# LECTURE 1

## THE GENERALIZED RAMANUJAN CONJECTURES AND L-FUNCTIONS

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R.C. $|\lambda(p)| \le 2$.

Since $\tau(n) = n^{11/2}\lambda(n)$.

So we can consider the question as one of the sizes of the Hecke eigenvalues (special to this setting in general one might consider R.C. as a bound for "periods").

First nontrivial bounds:

Rankin-Selberg: $|\lambda(p)| \le p^{3/10} + p^{-3/10}$
(the "trivial" bound is $p^{1/2}+p^{-1/2}$).

Idea: $$\sum_{n=1}^{\infty} \frac{|\lambda(n)|^2}{n^s}$$ " the Rankin Selberg L-function can be analytically continued much like the Riemann Zeta function. One then uses a Lemma of Landau....

- Deligne's proof of the Weil conjectures were inspired by this Rankin–Selberg method executed in the function field setting where analytic properties of "higher symmetric power functions" were known (GROTHENDIECK).

## Selberg's Eigenvalue Conjecture:

Firstly the ideas mentioned above all generalize to congruence surfaces

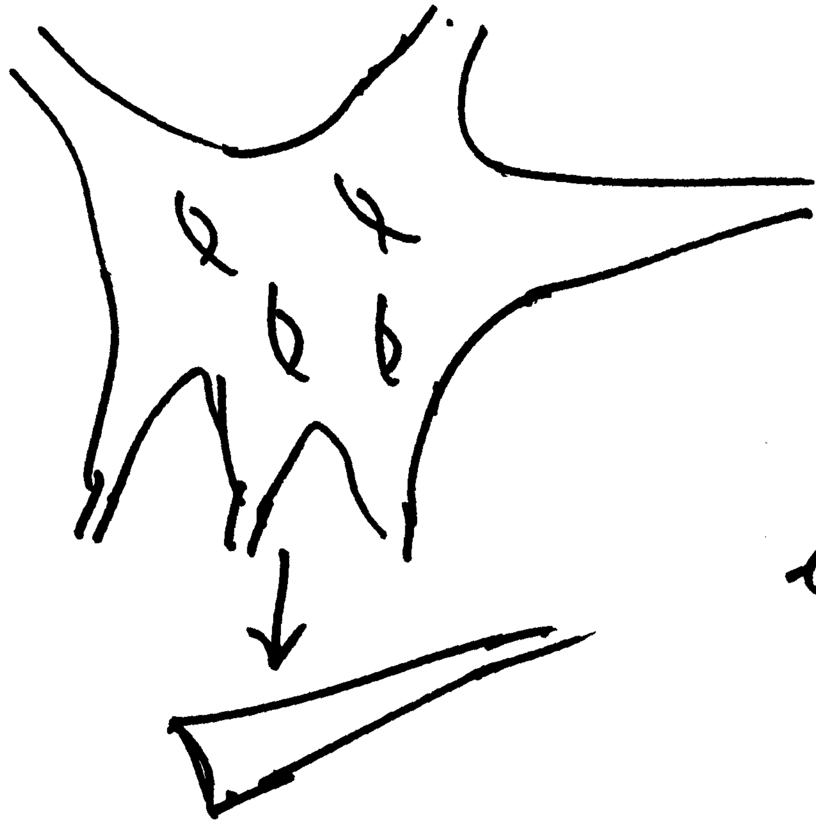
$$X(N) = \Gamma(N) \backslash \mathbb{H}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1 \pmod{N} \right\}.$$

In this case the space of forms has an inner product, Petersson inner product, for which $T_n$'s are selfadjoint $\Rightarrow$ diagonalized.

④

$X(N)$



covering

$X(1)$

Let $\Delta$ = divgrad w.r.t. the Laplacian on $\mathbb{H}$, line element

$$ds = \frac{|dz|}{y}$$ hyperbolic metric.

descends to $X(N)$.

$$\Delta\phi + \lambda\phi = 0 \quad \text{for } \phi \in L^2(X(N))$$

Laplace Spectrum of $X(N)$.

