+n_L(N-\bar{n}_B)\left(e^{-\hat{n}_L}-1\right)+\bar{n}_B(N-n_L)\left(e^{\hat{n}_L}-1\right)$$

Rescaling to density

$$\bar{n}_{A,B} \to N\bar{\rho}_{A,B}$$
$$n_i \to N\rho_i$$
$$\hat{n}_i \to \hat{\rho}_i$$
$$t \to N^{-1}t$$

Effective Hamiltonian ($N \gg 1$)

$$H(\boldsymbol{\rho},\hat{\boldsymbol{\rho}}) = \sum_{i=1}^{L-1} \left[\rho_i(1-\rho_{i+1})\left(e^{\hat{\rho}_{i+1}-\hat{\rho}_i}-1\right)+\rho_{i+1}(1-\rho_i)\left(e^{\hat{\rho}_i-\hat{\rho}_{i+1}}-1\right)\right]$$
$$+\rho_1(1-\bar{\rho}_A)\left(e^{-\hat{\rho}_1}-1\right)+\bar{\rho}_A(1-\rho_1)\left(e^{\hat{\rho}_1}-1\right)$$
$$+\rho_L(1-\bar{\rho}_B)\left(e^{-\hat{\rho}_L}-1\right)+\bar{\rho}_B(1-\rho_L)\left(e^{\hat{\rho}_L}-1\right)$$

Effective Hamiltonian $(N \gg 1)$

Large N limit

$$H(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}}) = \sum_{i=1}^{L-1} \left[\rho_i(1-\rho_{i+1})\left(e^{\hat{\rho}_{i+1}-\hat{\rho}_i}-1\right) + \rho_{i+1}(1-\rho_i)\left(e^{\hat{\rho}_i-\hat{\rho}_{i+1}}-1\right)\right]$$
$$+ \rho_1(1-\bar{\rho}_A)\left(e^{-\hat{\rho}_1}-1\right) + \bar{\rho}_A(1-\rho_1)\left(e^{\hat{\rho}_1}-1\right)$$
$$+ \rho_L(1-\bar{\rho}_B)\left(e^{-\hat{\rho}_L}-1\right) + \bar{\rho}_B(1-\rho_L)\left(e^{\hat{\rho}_L}-1\right)$$

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Continuum limit

$$X_{i+1} - X_i \sim L^{-1}$$
$$\rho_0 = \bar{\rho}_A, \quad \rho_{L+1} = \bar{\rho}_B$$
$$\hat{\rho}_0 = \hat{\rho}_{L+1} = 0$$

Hydrodynamic limit $(N \gg 1, L \gg 1)$

$$H(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}}) = \int_0^1 \mathrm{d}x \left[-(\partial_x \rho)(\partial_x \hat{\rho}) + \rho(1-\rho)(\partial_x \hat{\rho})^2\right]$$

$D(\rho) = 1$, $\sigma(\rho) = \rho(1-\rho)$: this model is indeed SEP-like.

$$\bar{n}_{A,B} \to N\bar{\rho}_{A,B}$$
$$n_i \to N\rho_i$$
$$\hat{n}_i \to \hat{\rho}_i$$
$$t \to N^{-1}t$$

$$i \to Lx$$
$$t \to L^2 t$$

Microscopic box dynamics → **Large N limit (discrete space)** → **Hydrodynamic limit (continuum)**

Objectives

- ✓ Develop a rescaling scheme which enables saddle-point techniques but does not directly lead to the hydrodynamic limit.
- ✓ Check if the rescaling scheme can recover the hydrodynamic limit under appropriate conditions.
- ✦ Obtain current CGFs assuming stationary saddle-point solutions.

Large N limit

$$\bar{n}_{A,B} \to N\bar{\rho}_{A,B}$$
$$n_i \to N\rho_i$$
$$\hat{n}_i \to \hat{\rho}_i$$
$$t \to N^{-1}t$$

$$i \to Lx$$
$$t \to L^2 t$$

Microscopic box dynamics → **Large N limit (discrete space)** → **Hydrodynamic limit (continuum)**

Other works using the large N limit

- Population dynamics
  Meerson & Sasorov, Phys. Rev. E **83**, 011129 (2011)
- Spin-j representation of SEP and KMP $(j = N/2 \to \infty)$
  Tailleur, Kurchan, Lecomte, J. Phys. A **41**, 505001 (2008)

Objectives

- ✓ Develop a rescaling scheme which enables saddle-point techniques but does not directly lead to the hydrodynamic limit.
- ✓ Check if the rescaling scheme can recover the hydrodynamic limit under appropriate conditions.
- ✦ Obtain current CGFs assuming stationary saddle-point solutions.

SEP-like model

Calculation of the current CGF

Rescaling

$$\bar{n}_{A,B} \to N\bar{\rho}_{A,B}$$
$$n_i \to N\rho_i$$
$$\hat{n}_i \to \hat{\rho}_i$$
$$t \to N^{-1}t$$

Saddle-point approximation

$$H_\lambda(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}}) = \sum_{i=1}^{L-1} \left[\rho_i(1-\rho_{i+1})\left(e^{\hat{\rho}_{i+1}-\hat{\rho}_i}-1\right) + \rho_{i+1}(1-\rho_i)\left(e^{\hat{\rho}_i-\hat{\rho}_{i+1}}-1\right)\right]$$
$$+\rho_1(1-\bar{\rho}_A)\left(e^{-\hat{\rho}_1}-1\right) + \bar{\rho}_A(1-\rho_1)\left(e^{\hat{\rho}_1}-1\right)$$
$$+\rho_L(1-\bar{\rho}_B)\left(e^{-\hat{\rho}_L+\lambda}-1\right) + \bar{\rho}_B(1-\rho_L)\left(e^{\hat{\rho}_L-\lambda}-1\right)$$

Stationary solution

Optimal profile

$$\rho_i(t) = \rho^*_{\lambda,i},\ \hat{\rho}_i(t) = \hat{\rho}^*_{\lambda,i}$$

Current CGF

$$\mu_L(\lambda) = H_\lambda(\boldsymbol{\rho}^*_\lambda, \hat{\boldsymbol{\rho}}^*_\lambda)$$

