# Lagrangian chaos and scalar mixing for models in fluid mechanics

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# Subject of this talk: Lagrangian flow





**Lagrangian flow**: flow of passive tracer particles in (incompressible) fluid in domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$.

- Velocity field $u(t, x), x \in \Omega$ evolves in time according to fluid mechanics model, e.g., Navier-Stokes
- Lagrangian flow $\phi^t : \Omega \to \Omega, t \geq 0$, solves ODE

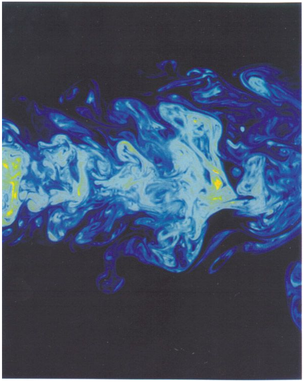
$$\frac{d}{dt}\phi^t(x) = u(t, \phi^t(x)) .$$

- Incompressibility ($\nabla_x \cdot u \equiv 0$) implies $\phi^t$ flow of volume-preserving diffeo's on $\Omega$

# Lagrangian flow



$$u(t,\cdot) : \Omega \to \mathbb{R}^d \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

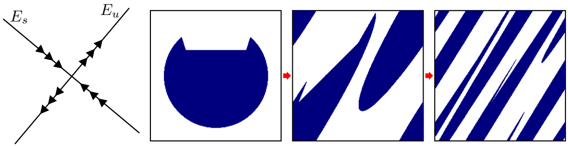


- At left: turbulent jet visualized by fluorescent dye.
- Stretching and folding mechanism should create hyperbolicity
- $\Rightarrow$ expect $\phi^t$ to be **chaotic**
- E.g.: ABC flow (stationary flow for Euler)

Image credit: K. R. Sreenivasan; taken from Shraiman & Siggia, "Scalar turbulence", *Nature* **405**, 639 - 646 (2008)

# Hyperbolicity: infinitesimal stretching and contracting & chaotic properties

Basic example: $F : \mathbb{T}^2 \to \mathbb{T}^2$, $F(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x \pmod{\mathbb{Z}^2}$.




Chaotic features:

- Sensitivity w.r.t. initial conditions: $d(F^n(p_1), F^n(p_2)) \gtrsim e^{\alpha n} d(p_1, p_2)$ when $p_1 - p_2 \notin E^s$
- Correlation decay: for $\phi, \psi \in C^1(\mathbb{T}^2, \mathbb{R})$,

$$\left| \int \phi \cdot \psi \circ F^n - \int \phi \int \psi \right| \leq C \|\phi\|_{C^1} \|\psi\|_{C^1} e^{-\beta n},$$

  $\alpha, \beta, C > 0$ constants.
- Same properties hold for all *uniformly hyperbolic* systems

# Comingled dynamics: hyperbolicity and ellipticity



Problem: stretching and folding should result in nonuniform hyperbolicity: much harder to study

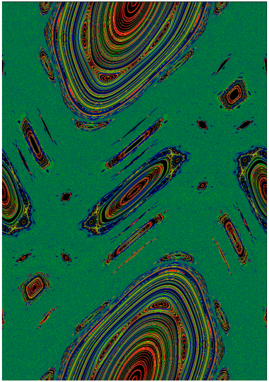


Image due to Wikipedia user Linas

- Chirikov standard map $F : \mathbb{T}^2 \to \mathbb{T}^2$ a **toy model** of stretching and folding.
- Duarte '95 and Gorodetski '12: convoluted comingling of elliptic islands and hyperbolic points in anti-integrable limit, a "predominantly hyperbolic" regime
- Standard map conjecture: $\{\lambda(p) > 0\}$ has positive area. **Wide open.**

# Lagrangian chaos



$$u(t, \cdot) : \Omega \to \mathbb{R}^d \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

## Question

When is $\phi^t$ **chaotic** in the sense of a positive Lyapunov exponent, i.e.,

$$\limsup_{t \to \infty} \frac{1}{t} \log \| D_x \phi^t \| > 0 \quad ?$$

[on a positive-volume set.]