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \cdots$$

Selberg's eigenvalue Conj. is the archimedian analogue of R.C.

<u>CONJ:</u> $\lambda_1(X(N)) \geq \frac{1}{4}$ $\left(\begin{matrix}\text{He proved} \\ \lambda_1 \geq 3/16\end{matrix}\right)$

Why $\frac{1}{4}$? The $L^2$ Spectrum of $\Delta$ on the freespace $L^2(\mathbb{H})$ is $\left[\frac{1}{4}, \infty\right)$.

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Very deep and powerful conjecture.

• <u>EG:</u> Consider the series (Raman., Hardy, ...)

$$\sum_{n=1}^{\infty} \frac{d(n)\,d(n+h)}{n^s}$$

if $h \neq 0$ this has an analytic continuation to $\mathbb{C}$ with ~~eigen~~ poles at $s = \frac{1}{2} + it_j$ where $t_j^2 + 1/4 = \lambda_j$.

An important more conceptual point of view which is critical when generalizing to other groups is the representation theoretic interpretation. (Gelfand, Piatetski-Shapiro, Satake, ...).

Work with $\mathbb{Q}$ instead of $\mathbb{Z}$ and bigger spaces

<u>Adeles:</u>

$$GL_2(\mathbb{A}) = \{ (g_\infty, g_2, g_3, \dots) : g_\infty \in GL_2(\mathbb{R}),\ g_p \in GL_2(\mathbb{Q}_p),\ g_p \in GL_2(\mathbb{Z}_p) \text{ for } p \text{ large enough} \}.$$

with restricted product topology

$$H_\chi = L^2(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}), \chi)$$

$\chi$ is a unitary central character (Dirichlet), $GL_2(\mathbb{Q})$ embedded diagonally.

• $L^2_{cusp}$ are $f$'s s.t.

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)\,dn = 0 \quad \text{for all } g.$$

$$N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}.$$

$G = GL_2(\mathbb{A})$ acts unitary on $L^2_{cusp}$ by

$$f(x) \to f(xg).$$

decomposes into a discrete sum of irreducible representations of $G(\mathbb{A})$ called "cusp forms"

$$L^2_{cusp}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}), \chi) = \bigoplus_{\pi} \pi$$

each $\pi$ is

$$\pi \cong \bigotimes_v \pi_v \quad , \quad \pi_v \text{ a (unitary) repr. of } GL_2(\mathbb{Q}_v).$$

Ramanujan + Selberg Conj $\Longleftrightarrow$

- $\pi$ automorphic cuspidal repr of $GL_2(\mathbb{A})$. Then each local ~~cons~~ factor $\pi_v$ is "tempered".

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<u>Remark:</u> $\pi_v$ being tempered means $\pi_v$ occurs in the decomposition of $L^2(GL_2(\mathbb{Q}_v))$, i.e in the "free space". The tempered spectrum for real groups was identified by Harish-Chandra.

- So one might say that the only spectrum that is excited arithmetically i.e. in the cuspidal automorphic spectrum of $GL_2$ is the tempered spectrum.

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In 1967 Langlands made a very bold extention of this conjecture to $G = GL_n$; it is one of the central problem.

<u>Generalized Ramanujan Conjecture:</u>

Let $K$ be a number field, $\pi$ an automorphic <u>cuspidal</u> repr occuring in $L^2(GL_n(K) \backslash GL_n(\mathbb{A}_K), \chi)$, $\pi \cong \bigotimes_v \pi_v$ (all places of $K$).
Then $\pi_v$ is tempered for all $v$.

Such a $\overline{\pi_v}$ has "Langlands parameters"

$$\mathrm{Ind}_P^{GL_n(K_v)} \tau \ , \text{ where } \underline{P} \text{ has}$$

blocks $n_1 + n_2 + \cdots + n_k = n$, $\tau = \tau_1[\sigma_1] \times \cdots \times \tau_k[\sigma_k]$
$\tau_j$ is tempered on $GL_{n_j}(K_v)$ and $\sigma_j \in \mathbb{R}$
Let $\sigma(\pi_v) = \max_j |\sigma_j|$, $\sigma(\pi_v) = 0 \iff \pi_v$ is tempered.