SEP-like model

Behaviors of the current CGF

Result for the large N limit

$\Phi(J) \sim J \ln J$

$$\mu_L(\lambda) = \begin{cases} (L+1)\,\sinh^2\left(\frac{1}{L+1}\operatorname{arcsinh}\sqrt{\omega}\right) & \text{for } \omega > 0 \\ -(L+1)\,\sin^2\left(\frac{1}{L+1}\arcsin\sqrt{-\omega}\right) & \text{for } \omega < 0 \end{cases}$$

Hydrodynamic result

$\Phi(J) \sim J^2$

$$\mu(\lambda) = \begin{cases} \frac{1}{L+1}\operatorname{arcsinh}^2\sqrt{\omega} & \text{for } \omega > 0 \\ -\frac{1}{L+1}\operatorname{arcsinh}^2\sqrt{-\omega} & \text{for } \omega < 0 \end{cases}$$

Auxiliary variable for both cases

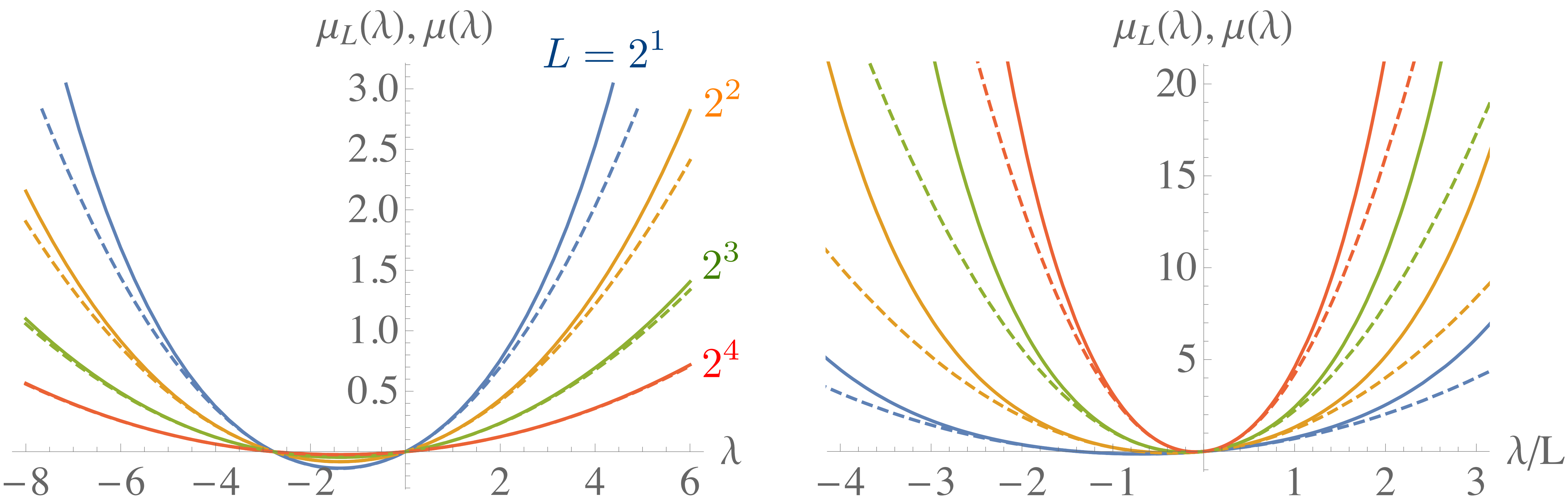
$$\omega(\lambda, \bar{\rho}_A, \bar{\rho}_B) = (1 - e^{-\lambda})[\bar{\rho}_B - e^{\lambda}\bar{\rho}_A + (e^{\lambda} - 1)\bar{\rho}_A\bar{\rho}_B]$$

SEP-like model

Behaviors of the current CGF

Comparison between the two limits

Dashed: hydrodynamic
Solid: large N limit



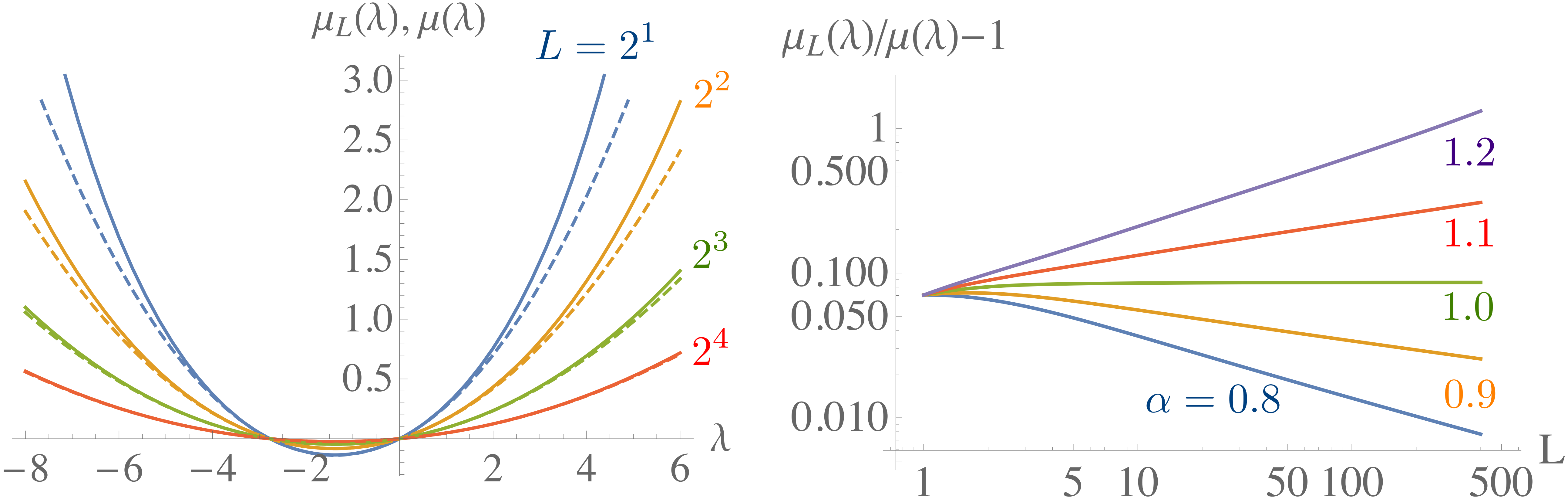
$$\mu_L(\lambda) - \mu(\lambda) = O(L^{-3})$$