Open question even when $u(t, x)$ given by stationary ABC flow

$$(\dot{x}, \dot{y}, \dot{z}) = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)$$

for any $A, B, C$, let alone when $u(t, \cdot)$ evolves according to a more 'realistic' fluids model such as driven Navier-Stokes.

# Setup: Stochastic fluid models



**Punchline: presence of noise makes verifying chaotic regimes tractable.**

Consider, e.g., 2D Navier-Stokes on $\Omega = \mathbb{T}^2$ with stochastic forcing:

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q \dot{W}_t \,, \quad \nabla \cdot u \equiv 0$$

where $QW_t$ is white-in-time, divergence free, and Sobolev in space

- 2D Navier-Stokes globally (mildly) well-posed for a.e. path realization
- Markov process $u_t = u(t, \cdot)$; unique stationary measure when $QW_t$ "sufficiently" nondegenerate
- Markov process $(u_t, x_t), x_t = \phi^t(x_0)$ on $H \times \mathbb{T}^2$

# Lagrangian Chaos (almost-surely positive Lyapunov exponent)



$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q\dot{W}_t\,, \quad \nabla \cdot u \equiv 0\,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

**Theorem (BBPS 2018, submitted)**

*If $QW_t$ satisfies certain nondegeneracy condition, then*

$$\lim_{t\to\infty} \frac{1}{t} \log |D_x \phi^t| = \lambda > 0 \qquad w.p.1$$

*for all initial $x \in \mathbb{T}^2$ and Sobolev regular vector fields $u_0$. Result also holds for 3D hyperviscous NSE, 2D & 3D Stokes and Galerkin-Navier-Stokes.*

Nondegeneracy needed is very mild: result valid for $u_t$ given by

$$u_t(x,y) = \begin{pmatrix} Z_1(t)\sin y + Z_2(t)\cos y \\ Z_3(t)\sin x + Z_4(t)\cos x \end{pmatrix},$$

$Z_i(t)$ independent Ornstein-Uhlenbeck processes (i.e. $u_t$ solves stochastic Stokes' equation)

# Almost-sure exponential mixing for Lagrangian flow



$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q \dot{W}_t \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

**Theorem (BBPS 2019, submitted)**

*Under the same conditions as previous theorem, for all $p \geq 1$, there exists a* **deterministic** $\gamma = \gamma(p) > 0$ *and a random constant $C = C(\omega, u_0, p)$ such that $\mathbb{P} \times \mu$ a.e. $(\omega, u_0)$ and arbitrary mean-zero $f, g \in H^1(\mathbb{T}^d)$, we have*

$$\left| \int f(x) \cdot g \circ \phi^t(x) \, dx \right| \leq C e^{-\gamma t} \|f\|_{H^1} \cdot \|g\|_{H^1}$$

*with $\mathbb{E} \int C^p d\mu(u_0) < \infty$.*

- A priori much stronger than simply having a positive Lyapunov exponent. Proof uses previous theorem as a lemma.
- Sometimes called *quenched correlation decay*: exponential decay of correlations **almost surely** (a.k.a. $H^{-1}$-decay)

# Implications for passive scalar advection



$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q \dot{W}_t \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

Motion of e.g., chemical concentration fluctuation $g(t,x) \in \mathbb{R}$ in the fluid:

$$\partial_t g + \underbrace{u \cdot \nabla g}_{\text{transport}} = \underbrace{\kappa \Delta g}_{\text{diffusion}} + \underbrace{\dot{\eta}_t}_{\text{random source}} \,, \quad g(0,x) = g_0(x)$$

where $\int g_0 dx = 0$.

