Supersymmetric SU(N) Yang-Mills QM
Canonical simulations at fixed fermion number

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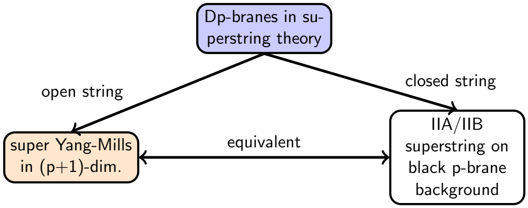
SYM gauge theories: dualities, black holes and all that

Gauge/gravity duality conjecture:

- $U(N)$ gauge theories as a low energy effective theory of N D-branes
- Dimensionally reduced large-N super Yang-Mills might provide a nonperturbative formulation of the string/M-theory
- Connection to black p-branes allows studying black hole thermodynamics through strongly coupled gauge theory:



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- Interesting physics:
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- Interesting expectations:
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    (depending on the fermion sector),
  - flat directions
- Interesting 'bosonisation':
  - fermion contribution decomposes into fermion sectors,
  - allows for a local fermion algorithm,
  - structure is the same as for QCD!

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  - anticommuting fermion fields $\overline{\psi}(t)$, $\psi(t)$, (complex 2-component spinors for $\mathcal{N} = 4$)
  - σ_i are the γ-matrices in d dimensions (Pauli matrices for $\mathcal{N} = 4$)
- all fields in the adjoint representation of SU(N)

Lattice regularisation

- Discretise the bosonic part:

$$S_B = \frac{1}{g^2} \sum_{t=0}^{L_t-1} \text{Tr} \left\{ D_t X_i(t) D_t X_i(t) - \frac{1}{2} \left[X_i(t), X_j(t) \right]^2 \right\}$$

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- Use Wilson term for the fermionic part,

$$S_F = \frac{1}{g^2} \sum_{t=0}^{L_t - 1} \mathrm{Tr} \left\{ \overline{\psi}(t) D_t \psi(t) - \overline{\psi}(t) \sigma_i \left[X_i(t), \psi(t) \right] \right\} ,$$

since

$$\partial^{\mathcal{W}} = \frac{1}{2}(\nabla^+ + \nabla^-) \pm \frac{1}{2} \nabla^+ \nabla^- \quad \overset{d=1}{\Longrightarrow} \quad \nabla^\pm$$

Lattice regularisation and reduced determinant

- Specifically, we have

$$S_F = \frac{1}{2g^2} \sum_{t=0}^{L_t - 1} \left[-\overline{\psi}^a_\alpha(t) W^{ab}_{\alpha\beta}(t) e^{+\mu L_t} \psi^b_\beta(t+1) + \overline{\psi}^a_\alpha(t) \Phi^{ac}_{\alpha\beta}(t) \psi^c_\beta(t) \right]$$

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- Dimensional reduction of determinant at finite density $\mu \neq 0$:

$$\det \mathcal{D}_{p,a}[U, X_i; \mu] = \det \left[\prod_{t=0}^{L_t-1} \Phi(t) W(t) \mp e^{+\mu L_t} \right]$$

Fugacity expansion

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- For $n_f = 0$ and $n_f = 2(N^2 - 1) \equiv n_f^{\max}$ (quenched):

$$\det \mathcal{D}_{n_f}[U, X_i] \geq 0 \quad \text{positive}$$

Fugacity expansion and transfer matrices

- Canonical determinants are expressed in terms of elementary symmetric functions S_k of order k of $\{\tau_i\}$:

$$\det \mathcal{D}_{n_f}[U, X_i] = S_{n_f^{\max} - n_f}(\mathcal{T})$$

where

$$S_k(\mathcal{T}) \equiv S_k(\{\tau_i\}) = \sum_{1 \le i_1 < \cdots < i_k \le n_f^{\max}} \prod_{j=1}^{k} \tau_{i_j} \,.$$

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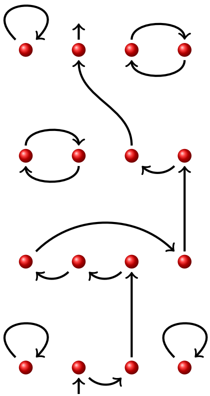
- Proof via fermion loop formulation:

$\Rightarrow$ explicit construction in each fermion sector

Fermion loop formulation $\Leftrightarrow$ all order hopping expansion

- Configurations can be classified according to the number of propagating fermions n_f:

$$n_f = 1$$



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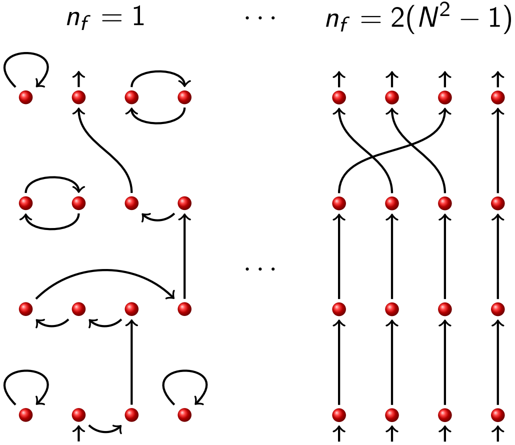