SEP-like model

Behaviors of the current CGF

Comparison between the two limits

Dashed: hydrodynamic
Solid: large N limit



$$\mu_L(\lambda) - \mu(\lambda) = O(L^{-3})$$

$$\lim_{L\to\infty} \mu_L(\bar{\lambda}L^{\alpha}) \neq \lim_{L\to\infty} \mu(\bar{\lambda}L^{\alpha}) \text{ if } \alpha \geq 1$$

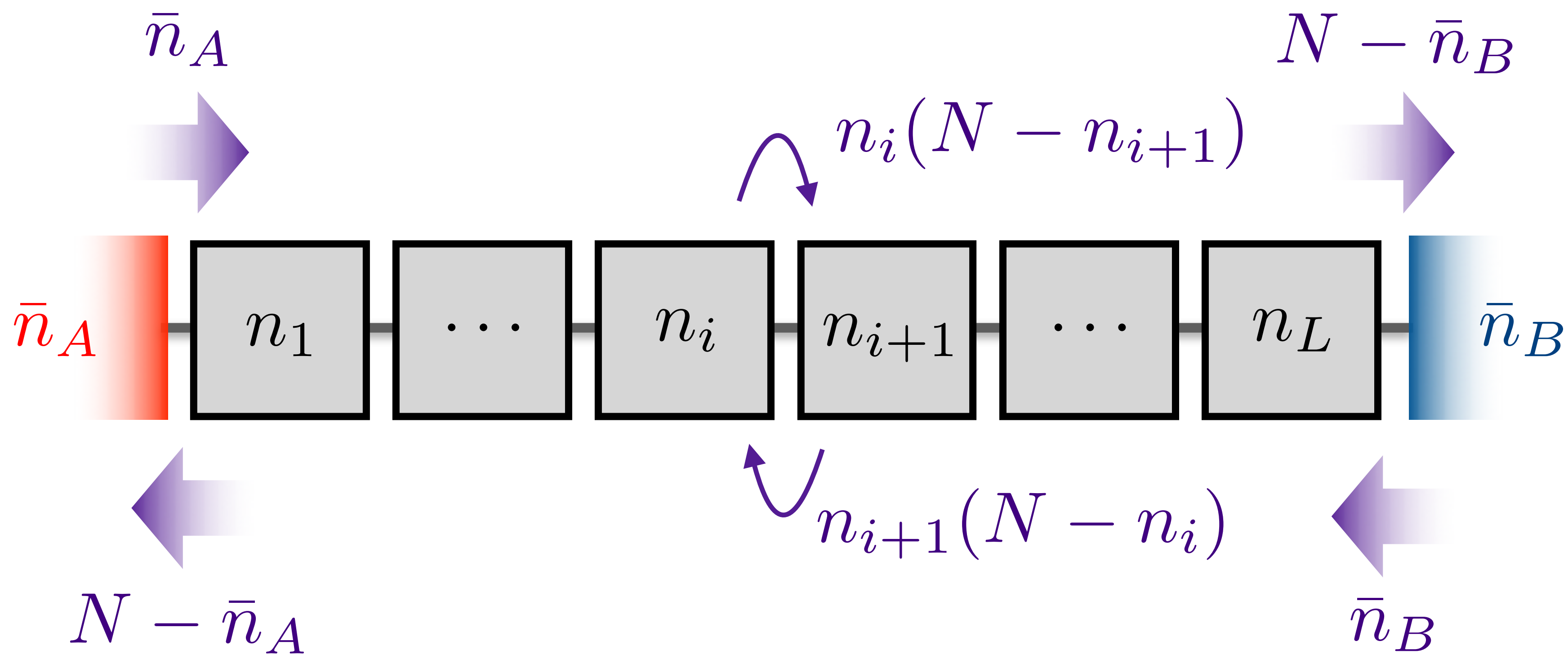
Since $J = \frac{\partial \mu_L}{\partial \lambda} \sim e^{\bar{\lambda}L^{\alpha-1}}$, non-hydrodynamic behaviors are observed for J stronger than $O(1)$.

Since $J = \frac{\partial \mu_L}{\partial \lambda} \sim e^{\bar{\lambda} L^{\alpha - 1}}$, non-hydrodynamic behaviors are observed for J stronger than $O(1)$.

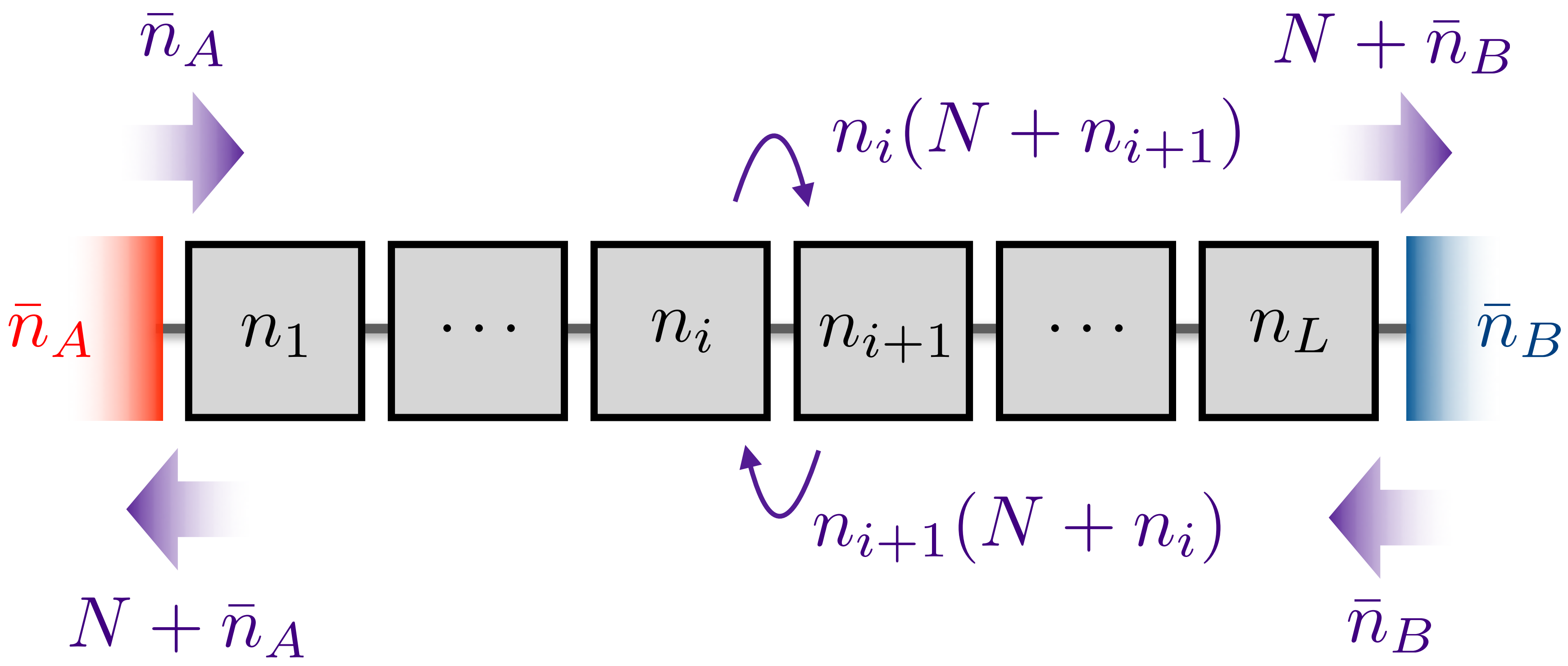
Why $O(1)$, instead of somewhere between $O(1/L)$ and $O(1)$?

