Diophantine Inheritance and dichotomy for p-adic measures

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Based on a joint work with Anish Ghosh (Tata Institute)

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- $\omega(\mathbf{x}) := \sup_{\mathbf{x} \in \mathcal{W}_v} v$.

- For every $\mathbf{x} \in \mathbb{R}^n$, $\omega(\mathbf{x}) \geq n$.

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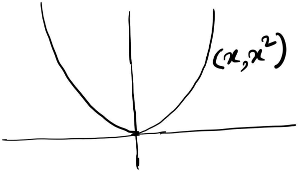

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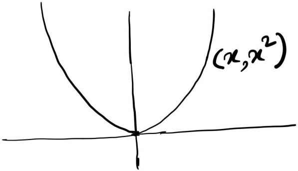
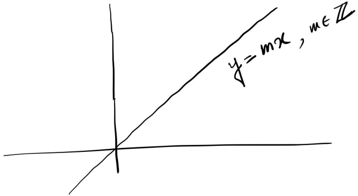


- $\lambda\{x \in \mathbb{R} \mid (x, x^2) \text{ is VWA } \} =?$

Mahler's conjecture



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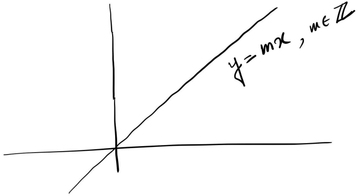

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- In 1932 Mahler conjectured that a.e. point on the Veronese curve $(x, x^2, x^3, \ldots, x^n)$ is *not* very well approximable.

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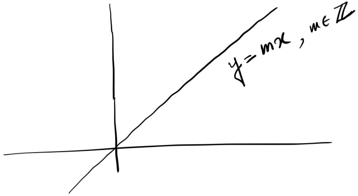

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- This was settled by Sprindžuk in 1964.

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- $\omega(\mathbf{f}_\star\lambda) = n$ for "nondegenerate" manifold $\mathbf{f} : U \subset \mathbb{R}^d \to \mathbb{R}^n$.

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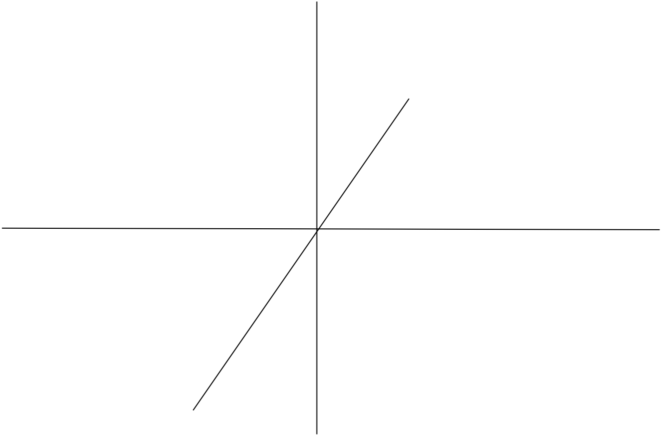
- Kleinbock-Lindenstrauss-Weiss improved that to geometric condition on measures to study Diophantine property.

Affine space



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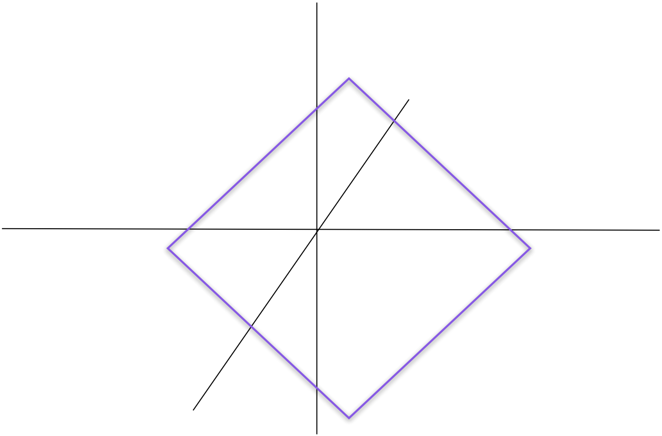

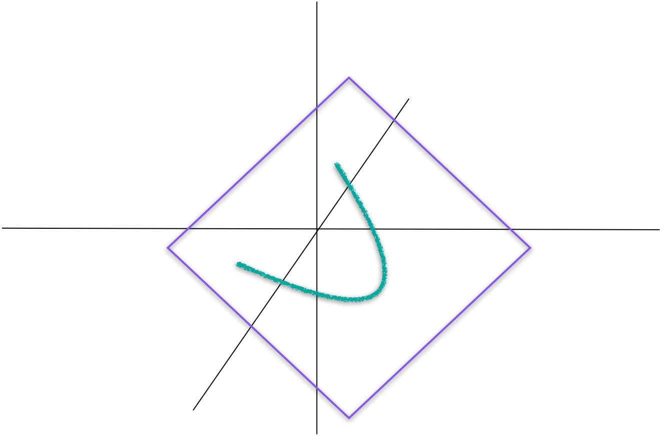

Inheritance

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- In this case both directions of Dani correspondence play role.
- Kleinbock later generalised this and proved $\omega(\mathcal{L}) = \omega(\mathcal{M})$ for nondegenerate manifolds of $\mathcal{L}$.