- At $\kappa = 0$, $g_t(x) = g_0((\phi^t)^{-1}(x))$ and $\nabla g_t = (D_x \phi^t)^{-\top} \nabla g_0$.
- Lagrangian chaos & mixing $\Rightarrow$ cascade of power spectrum of $g_t$ towards higher modes (up until dissipative range where $\kappa\Delta$ predominates)
- **Toy model** of hydrodynamic turbulence in NSE ($\nu \to 0$)

# Corollary 1: Yaglom's Law



$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q \dot{W}_t \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$
$$\partial_t g + u \cdot \nabla g = \kappa \Delta g + \dot{\eta}_t \,, \qquad g(0, x) = g_0(x)$$

In 1949, Isaak Yaglom predicted the following analogue of the Kolmogorov 4/5 law for passive scalars:

$$\mathbf{E}\left( |\delta_\ell g|^2 \delta_\ell u \cdot \frac{\ell}{|\ell|} \right) \sim_{\substack{\kappa \to 0 \\ \ell \to 0}} -\frac{4}{d} \epsilon |\ell| \,,$$

where $\delta_\ell h(x) = h(x + \ell) - h(x)$ and $\epsilon = \frac{1}{2} \mathbf{E} \|\eta\|^2$.

**Theorem (BBPS 18)**

*Let $(u, g^\kappa)$ be statistically stationary, $\int g^\kappa dx \equiv 0$. Then $\exists \ell_D = \ell_D(\kappa), \lim_{\kappa \to 0} \ell_D(\kappa) = 0$, such that*

$$\lim_{\kappa \to 0} \frac{1}{\ell_D} \mathbf{E} \int_{\mathbb{T}^d} \int_{\mathbb{S}^{d-1}} |\delta_{\ell_D \mathbf{n}} g^\kappa|^2 \delta_{\ell_D \mathbf{n}} u \cdot \mathbf{n} \; dS(\mathbf{n}) dx = -\frac{4}{d} \epsilon$$

# Corollary 2: Batchelor's Law

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q \dot{W}_t \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

$$\partial_t g + u \cdot \nabla g = \kappa \Delta g + \dot{\eta}_t \,, \qquad g(0,x) = g_0(x), \int g_0 dx = 0$$

Let $\Pi_{\leq N} g$ be projection onto span of Fourier modes $\sin(k \cdot x), \cos(k \cdot x)$, $|k|_\infty \leq N$

**Theorem (BBPS 19, in prep)**

*Let $(u, g^\kappa)$ be statistically stationary. Then,*

$$\mathbf{E} \| \Pi_{\leq N} g^\kappa \|_{L^2}^2 \approx \log N \quad \textit{for} \quad 1 \ll N \lesssim \kappa^{-1/2}$$

- Analogue of 5/3 law for power spectrum in hydrodynamic turbulence
- To our knowledge, these constitute first-ever rigorous proof of a universal turbulent scaling law or power spectrum for a fluid evolving according to NSE (c.f. Kraichnan model)
- Requires much add'l work: must study stochastic representation $\dot{\phi}^t(x) = u(t, \phi^t(x)) + \sqrt{\kappa} \dot{W}_t$ of $\kappa \Delta g$

# Ideas from the proof





Diverse array of tools needed:

- Dynamics:
  - Multiplicative ergodic theorem
  - Furstenberg criterion / rigidity of zero Lyapunov exponent cocycles (c.f. Invariance Principle after Avila-Viana)
- Stochastic PDE:
  - Regularity theory for SPDE: strong Feller property
    - Malliavin calculus / nonadapted stochastic calculus
  - Harris's Theorem & Lyapunov *functions*: conditions for mixing of Markov processes in infinite dimensions

# How to prove $\lambda > 0$?



For $t \in \mathbb{N}$,

$$D_{x_0}\phi^t_{u_0} = D_{x_{t-1}}\phi^1_{u_{t-1}} \circ \cdots \circ D_{x_0}\phi^1_{u_0},$$

where $u_t = u(t, \cdot)$ and $x_t = \phi^t(x_0)$. Note: $(u_t, x_t)$ Markovian.

A simplified model:

- Random products of IID $2 \times 2$ matrices $A_1, A_2, \cdots$ of determinant 1, $A^n = A_n \circ \cdots \circ A_1$.
- Lyapunov exponent $\eta = \lim_{n\to\infty} \frac{1}{n} \log |A^n|$ exists and constant wp1. Note $\eta \geq 0$.