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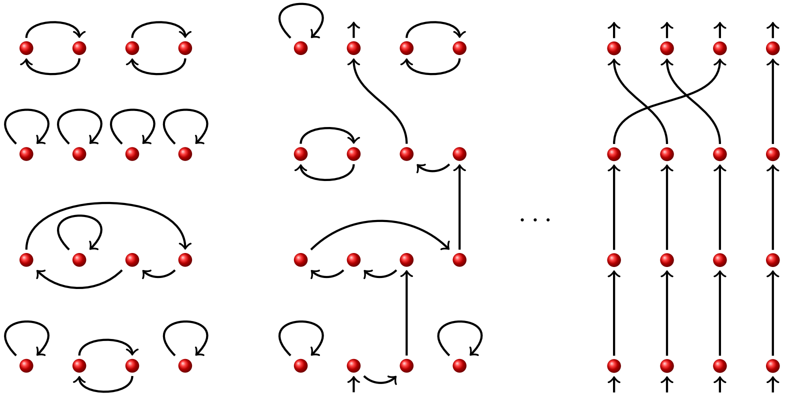

Transfer matrices

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- Finally one can proof by linear algebra

$$\sum_{B} \det \mathcal{T}^{\not{B}\not{B}} = S_{n_f^{\max} - n_f}(\mathcal{T})\,.$$

Summary ...so far

- Canonical determinants are directly given by transfer matrices

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- Remarks:
  - $\mathcal{T}$ describes the dimensionally reduced effective action for W,
  - our result allows for local fermion algorithm,
  - allows canonical simulations at fixed n_f.

Canonical simulations at fixed n_f

- Evaluation of the sum of principal minors

$$Z_{n_f} = \int \mathcal{D}U\,\mathcal{D}X\; e^{-S[U,X]} \sum_{B} \det \mathcal{T}[U,X]^{\bcancel{B}\bcancel{B}}$$

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- Efficient stochastic evaluation of $\sum_B$:
  - treat index set B as dynamical degree of freedom
  - update $B \to B'$ using Fisher-Yates reshuffling
  - new random set B' is accepted with probability

$$p_{B\to B'} = \min[1, A_{B\to B'}] \quad \text{with} \quad A_{B\to B'} = \left| \frac{\det \mathcal{T}[U,X]^{\not{B}'\not{B}'}}{\det \mathcal{T}[U,X]^{\not{B}\not{B}}} \right| .$$

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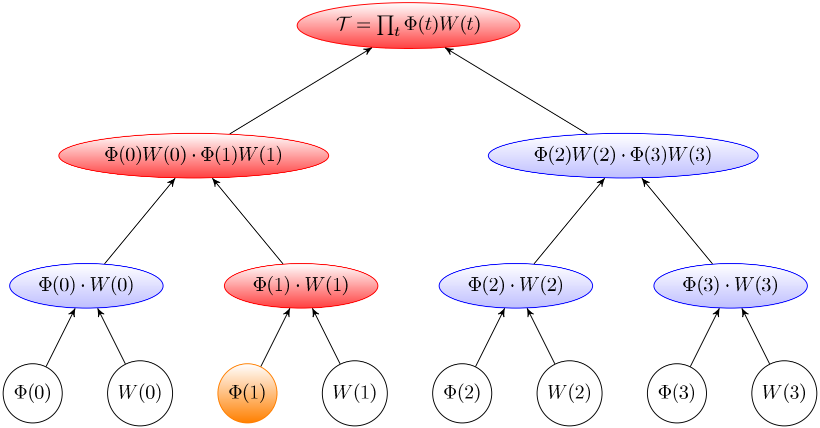

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  - after each change of $\Phi(t)$ or $W(t)$
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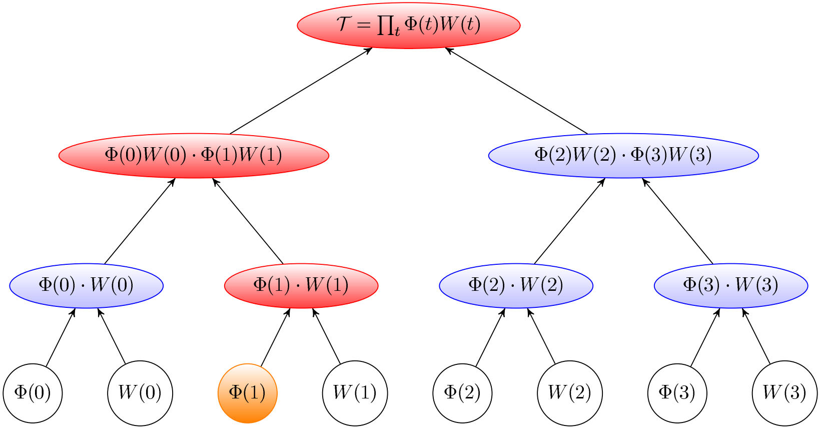
    $\Rightarrow$ only $\mathcal{O}(\log L_t)$ matrix multiplications