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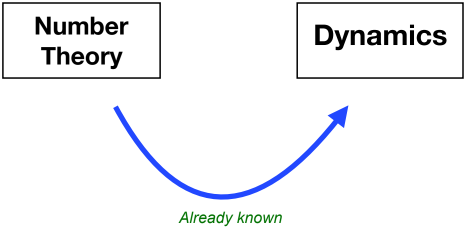
- Later Kleinbock asked the more general question i.e. whether $\omega(\mathcal{L}) = \omega(\mathcal{M})$ holds for nondegenerate manifolds $\mathcal{M}$ of affine subspace $\mathcal{L} \subset \mathbb{Q}_p^n$.
- We could answer these questions.

p-adic Diophantine Approximation



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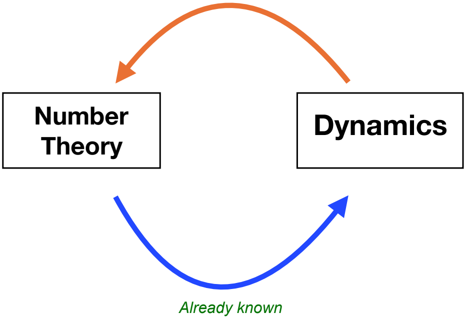

Main theorem

- We introduce a new p-adic Diophantine approximation which is better suited to homogeneous dynamics, and which we show to be closely related to the exponent considered by Kleinbock and Tomanov.

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Theorem (DG, 2019)

Let $\mathcal{L}$ be an affine subspace of $\mathbb{Q}_p^n$, and let $\mathcal{M}$ be a submanifold of $\mathcal{L}$ which is nondegenerate in $\mathcal{L}$. Then

$$\omega(\mathcal{M}) = \omega(\mathcal{L}) = \inf\{\omega(\mathbf{y}) \mid \mathbf{y} \in \mathcal{L}\} = \inf\{\omega(\mathbf{y}) \mid \mathbf{y} \in \mathcal{M}\}.$$

Approximation by $\mathbb{Z}$

Definition

$\mathbf{y} \in \mathbb{Q}_p^n$ is v$-\mathbb{Z}$-approximable if there are infinitely many $\tilde{\mathbf{q}} = (q_0, \mathbf{q}) \in \mathbb{Z}^{n+1}$ such that

$$|q_0 + \mathbf{q} \cdot \mathbf{y}|_p \leq \|\tilde{\mathbf{q}}\|_\infty^{-v}. \tag{1.1}$$

- $w(\mathbf{y}) = \sup\{v \text{ appearing in } 1.1\}$.

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- In view of p-adic Dirichlet's theorem one have $w(\mathbf{y}) \geq n+1$.
- A vector is very well approximable if $w(\mathbf{y}) > n+1$.

Approximation by $\mathbb{Z}[\frac{1}{p}]$

Definition

$\mathbb{Z}[1/p]$ **approximable vectors:** $\mathbf{y} \in \mathbb{Q}_p^n$ is v$-\mathbb{Z}[1/p]$-approximable if there exists $\tilde{\mathbf{q}} = (q_0, \mathbf{q}) \in \mathbb{Z}[1/p]^{n+1}$ with unbounded $\|\mathbf{q}\|_p \|\tilde{\mathbf{q}}\|_\infty$ such that

$$|q_0 + \mathbf{q} \cdot \mathbf{y}|_p < \frac{1}{(\|\mathbf{q}\|_p \|\tilde{\mathbf{q}}\|_\infty)^v \|\tilde{\mathbf{q}}\|_\infty}. \tag{1.2}$$

- $w_p(\mathbf{y}) = \sup\{v \text{ appearing in } 1.2\}$.

Relation between two exponents

- The following proposition gives a nice relation between these two types of approximation.

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For any $\mathbf{y} \in \mathbb{Q}_p^n$ *we have*

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- This $\mathbb{Z}[\frac{1}{p}]$ approximation turns out to be directly connected to dynamics.

Connection to dynamics

- For any $\mathbf{y} \in \mathbb{Q}_p^n$ we associate a lattice $u_{\mathbf{y}}\mathcal{D}^{n+1}$ in $(\mathbb{Q}_p \times \mathbb{R})^{n+1}$, where $u_{\mathbf{y}}$ is defined as

$$u_y^p = \begin{bmatrix} 1 & \mathbf{y} \\ 0 & I_n \end{bmatrix}.$$

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- For $t \in \mathbb{N}$ define

$$g_t^p = \begin{bmatrix} p^{-t} & 0 \\ 0 & I_n \end{bmatrix} \text{ and } g_t^\infty = \text{diag}(p^{-\frac{t}{n+1}}, \cdots, p^{-\frac{t}{n+1}}).$$

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- $\omega_p(\mathbf{y}) \geq v$, where $v > n$.
- For every $d < c$ there exists arbitrarily large $t \in \mathbb{N}$ such that $\delta(g_t u_{\mathbf{y}} \mathcal{D}^{n+1}) \leq p^{-dt}$, where $c = \frac{v-n}{(n+1)(v+1)}$.

More theorems

Theorem (DG, 2019)

Let $\mathcal{L}$ be an affine subspace, parametrized by a matrix A. If all the rows (resp. columns) are rational multiples of one row (resp. column) then one has

$$w_p(\mathcal{L}) = \max(n, w_p(A)) \text{ and } w(\mathcal{L}) = \max(n+1, w(A)).$$

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Theorem (DG, 2019)

Suppose $\mathcal{M}$ is a connected analytic manifold of $\mathbb{Q}_p^n$. Let $v \geq n$ and suppose $w_p(\mathbf{y}) \leq v$ for some $\mathbf{y} \in \mathcal{M}$ then for almost every $\mathbf{y} \in \mathcal{M}, w_p(\mathbf{y}) \leq v$.

- So if one point is not very well approximable then almost every point is not very well approximable.

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