Before trying to answer this question,
let's check a different class of model first.

SEP-like dynamics



KMP-like dynamics



Features

- Attraction instead of excluded volume repulsion.
- N is no longer an upper bound on the number of particles.
- Depending on the reservoir conditions, the occupancy of each site can be much larger than N.

Idea: consider the case when $\bar{n}_{A,B} \sim N^2$.

Propagator for the number configuration

$$P[\mathbf{n}_\mathsf{f}, t_\mathsf{f}|\mathbf{n}_\mathsf{i}, t_\mathsf{i}] = \int D\mathbf{n}D\hat{\mathbf{n}} \; \exp\left\{-\int_{t_\mathsf{i}}^{t_\mathsf{f}} \mathrm{d}t \; [\hat{\mathbf{n}} \cdot \dot{\mathbf{n}} - H(\mathbf{n}, \hat{\mathbf{n}})]\right\}$$

Rescaling to density

$$\bar{n}_{A,B} \to N^2 \bar{\rho}_{A,B}$$

$$n_i \to N^2 \rho_i$$

$$\hat{n}_i \to N^{-1} \hat{\rho}_i$$

$$t \to N^{-1} t$$

Propagator for the density configuration

$$P[\boldsymbol{\rho}_\mathsf{f}, t_\mathsf{f}|\boldsymbol{\rho}_\mathsf{i}, t_\mathsf{i}] = \int \mathcal{D}\boldsymbol{\rho}\mathcal{D}\hat{\boldsymbol{\rho}} \; \exp\left\{-\underline{N} \int_{t_\mathsf{i}}^{t_\mathsf{f}} \mathrm{d}t \, [\hat{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}} - H(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}})]\right\}$$

$N \gg 1$ enables the saddle-point approximation

Pre-Hamiltonian

KMP-like model

$$H_\lambda(\mathbf{n},\hat{\mathbf{n}}) \equiv \sum_{i=1}^{L-1}\left[n_i(N+n_{i+1})\left(e^{\hat{n}_{i+1}-\hat{n}_i}-1\right)+n_{i+1}(N+n_i)\left(e^{\hat{n}_i-\hat{n}_{i+1}}-1\right)\right]$$
$$+\,n_1(N+\bar{n}_A)\left(e^{-\hat{n}_1}-1\right)+\bar{n}_A(N+n_1)\left(e^{\hat{n}_1}-1\right)$$
$$+\,n_L(N+\bar{n}_B)\left(e^{-\hat{n}_L+\lambda}-1\right)+\bar{n}_B(N+n_L)\left(e^{\hat{n}_L-\lambda}-1\right)$$

Rescaling to density

$$\bar{n}_{A,B} \to N^2\bar{\rho}_{A,B}$$
$$n_i \to N^2\rho_i$$
$$\hat{n}_i \to N^{-1}\hat{\rho}_i$$
$$t \to N^{-1}t$$

Effective Hamiltonian $(N \gg 1)$

$$H_\lambda(\boldsymbol{\rho},\hat{\boldsymbol{\rho}}) \equiv \sum_{i=1}^{L-1}\left[(\hat{\rho}_{i+1}-\hat{\rho}_i)(\rho_i-\rho_{i+1})+(\hat{\rho}_i-\hat{\rho}_{i+1})^2\rho_i\rho_{i+1}\right]$$
$$+\,\hat{\rho}_1(\bar{\rho}_A-\rho_1)+\hat{\rho}_1^2\,\rho_1\bar{\rho}_A+(\hat{\rho}_L-\lambda)(\bar{\rho}_B-\rho_L)+(\hat{\rho}_L-\lambda)^2\,\rho_L\bar{\rho}_B$$

Effective Hamiltonian $(N \gg 1)$

KMP-like model

$$H_\lambda(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}}) \equiv \sum_{i=1}^{L-1} \left[(\hat{\rho}_{i+1} - \hat{\rho}_i)(\rho_i - \rho_{i+1}) + (\hat{\rho}_i - \hat{\rho}_{i+1})^2 \rho_i \rho_{i+1} \right]$$

$$+ \hat{\rho}_1 (\bar{\rho}_A - \rho_1) + \hat{\rho}_1^2 \rho_1 \bar{\rho}_A + (\hat{\rho}_L - \lambda)(\bar{\rho}_B - \rho_L) + (\hat{\rho}_L - \lambda)^2 \rho_L \bar{\rho}_B$$

Diffusive scaling

$$i \to Lx, \quad t \to L^2 t$$

Continuum limit

$$X_{i+1} - X_i \sim L^{-1}$$

$$\rho_0 = \bar{\rho}_A, \quad \rho_{L+1} = \bar{\rho}_B$$

$$\hat{\rho}_0 = 0, \quad \hat{\rho}_{L+1} = \lambda$$

Hydrodynamic limit $(N \gg 1, L \gg 1)$

$$H(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}}) = \int_0^1 \mathrm{d}x \left[-(\partial_x \rho)(\partial_x \hat{\rho}) + \rho^2 (\partial_x \hat{\rho})^2 \right]$$

$D(\rho) = 1$, $\sigma(\rho) = \rho^2$: this model is indeed KMP-like.