Canonical simulations at fixed n_f

- Simulations for $N = 2$ with $n_f^{\max} = 2(N^2 - 1) = 6$:
  - SU(2) adjoint $\Rightarrow$ sectors $n_f = \{0, 1, \ldots, 6\}$

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- Open questions:
  - sign problem: $\det \mathcal{D}_{n_f}[U, X_i] \geq 0 \quad \forall n_f$?
  - flat directions?

    $\Rightarrow$ regularize with a term $\propto m^2 X^2$

Canonical determinants for SU(2)

- No sign problem in sectors $n_f = 0, 2, 3, 4$ and 6:
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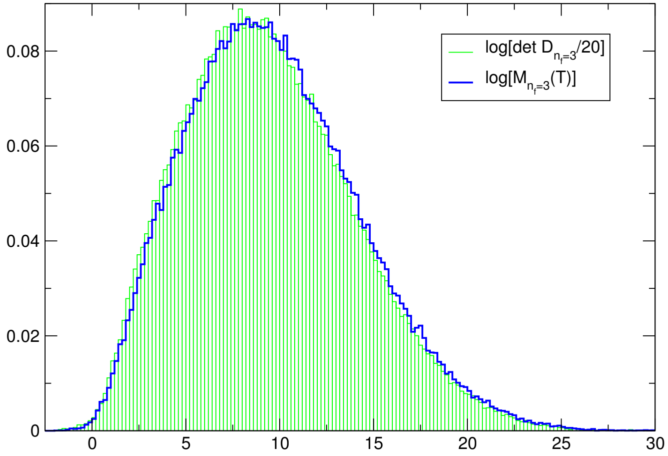
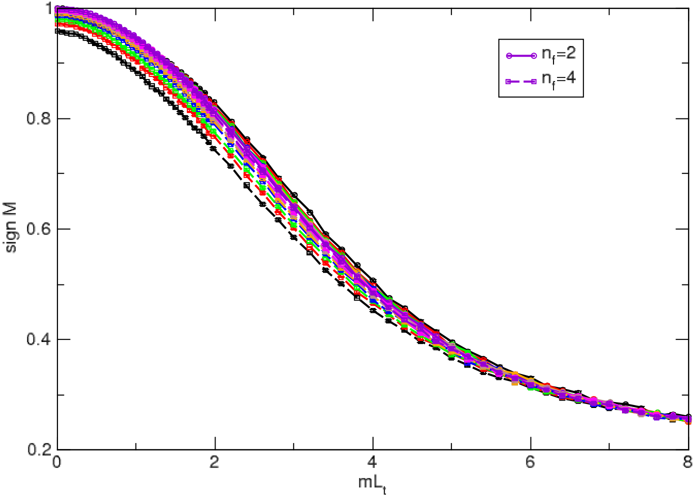
  - e.g. for $n_f = 3$ at $\beta = 1.2$ on $L_t = 24$:



Canonical determinants for SU(2)

- No sign problem in sectors $n_f = 0, 2, 3, 4$ and 6:
  - sign of minor at $\beta = 2.0$ at $n_f = 2$ and $n_f = 4$:



Canonical determinants for SU(2)