**Theorem (Furstenberg '68)**

*If $\eta = 0$ then 2 cases:*

(a) *$\exists$ deterministic inner product $\langle \cdot, \cdot \rangle$ with respect to which $A_1$ is almost-surely an isometry.*

(b) *$\exists$ deterministic lines $\{L_i\}_{i=1}^p, p \in \{1, 2\}$ such that $A_1(\cup_{i=1}^p L_i) = \cup_{i=1}^p L_i$.*

# Prove $\lambda > 0$ by contradiction

$$u(t,\cdot) : \mathbb{T}^d \to \mathbb{R}^d , \quad \dot{\phi}^t(x) = u(t, \phi^t(x)) , \quad u_t = u(t,x) , \quad x_t = \phi^t(x_0) .$$

In our setting:

**Proposition (with J. Bedrossian & S. Punshon-Smith)**

*Fix $d = 2$. If $\lambda = 0$, 2 cases:*

(a) *$\exists$ deterministic, continuously-varying family of inner products $\langle \cdot, \cdot \rangle_{u,x}$ such that $D_{x_0}\phi^t$ an isometry $\langle \cdot, \cdot \rangle_{u_0,x_0} \to \langle \cdot, \cdot \rangle_{u_t,x_t}$.*

(b) *$\exists$ deterministic, continuously-varying families of lines $L^i(u,x), i \leq p, p = 1,2$ such that*

$$D_{x_0}\phi^t\Big( \cup_{i=1}^p L^i(u_0, x_0)\Big) = \cup_{i=1}^p L^i(u_t, x_t)$$

In both cases, $\lambda = 0$ implies **degeneracy** in law of $D_x\phi^t$.

Note: Many such generalizations exist: c.f. Ledrappier, Virtser, Royer, Baxendale, Carverhill

# Ingredients from SPDE

$$u(t,\cdot):\mathbb{T}^d \to \mathbb{R}^d\,, \quad \dot{\phi}^t(x) = u(t,\phi^t(x))\,, \quad u_t = u(t,x)\,, \quad x_t = \phi^t(x_0)\,.$$

**Definition**

Let $(z_t)$ be a Markov process on a Polish space $Z$. We say it has the strong Feller property if for all bounded measurable $\phi : Z \to \mathbb{R}$, have

$$z \mapsto \mathbf{E}\Big(\phi(z_t)|z_0 = z\Big)$$

is continuous for all $t > 0$.

- We require Strong Feller for $z_t = (u_t, x_t)$ process to check continuity of 'deterministic' families (inner products or line bundles)
- For finite-dimensional processes: Hörmander's condition.
- In infinite-dimensions: Malliavin calculus with nonadapted controls
  - Necessary to force infinitely many Fourier modes

# Almost-sure correlation decay: two-point motion



- Consider the *two*-point motion $(u_t, x_t, y_t)$ with $(x \neq y)$

$$\partial_t x_t = u_t(x_t)$$
$$\partial_t y_t = u_t(y_t).$$

This Markov process lives in $\mathbf{H} \times \mathcal{D}^c$ where $\mathcal{D} = \{(x,y) \in \mathbb{T}^{2d} : x = y\}$.

- *Geometric ergodicity* of this two-point motion implies the desired mixing: for some $\mathcal{V} \in L^1(\mu \times Leb \times Leb)$,

$$\left| \mathbf{E}\varphi(u_t, x_t, y_t) - \int_{\mathbb{T}^d \times \mathbb{T}^d} \int_{L^2} \varphi(u,x,y)\mu(du)dxdy \right| \lesssim \mathcal{V}(u_0, x_0, y_0) e^{-\gamma t} \|\varphi\|_{L^\infty}.$$

Basic idea why: apply Borel-Cantelli after the following $L^2$ trick (Dolgopyat-Kaloshin-Koralov '04, Ayyer-Liverani-Stenlund '07)

$$\mathbb{P} \times \mu \left( \left| \int f \circ \phi^n g dx \right| > e^{-qn} \right) \leq e^{2qn} \int \left| \mathbf{E}_{u,x,y} f(x_n) f(y_n) g(x) g(y) \right| dxdyd\mu$$
$$\lesssim \|f\|_{L^\infty}^2 \|g\|_{L^\infty}^2 e^{(2q-\gamma)n}.$$

More quantitative control on $D$ requires regularity of $f, g$ and a more complicated argument.