KMP-like model

Calculation of the current CGF

Rescaling

$$\bar{n}_{A,B} \to N^2 \bar{\rho}_{A,B}$$
$$n_i \to N^2 \rho_i$$
$$\hat{n}_i \to N^{-1} \hat{\rho}_i$$
$$t \to N^{-1} t$$

Current CGF

$$\mu_L(\lambda) = H_\lambda(\boldsymbol{\rho}^*_\lambda, \hat{\boldsymbol{\rho}}^*_\lambda)$$

Optimal profile

$$\rho_i(t) = \rho^*_{\lambda,i},\ \hat{\rho}_i(t) = \hat{\rho}^*_{\lambda,i}$$

Saddle-point approximation

Stationary solution

$$H_\lambda(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}}) \equiv \sum_{i=1}^{L-1} \left[(\hat{\rho}_{i+1} - \hat{\rho}_i)(\rho_i - \rho_{i+1}) + (\hat{\rho}_i - \hat{\rho}_{i+1})^2 \rho_i \rho_{i+1} \right]$$
$$+ \hat{\rho}_1(\bar{\rho}_A - \rho_1) + \hat{\rho}_1^2 \rho_1 \bar{\rho}_A + (\hat{\rho}_L - \lambda)(\bar{\rho}_B - \rho_L) + (\hat{\rho}_L - \lambda)^2 \rho_L \bar{\rho}_B$$

KMP-like model

Behaviors of the current CGF

Result for the large N limit

$$\mu_L(\lambda) = \begin{cases} -(L+1)\,\text{sinh}^2\left(\frac{1}{L+1}\text{arcsinh}\sqrt{\omega}\right) & \text{for } \omega > 0 \\ (L+1)\,\text{sin}^2\left(\frac{1}{L+1}\,\text{arcsin}\,\sqrt{-\omega}\right) & \text{for } \omega < 0 \end{cases}$$

Hydrodynamic result

$$\mu(\lambda) = \begin{cases} -\frac{1}{L+1}\text{arcsinh}^2\sqrt{\omega} & \text{for } \omega > 0 \\ \frac{1}{L+1}\text{arcsinh}^2\sqrt{-\omega} & \text{for } \omega < 0 \end{cases}$$

Auxiliary variable for both cases

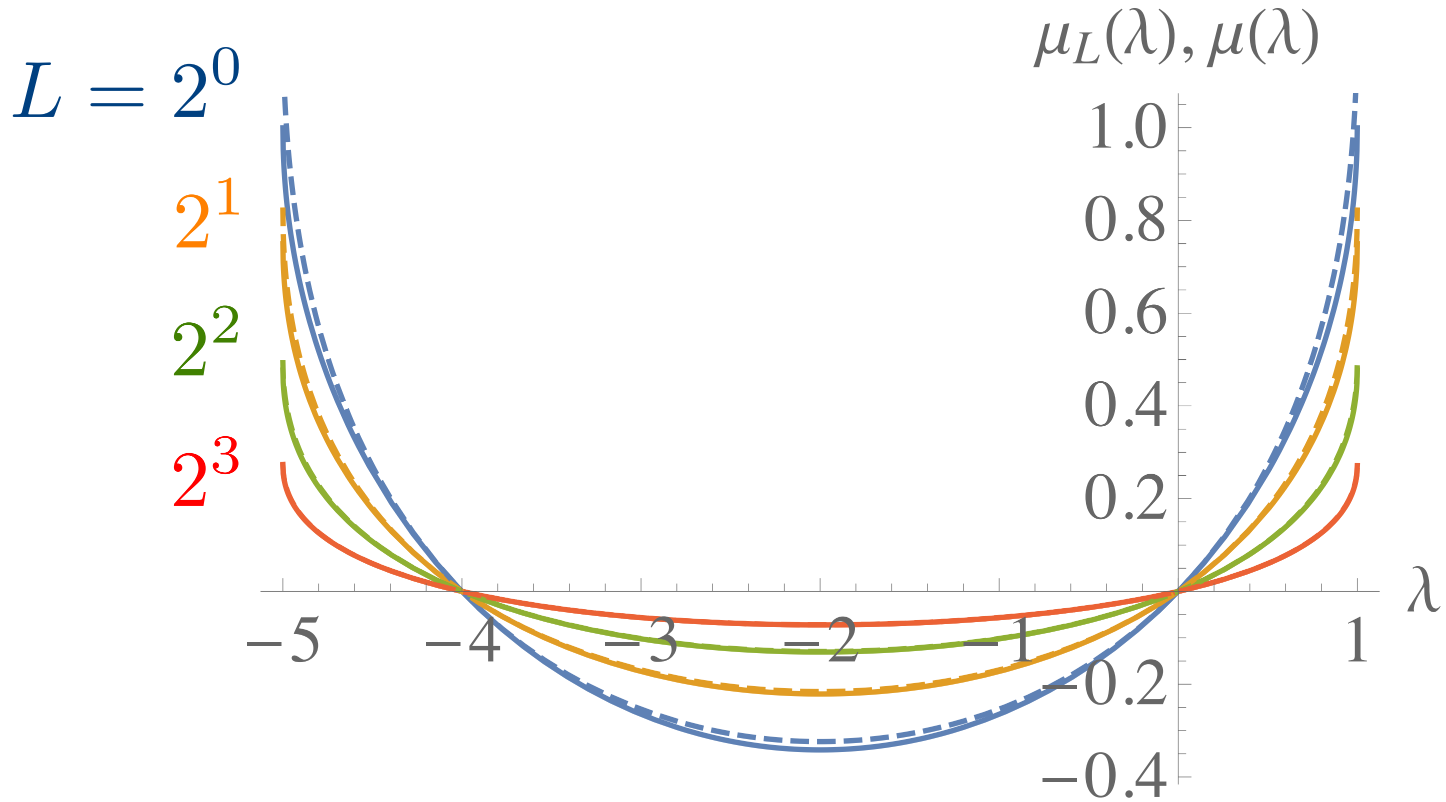
$$\omega(\lambda, \bar{\rho}_A, \bar{\rho}_B) = \lambda(\bar{\rho}_B - \bar{\rho}_A) - \lambda^2 \bar{\rho}_A \bar{\rho}_B$$

KMP-like model

Behaviors of the current CGF

Comparison between the two limits

Dashed: hydrodynamic
Solid: large N limit



Each pair converge like L^{-3} over the entire domain.