- No sign problem in sectors $n_f = 0, 2, 3, 4$ and 6:
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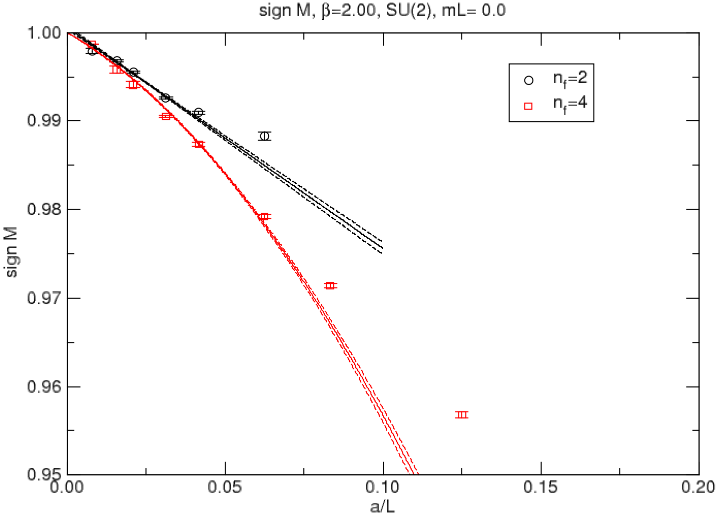
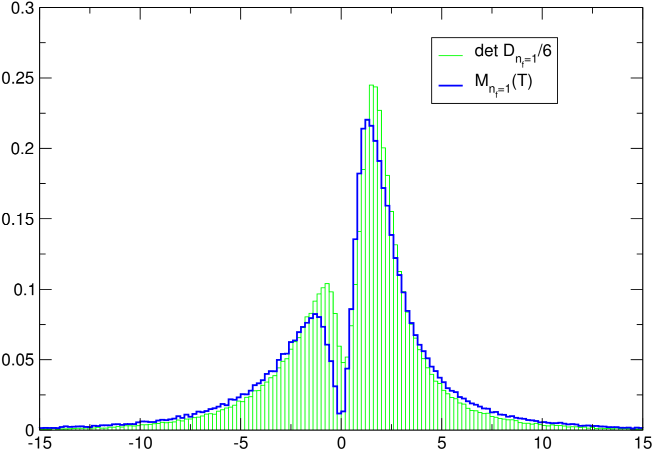


Canonical determinants for SU(2)

- Significant negative contributions in sectors $n_f = 1$ and 5:
  - e.g. for $n_f = 1$ at $\beta = 1.2$ on $L_t = 24$:



Canonical determinants for SU(2)

- Significant negative contributions in sectors $n_f = 1$ and 5:
  - absolutely crucial for reweighting to other sectors,
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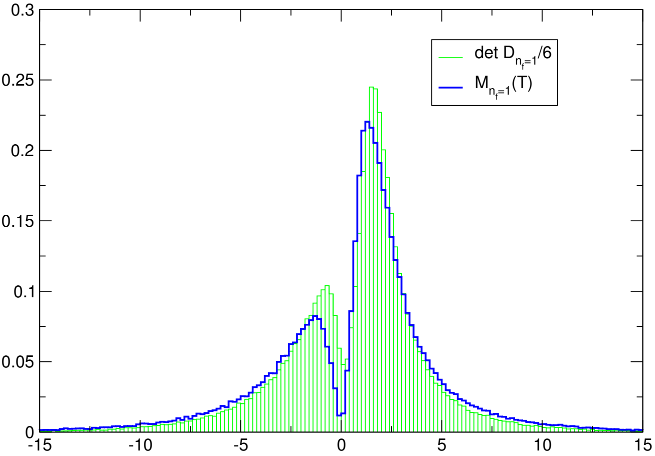
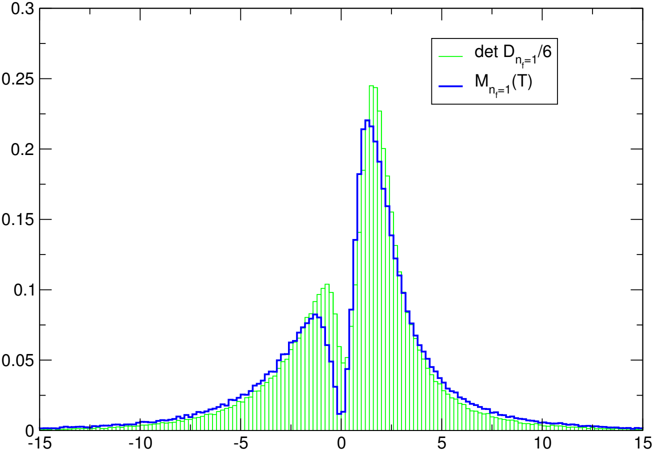


Canonical determinants for SU(2)

- Significant negative contributions in sectors $n_f = 1$ and 5:
  - absolutely crucial for reweighting to other sectors,
  - distribution remains fixed in the continuum limit $L_t \to \infty$



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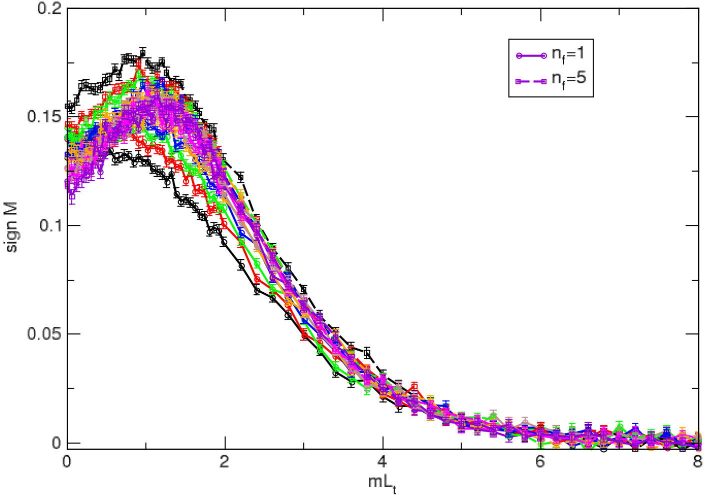