# Harris' theorem via Goldys/Maslowski

- Goldys-Maslowski '04: convenient framework for checking the conditions for Harris' theorem for e.g. dissipative semilinear parabolic SPDE.
- Let $Z_t$ be a Markov process on a Polish space $\mathcal{Z}$ with transition kernels $Q_t(z, K)$. Suppose:
  - **strong Feller**: $\forall t > 0$ and bounded (measurable) $\psi : \mathcal{Z} \to \mathbb{R}$, $z \mapsto Q_t \psi(z)$ is continuous on $\mathcal{Z}$.
  - **Topological irreducibility**: $\forall$ open $U \subset \mathcal{Z}$, $Q_t(z, U) > 0$ for all $t > 0, z \in \mathcal{Z}$.
  - **Drift condition**: $\exists \mathcal{V} : \mathcal{Z} \to [1, \infty)$ and constants $k, \alpha, c > 0$ such that

$$Q_t \mathcal{V} \leq k e^{-\alpha t} \mathcal{V} + c$$

  - **Uniform lower bounds**: $\forall r > 1$, $\exists$ a compact $K \subset \mathcal{Z}$ and a $t_0 = t_0(r) > 0$ such that

$$\inf_{z : \mathcal{V}(z) \leq r} Q_{t_0}(z, K) > 0 \, .$$

- Checking the lower bound usually isn't hard for parabolic equations: basically we just need $\mathcal{V}(z) \to \infty$ as $z \to \infty$.

# Harris' theorem via Goldys/Maslowski '04

**Theorem (Goldys/Maslowski '04)**

*Then, the Markov process $(Z_t)$ admits a unique stationary measure $m$, with respect to which $(Z_t)$ is geometrically ergodic in $C_{\mathcal{V}}$. That is, for all $\psi \in C_{\mathcal{V}}$, we have that*

$$\left| Q_t \psi(z) - \int \psi dm \right| \lesssim C \mathcal{V}(z) e^{-\beta t} \|\psi\|_{C_{\mathcal{V}}} \quad \text{for all } t > 0 ,$$

*where*

$$\|\psi\|_{C_{\mathcal{V}}} := \sup_{z \in \mathcal{Z}} \frac{|\psi(z)|}{\mathcal{V}(z)}$$

- Strong Feller follows from some Malliavin calculus and Hörmander bracket conditions (not too different from a lemma in Lagrangian chaos).
- Irreducibility follows from an elementary approximate control argument (also not too different from a lemma in BBPS 18).
- Chief new difficulty: the drift condition $\mathcal{V}$!!!

# Drift condition



- The diagonal $\mathcal{D}$ is being treated as part of "infinity", so $\mathcal{V}$ needs to imply that particles close together separate exponentially fast...
- When $d(x_t, y_t) \ll 1$ we expect

$$w_t = x_t - y_t \approx D_x \phi^t(x) w. \quad (1)$$

- Previously: proved $\exists \lambda_1 > 0 \ \forall u, x$, $\mathbf{P}$-a.e. $\lim_{t\to\infty} \frac{1}{t} \log |D_x \phi^t| = \lambda_1$. Indeed, each given $v_0 \in \mathbb{R}^2$ grows at $\lambda_1$ with probability 1!
- Implies repulsion for linearized dynamics near diagonal $\mathcal{D}$.
- It then makes sense to look for a drift condition of the form:

$$\mathcal{V}(u, x, y) = |d(x, y)|^{-p} \psi_p \left( u, x, \frac{x - y}{|x - y|} \right) \chi(d(x, y)) + c_0 V(u)$$
$$=: h_p(u, x, y) + c_0 V(u)$$

where $V$ satisfies a drift condition on the Navier-Stokes equation:

$$V(u) = (1 + \|u\|_{\mathbf{H}}^2)^{\beta} \exp\left(\eta \|\nabla u\|_{L^2}^2\right).$$