The domain is bounded by

$$-\frac{1}{\bar{\rho}_B} < \lambda < \frac{1}{\bar{\rho}_A}$$

Same exponential tails are observed for both cases.
Hydrodynamic and non-hydrodynamic regimes are undistinguishable.

SEP-like model

Since $J = \frac{\partial \mu_L}{\partial \lambda} \sim e^{\bar{\lambda} L^{\alpha - 1}}$, non-hydrodynamic behaviors are observed for J stronger than $O(1)$.

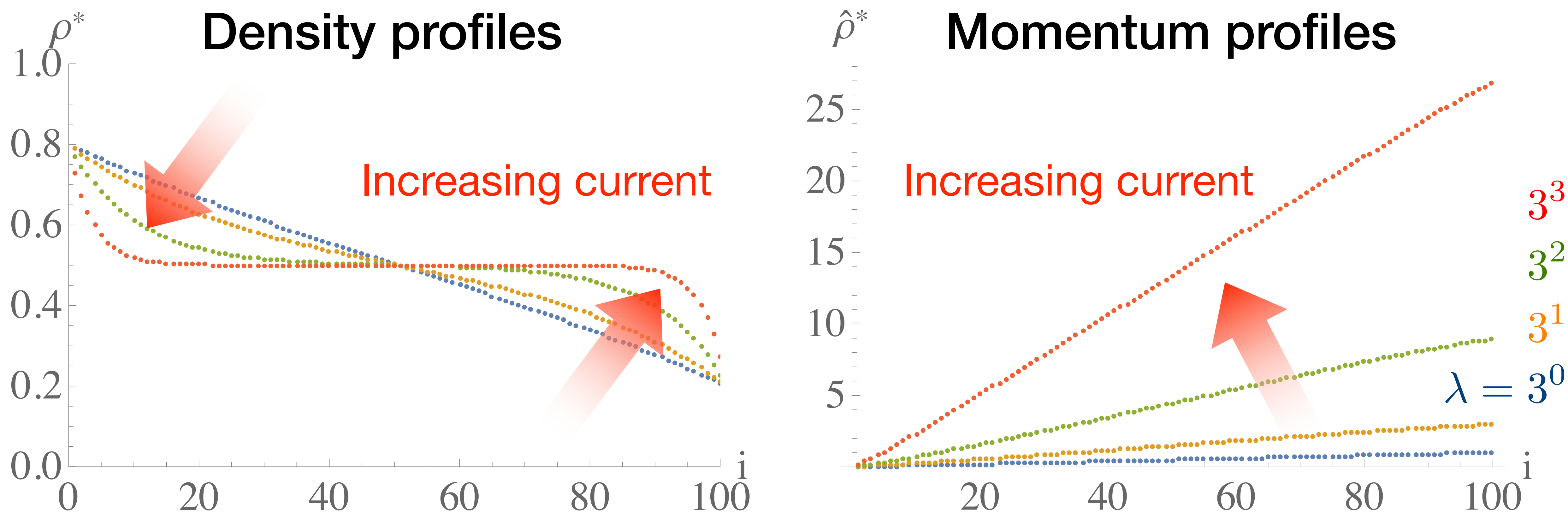
KMP-like model

Hydrodynamic and non-hydrodynamic regimes are undistinguishable.

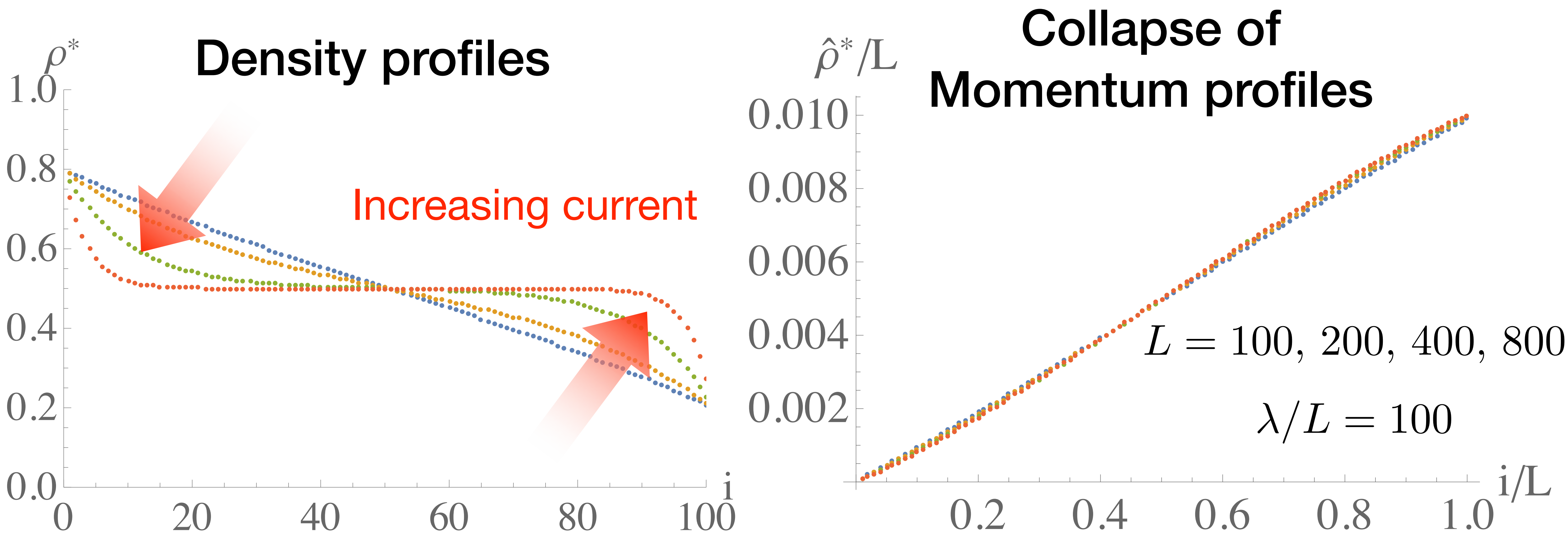
Why so different?