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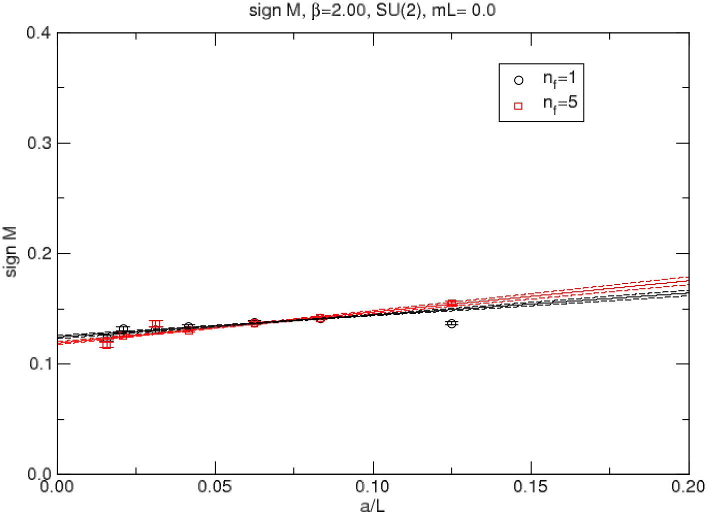
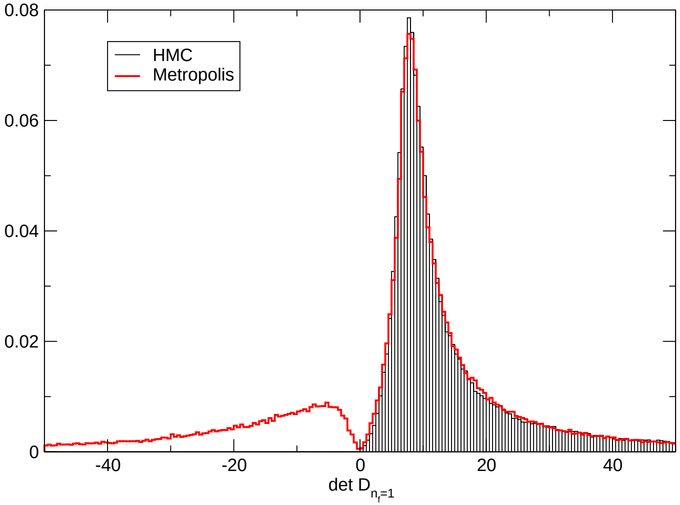


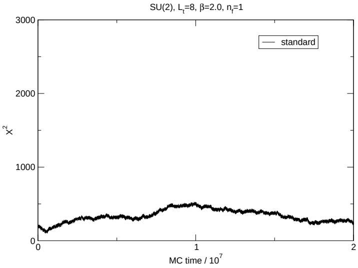
Canonical determinants for SU(2)

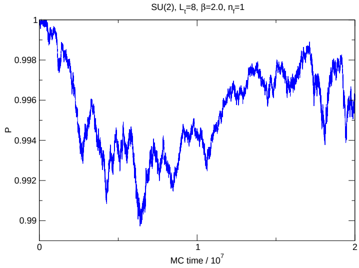
- Significant negative contributions in sectors $n_f = 1$ and 5:
  - gets better towards higher temperatures,
  - $n_f = 1$ at $L_t = 48$ and $\beta = 0.6$:



Flat directions and metastabilities

- System may suffer from running away along flat directions, where $[X_i, X_j] \sim 0$:
  - there exist metastable states along the flat directions
  - X^2 can become arbitrarily large, gauge field freezes
  - algorithm suffers from critical slowing down in those metastable states





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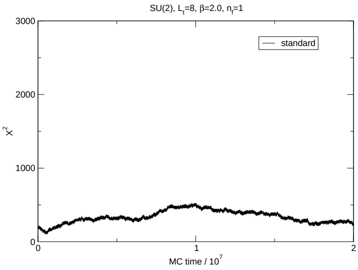

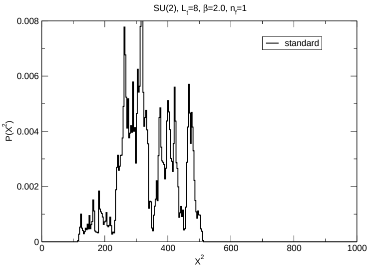

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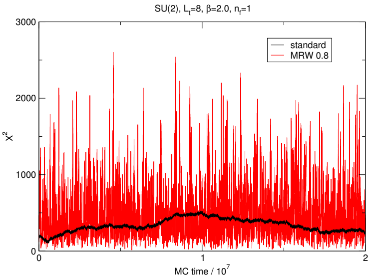
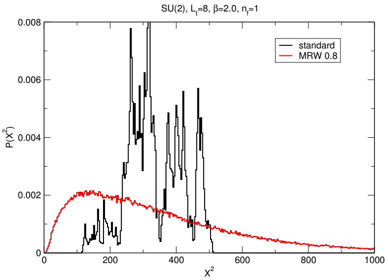
- Multiplicative random walk update:
  - update X *collectively* by rescaling with a random factor R:

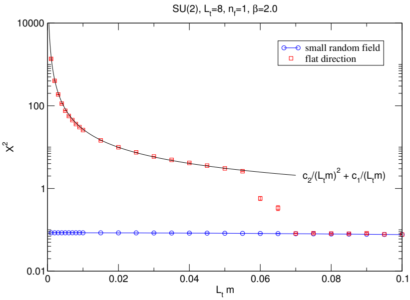


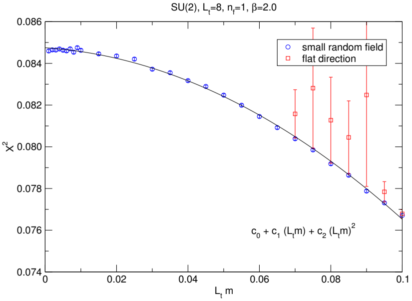


Flat directions and metastabilities

- Metastable phase along flat directions may persist for small m:
  - X^2 diverges in the limit $m \to 0$

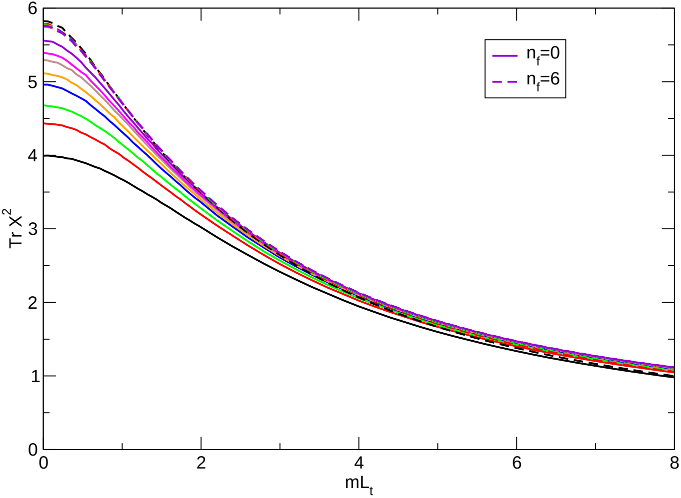
- Simulations starting from small random field configurations seem stable:
  - X^2 well behaved in the limit $m \to 0$





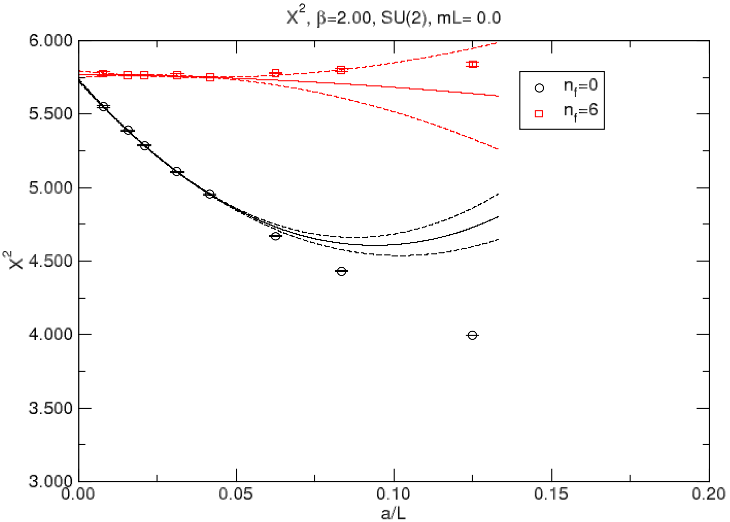
Canonical simulations at fixed n_f for SU(2)

Moduli of X^2 at $\beta = 2.0$ at $n_f = 0$ and $n_f = 6$:



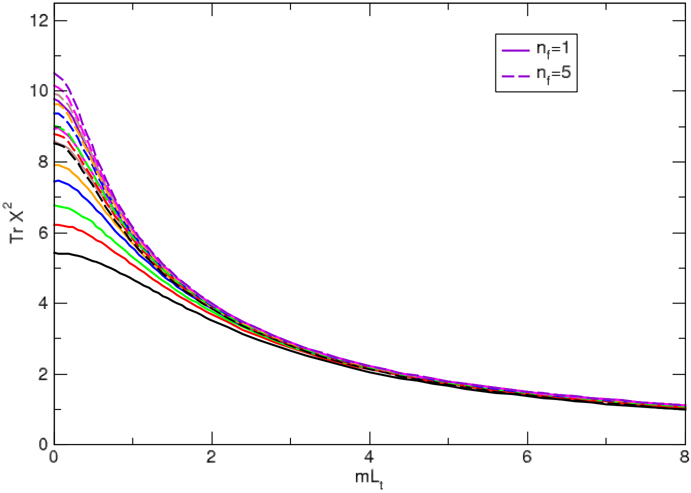
Canonical simulations at fixed n_f for SU(2)

Continuum limit at $mL_t \to 0$ at $\beta = 2.0$ at $n_f = 0$ and $n_f = 6$:



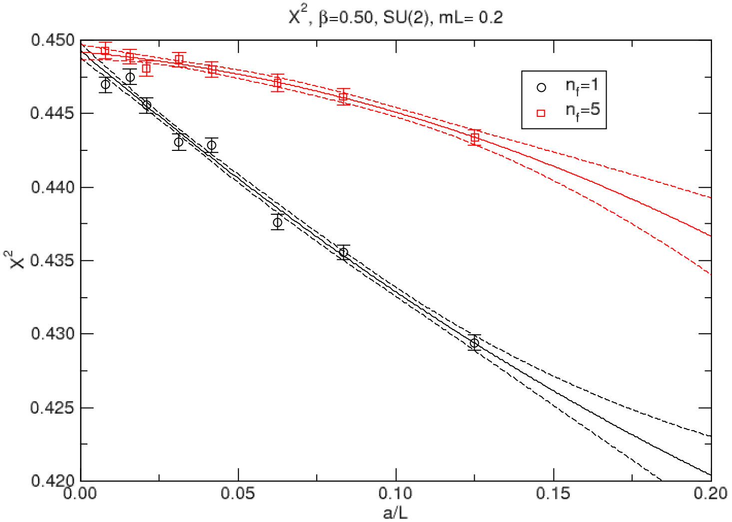
Canonical simulations at fixed n_f for SU(2)

Moduli of X^2 at $\beta = 2.0$ at $n_f = 1$ and $n_f = 5$:



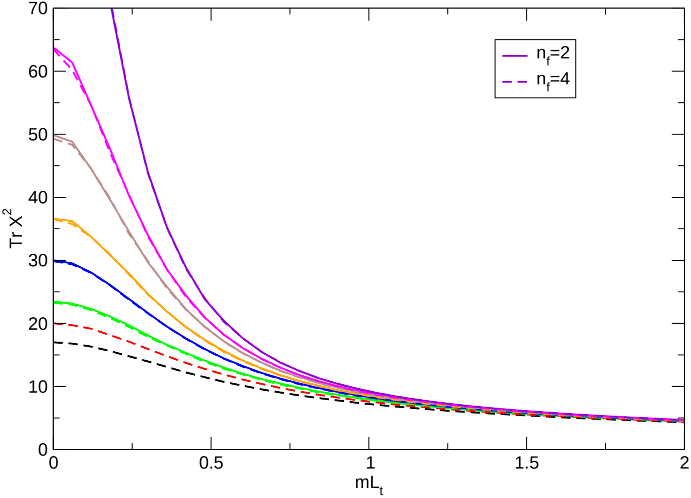
Canonical simulations at fixed n_f for SU(2)

Continuum limit at $mL_t \to 0$ at $\beta = 0.5$ at $n_f = 1$ and $n_f = 5$:



Canonical simulations at fixed n_f for SU(2)

Moduli of X^2 at $\beta = 2.0$ at $n_f = 2$ and $n_f = 4$: sectors degenerate



Canonical simulations at fixed n_f for SU(2)