- There is a local bound since $\pi_v$ is "generic" (globally cusp $\Rightarrow$ locally gen). one has

$$|\sigma(\pi_v)| < \frac{1}{2}.$$

$\boxed{\text{Jacquet-Shalika Vogan, Tadic}}$

<u>Note:</u> for $n=2$ this say very little but for $n \geq 3$ it is nontrivial.

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Thanks to the development of Whittaker functions and the Rankin-Selberg L-functions for $GL_n$ by Jacquet - Shalika - Piatetski Shapiro, on has a full understanding of the global analytic properties of

$L(s, \pi \times \pi')$.

$\pi, \pi'$ on aut cusp $GL_n$ $GL_m$ resp.

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Using the Rankin-Selberg L-fns by a new technique of deforming them into twists by finite characters and questions of their zeros in families lead to:

__Theorem__ (Luo-Rudnick-S 1995 for $\pi_v$ unramified and Mueller-Speh in general 2005?)

$$\sigma(\pi_v) \leq \frac{1}{2} - \frac{1}{n^2+1}.$$

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For $n = 2, 3$ and $4$ one can do better using certain hard earned functorial lifts for $\mathrm{Sym}^2$, $\mathrm{Sym}^3$ and $\mathrm{Sym}^4$ for $GL_2$ (Kim-Shahidi, Cogdell-Kim-Piatetski Shapiro-Shahidi) + techniques of Duke-Iwaniec
$\Rightarrow$ Kim-S (2002) for $K = \mathbb{Q}$

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Recently certain difficulties associated with units in $O_K$, general $K$, were resolved using Grossenkaracherre cleverly $\Rightarrow$

<u>**THEOREM**</u> (BLOMER-BRUMLEY 2011):

$n=2$ ; $\sigma(\pi_v) \leq \frac{7}{64}$

$n=3$ ; $\sigma(\pi_v) \leq \frac{5}{14}$

$n=4$ ; $\sigma(\pi_v) \leq \frac{9}{22}$.

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In particular towards Selberg's eigenvalue conj one has

$$\lambda_1(X(N)) \geqslant \frac{1}{4} - \left(\frac{7}{64}\right)^2 .$$

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So the G.R.C. is not known even for $GL_2/\mathbb{Q}$. We noted that Deligne proved R.C. for holomorphic forms.

There is a striking extention to $GL_n$ for certain forms.

A series of recent works using

- the trace formula
- shimura varieties
- Deligne's proof of the Weil Conj.

KOTTWITZ, CLOZEL, HARRIS-TAYLOR, SHIN (2011)
CARAIANI (2011)

THEOREM: If $K$ is totally real (resp CM) and $\pi$ is automorphic cuspidal on $GL_n/K$ and is self dual (resp conj. self dual) and $\pi_\infty$ is algebraic cohomological, then $\pi_v$ satisfies GRC at all finite places.

FOR MORE GENERAL GROUPS $G/K$, THE 'NAIVE' RAMANUJAN CONJ (i.e. tempered) FAILS, first noted by Saito-Kurokawa, Howe-Piatetski Shapiro.

Its formulation is difficult but at least for classical groups thanks to recent work of ARTHUR as well as developments around the fundamental lemma, there is now a precise formulation of GRC and the trace formula allows one to reduce the formulated Conjecture to $GL_n$. In this way the bounds known for $GL_n$ give very sharp bounds for these $G$'s.

# SUBCONVEXITY

AS MENTIONED AT THE BEGINNING IN $GL_2$ WE CAN INTEPRET R.C IN TERMS OF SIZES OF HECKE EIGENVALUES OR PERIODS. IN GENERAL THE LATTER IS MORE DIFFICULT AND IN VARIOUS SPECIFIC CASE IT IS RELATED TO ESTIMATING L-FUNCTIONS ON THEIR CRITICAL LINE.