Let's check optimal profiles supporting large current fluctuations.

Optimal profiles of the SEP-like model



Optimal profiles of the SEP-like model

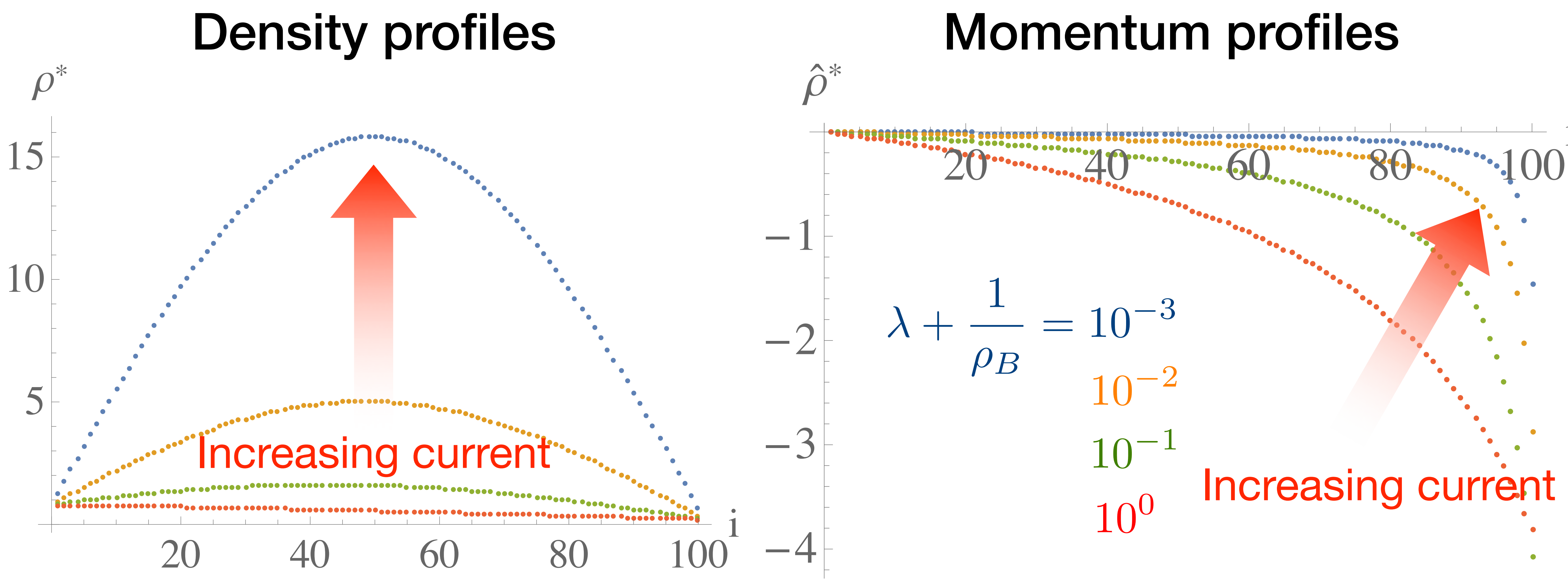


$$j = -\partial_x \rho + 2\rho(1-\rho)\underline{\partial_x \hat{\rho}}$$

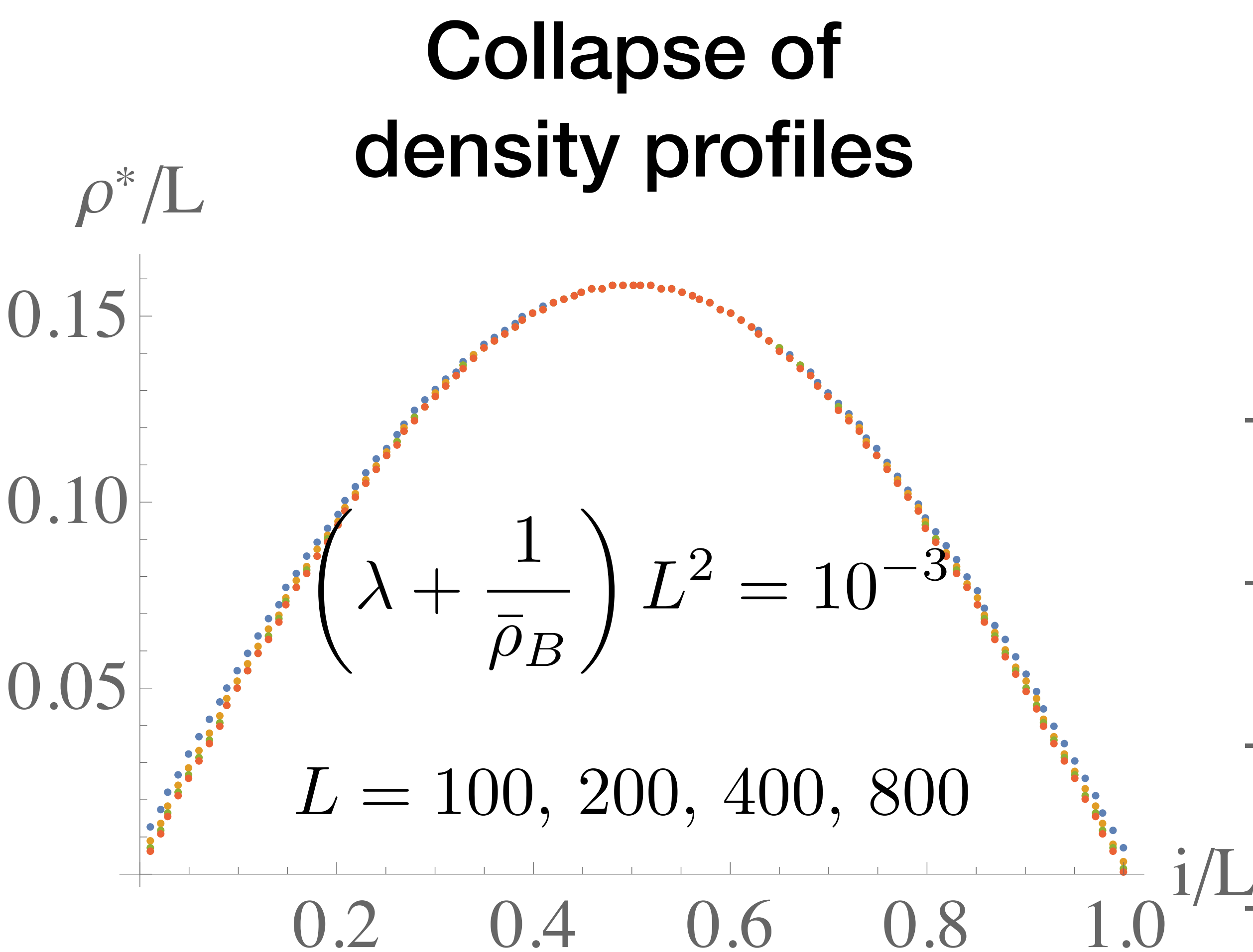
Large current supported by large momentum gradient