Inclusion of $\mathbf{H}$ is not a trivial extension of existing work. The actual proof uses a range of $\beta, \eta$'s and compact embedding arg's

# Twisted semigroup

- We construct $\psi_p$ as the dominant eigenfunction of the "twisted" semi-group acting on observables of the $(u_t, x_t, v_t)$:

$$\hat{P}^p_t \psi(u, x, v) = \mathbf{E}_{u,x,v} |D\phi_t v|^{-p} \psi(u_t, x_t, v_t)$$

  Seek $\psi_p$ such that $\hat{P}^p_t \psi_p = e^{-\Lambda(p) t} \psi_p$ for $\Lambda(p) > 0$.
- Note that due to the unbounded/infinite dimensional phase space, it is not obvious that $\widehat{P}^p_t$ is even *bounded* $C_V \to C_V$ for any $t > 0$.
- Baxendale-Strook 1988 used the twisted semigroup to study large deviations away from the diagonal for the two-point motions.
- $\Lambda(p)$ is the *moment Lyapunov function*[1]

$$\Lambda(p) = -\frac{1}{t} \lim_{t \to \infty} \log \mathbf{E} |D_x \phi^t|^{-p}.$$

- We eventually verify $\Lambda(p) \approx p\lambda_1$ for $0 < p \ll 1$, where $\lambda_1$ is the top Lagrangian Lyapunov exponent.

[1] This comes up in studying large deviations in convergence of Lyapunov exponents; c.f. Arnold et al. '80s.

# Spectral perturbation and the drift condition



- **Step 1:** construct $\psi_p \in C_V$, $\partial_v \psi_p \in C_V$, $\psi_p > 0$
  - Obtain a spectral gap for $\widehat{P}^0_t$ in the corresponding spaces; main step is a gradient bound / "Lasota-Yorke" estimate[2]
  - Spectral theoretic perturbation argument gives $\psi_p$ as the dominant eigenvector of $0 < p \ll 1$.
  - Lagrangian $\lambda_1 > 0 \Rightarrow$ eigenvalue strictly negative
- **Step 2:** obtain drift condition for two point motion by linearized approximation:
  - Key lemma is an estimate based on infinitesimal generators:

$$\mathcal{L}^{(2)} h_p \le \mathcal{L}^{(Lin)} h_p + CV(u) \le -\Lambda(p) h_p + C' V(u),$$

    where $\mathcal{L}^{(2)}$ is the generator for two-point motion and $\mathcal{L}^{(Lin)}$ for linearized motion.
  - Error is absorbed by taking advantage of the drift conditions for NSE, giving

$$\widehat{P}^{(2)}_t \mathcal{V} \le e^{-\gamma t} \mathcal{V} + C''.$$

[2] $\| D_v \widehat{P}^0_t \phi \|_{L^\infty} \lesssim V(u) \left( \|\phi\|_{L^\infty} + e^{-\alpha t} \| D_v \phi \|_{L^\infty} \right)$; this is proved via Malliavin calculus a la Hairer/Mattingly '06 proof of asymptotic strong Feller for 2D NSE with degenerate forcing.

# Recap and conclusion



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$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q \dot{W}_t \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$
$$\partial_t g + u \cdot \nabla g = \kappa \Delta g + \dot{\eta}_t \,, \qquad g(0,x) = g_0(x)$$

- Using stochastic framework, have shown Lagrangian flow $\phi^t$ is chaotic (pos. LE, exponential correlation decay with probability 1) when $u_t$ evolves by Navier-Stokes or other models
- Consequences for passive scalar turbulence: quantitative control on formation of small scales for concentration density of chemicals being passively advected by flow

**Thanks for your attention!**