Fix $K/\mathbb{Q}$ and $n$. $\pi$ an automorphic cusp form on $GL_n/K$ and $L(s,\pi)$ its <u>standard</u> L-function (its finite part!) i.e.

$$L(s,\pi) = \prod_{v<\infty} L(s,\pi_v).$$

$\pi$ has a conductor $N_\pi \geq 1$ measuring it ramification.

The analytic conductor

$$C_\pi = N_\pi \cdot C(\pi_\infty)$$

includes also the "complexity" of $\pi_\infty$.

We are interested in the size of

$L(\frac{1}{2}, \pi)$ (or $L(\frac{1}{2}+it, \pi)$),

in terms of its analytic conductor.

For example if $\pi$ is the trivial representation of $GL_1/\mathbb{Q}$ then

$$L(s,\pi) = \zeta(s).$$

$L(\frac{1}{2}+it, \pi)$, $C_\pi = (|t|+1)$

convex bound

$\zeta(\frac{1}{2}+it) \ll |t|^{1/4}$ and (essentially) Weyl prove a bound $\ll |t|^{1/6}$!

## GRAND LINDELOFF HYPOTHESIS

$$L(\tfrac{1}{2},\pi) \ll_\varepsilon C_\pi^{\varepsilon} \quad \text{for } \varepsilon > 0.$$

NB: GLH follows from GRH

## CONVEX BOUND (IWANIEC-MOLTENI)

$$L(\tfrac{1}{2},\pi) \ll_\varepsilon C_\pi^{1/4+\varepsilon}.$$

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A subconvex bound for a family $\mathcal{F}$ of $\pi$'s (eg the universal family) is a bound of the form, there is $\delta > 0$ s.t. $L(\tfrac{1}{2},\pi) \ll C_\pi^{1/4-\delta}$ for $\pi \in \mathcal{F}$.

These turn out to be decisive in numerous problems.

For example by a formula of Waldspurger, Fourier coeff of holomorphic cusp forms of $1/2$-integral weight are expressed as values ~~holomorphic~~ at $1/2$ of L-functions of holomorphic cusp forms of even weight.

- The R.C. for such forms of $1/2$-integral weight $\Longleftrightarrow$ Lindelof Hypothesis for a suitable family.

Indeed one can prove the Duke-Schulze Pillot theorem on

$$x_1^2 + x_2^2 + 10\, x_3^2.$$

by proving a subconvex estimate for certain $\mathcal{F}$'s on $GL_2/\mathbb{Q}$ (DUKE-FRIEDLANDER-IWANIEC)

The Hilbert problem for ternary integral forms over K was proven by establishing a subconvex estimate for Hilbert hol) $\pi$'s.

THE PROOF FOR THESE L-FUNCTIONS OF HOL FORMS USES GRC FOR $GL_2/K$ (BOUNDS).

Thanks to works of many people today we know a subconvex bound quite generally for $GL_1$ and $GL_2/K$

· For the universal family $GL_2/K$ this is due Michel and Venkatesh. (2010 PIHES).

· VARIOUS SUBCONVEXITY RESULTS ARE KNOWN FOR RANKIN SELBERG L-FUNCTIONS OF $GL_2^2$ FORMS, BUT ~~THE~~ ANY GENUINE EVEN FOR $GL_3$ IS OPEN.

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WEAK SUBCONVEXITY (SOUNDARARAJAN 2010):

For $\mathcal{F}$ on $GL_n/\mathbb{Q}$ this is a bound

$$L(\tfrac{1}{2},\pi) \ll_{\varepsilon} \frac{C_\pi^{1/4}}{(\log C_\pi)^{1-\varepsilon}}, \pi \in \mathcal{F}.$$

AS A CONSEQUENCE

THEOREM (HOLOWINSKI SOUNDARARAJAN)

$$X = \Gamma \backslash \mathbb{H} \quad (\Gamma = SL_2(\mathbb{Z}))$$

and $f$ varies over holomorphic cusp forms of weight $k$ (Hecke-eigenforms) Then the zeros of $f$ ~~...~~ become equidistributed in $X$ w.r.t. hyperbolic area form as $k \to \infty$

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