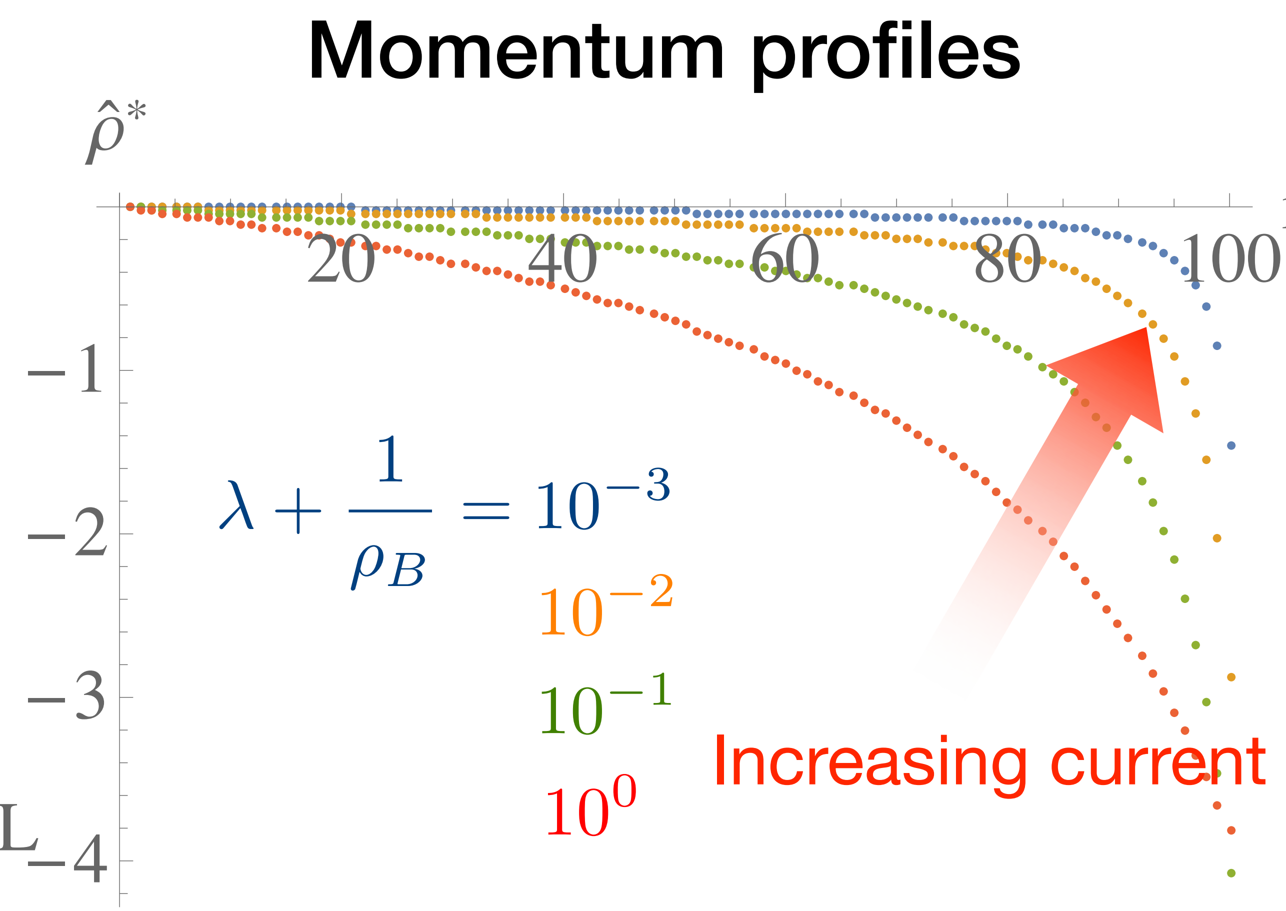
Since $\partial_x \hat{\rho} \sim L^{\alpha-1}$ for $\lambda \sim L^{\alpha}$,
$\alpha = 1$ is the borderline for small gradient.

Optimal profiles of the KMP-like model



Optimal profiles of the KMP-like model





$$j = -\partial_x \rho + 2\rho^2 \partial_x \hat{\rho}$$

Large current supported by large density values

The large current is dominated by the non-gradient second term and is blind to the diverging density gradient.

"Deterministic component" "Stochastic component"

SEP-like model

$$j = -\partial_x \rho + 2\rho(1-\rho)\underline{\partial_x \hat{\rho}}$$

Large current supported by large momentum gradient

KMP-like model

$$j = -\partial_x \rho + \underline{2\rho^2}\partial_x \hat{\rho}$$

Large current supported by large density values

Stochastic component plays a dominant role in both cases.

Hydrodynamic description breaks down only when the stochastic component requires a large gradient to support a large current fluctuation.

Meerson & Sasorov, Phys. Rev. E **89**, 010101 (2014)

Summary and future works

Summary

- We formulated the large N limit to investigate the possibility of non-hydrodynamic tail behaviors of current fluctuations.
- Under the assumption of stationary saddle-point solutions, we obtained expressions for current CGFs of SEP-like and KMP-like models.
- The hydrodynamic description breaks down for the SEP-like model, but it remains valid for the KMP-like model.

Future works

- The validity of the stationary saddle-point solution remains to be established more rigorously.
- The large N limit might be useful for making minimal models of nonequilibrium systems.