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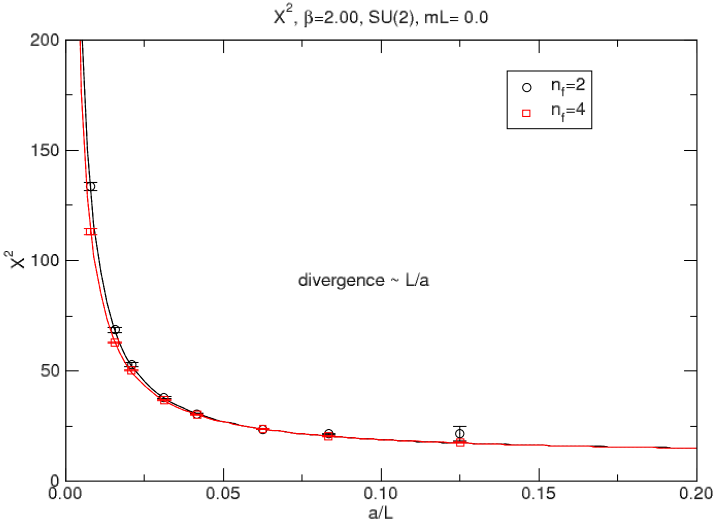
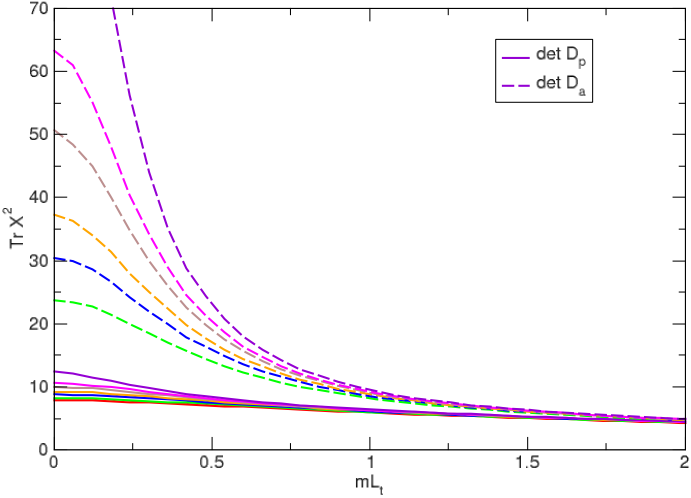


Canonical simulations at fixed n_f for SU(2)

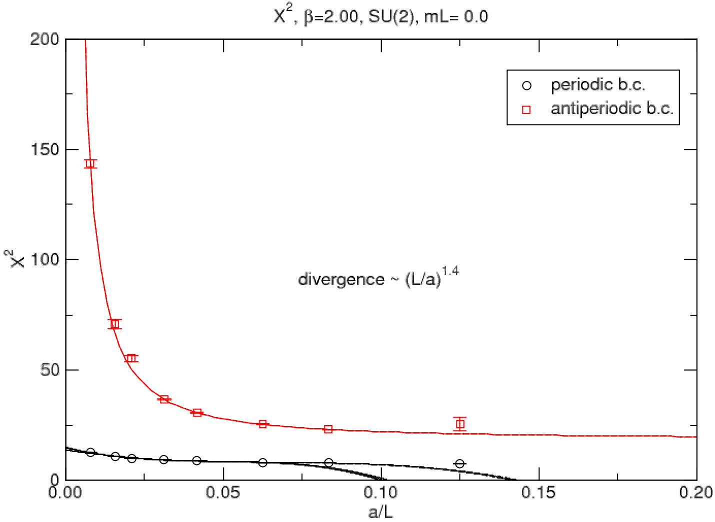
Moduli of X^2 at $\beta = 2.0$ for periodic and thermal b.c.:

$\Rightarrow$ divergent contributions from $n_f = 2, 3, 4$ cancel in det D_p



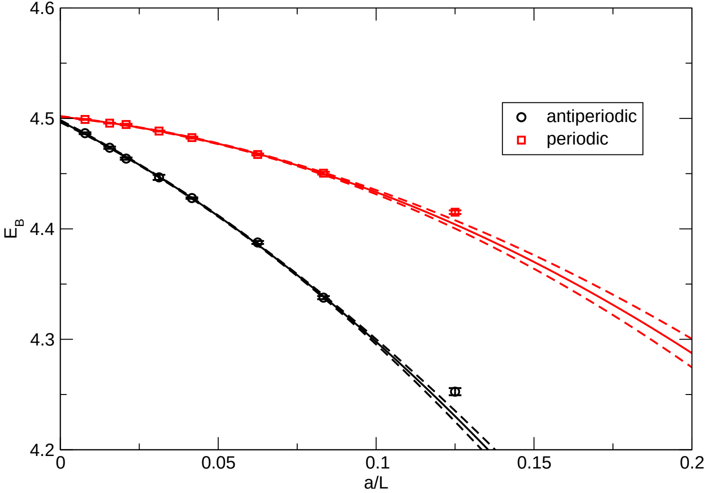
Canonical simulations at fixed n_f for SU(2)

Continuum limit in the limit $mL_t \to 0$ at $\beta = 2.0$ for (a-)per. b.c.:



Canonical simulations at fixed n_f for SU(2)

Other quantities remain finite, e.g. E_B at $\beta = 0.5$:



Conclusions

- Supersymmetric Yang-Mills SU(N) gauge theories on the lattice:
  - $\mathcal{N} = 4$ in $d = 0 + 1$ well understood & under control
  - nonperturbative quantitative statements possible
- Complete description of the phase structure including flat directions
  - $\Rightarrow$ study mass spectra in each sector next
- Interpretation of the divergencies:
  - thermodynamics of black holes (evaporation,...?)
  - large-N limit

Canonical formulation is crucial:

- solves (or avoids) the fermion sign problem
- formalism and techniques also applicable to QCD!