THIN GROUPS

$SL_n(\mathbb{Z})$, $n \times n$ integer matrices with det equal to 1.

It and various of its subgroups come up everywhere

automorphic forms, number theory, geometry, ...

[MORE GENERALLY ALLOW FINITELY GENERATED SUBGROUPS OF $GL_n(K)$, K a NUMBER FIELD]

(1) STRONG APPROXIMATION

chinese remainder theorem (elementary)

$q \geq 1$

$$SL_n(\mathbb{Z}) \xrightarrow[\text{reduce mod } q]{\pi_q} SL_n(\mathbb{Z}/q\mathbb{Z})$$

is onto.

More generally if G is a matrix algebraic group defined $/\mathbb{Q}$, $\Gamma = G(\mathbb{Z})$ the $\mathbb{Z}$ points of G

Then the image

$$G(\mathbb{Z}) \longrightarrow G(\mathbb{Z}/q\mathbb{Z})$$

is well understood – strong approximation (G semisimple).

Less well known is and less elementary is

(2) SUPERSTRONG APPROXIMATION (EXPANSION)

$\Gamma = SL_n(\mathbb{Z})$, take a finite symmetric set of generators S of Γ

$$s \in S \iff s^{-1} \in S$$

Form the connected Cayley graphs

$$X_q = (SL_n(\mathbb{Z}/q\mathbb{Z}), S) \quad , q \geq 1$$

x is joined to sx, $x \in S$.

$|S|$-regular connected graph on $|SL_n(\mathbb{Z}/q\mathbb{Z})|$ vertices.

- X_q is an expander family.

I.E. A_q the adjacency operator on $l^2(X_q)$

λ_0 = biggest eigenvalue = $|S|$

there is $\varepsilon_0 > 0$ s.t.

$$\lambda_1(A_q) \le |S| - \varepsilon_0 \text{, for all } q.$$

This asserts that the graphs X_q are highly connected (sparse) and this is critical to many applications

THIS HAS ITS ROOTS IN SELBERG'S THEOREM THAT FOR THE HYPERBOLIC SURFACES $\Gamma(q)\backslash\mathbb{H}$, $\Gamma(q) = \{\gamma \in \Gamma : \gamma \equiv I(q)\}$

$$\lambda_1(\Gamma(q)\backslash\mathbb{H}) \ge \frac{3}{16}$$

λ_1 = smallest eigenvalue of the Laplacian.

THIS EXPANSION IS KNOWN FOR GENERAL $G(\mathbb{Z})$'s, G SEMISIMPLE $/\mathbb{Q}$
(BURGER / S, CLOZEL "PROPERTY TAU")

—

If $\Gamma \leq SL_n(\mathbb{Z})$ let

$$G = Zcl(\Gamma), \quad \text{Zariski closure.}$$

So if Γ is finite index in $G(\mathbb{Z})$ then strong and superstrong approximation hold, however if Γ is of infinite index then the techniques used above (automorphic forms) don't apply.

<u>DEFINITION</u> Γ IS THIN IF IT IS OF INFINITE INDEX IN $G(\mathbb{Z})$, $G = Zcl(\Gamma)$.

Remarkably both strong and superstrong approximation continue to hold for thin groups!

- Strong approximation for thin groups is due to Matthews-Weisfeiller-Vaserstein ....

<u>Eg</u>: $\Gamma \leq SL_n(\mathbb{Q})$ finitely generated, $Zcl(\Gamma) = SL_n$.

Then there is a $q_0 = q_0(\Gamma) < \infty$ such that

$$\Gamma \longrightarrow SL_n(\mathbb{Z}/q\mathbb{Z}) \text{ is onto}$$

for $(q, q_0) = 1$.

[so outside a finite set of places Γ doesn't know its thin as far as these quotients]

Given this one can ask (Lubotzky 90's) are the X_q's expanders in this thin setting?

The answer is yes thanks to works of many people

[S-X] Sarnak-Xue (1994)

[Ga] Gamburd (1999)

[H] Helfgott (2006)

[B-G-1] Bourgain-Gamburd (2006)

[B-G-S] Bourgain-Gamburd-S (2009)

[P-S] Pyber-Szabo (2010)

[B-G-T] Breuillard-Green-Tao (2010)

[V] Varju (2011).

Salehi-Varju (2011)

Almost final version

THE FUNDAMENTAL EXPANSION THEOREM:

Let $\Gamma \leq SL_n(\mathbb{Q})$ be finitely gen. with set S. Then the congruence graphs $(\pi_q(\Gamma), S)$, for q squarefree & coprime to $q_0(\Gamma)$ form an expander family iff G^0 the connected component of $G = Zcl(\Gamma)$ is perfect (i.e. $[G,G] = G$). Moreover the determination of the expansion constant is effective - if not feasible.

SOME APPLICATIONS

(1) Affine sieve and diophantine analysis on orbits of thin groups:

(This was the impetus for developing the expansion theory)

- One can now execute a BRUN combinatorial sieve in the most general setting where one expects to find primes (or almost primes) SALEHI / S.

EXAMPLE (special features)

INTEGRAL APOLLONIAN PACKINGS

$$F(x_1,x_2,x_3,x_4) = 2(x_1^2+x_2^2+x_3^2+x_4^2) - (x_1+x_2+x_3+x_4)^2$$

"DESCARTES FORM"

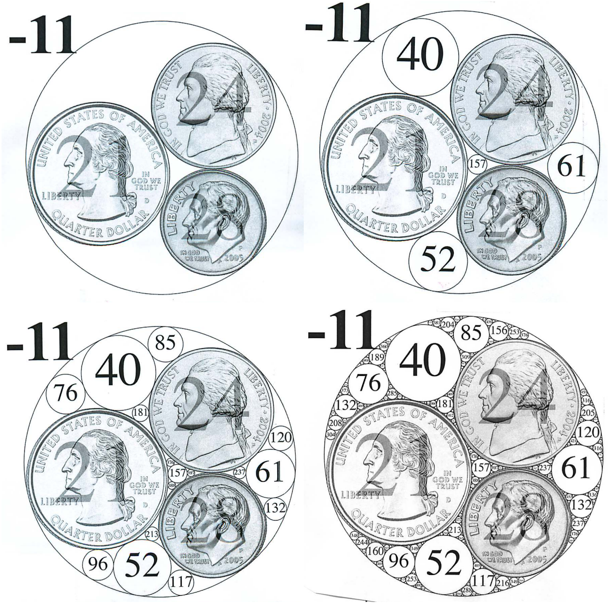
$G = O_F$ the orthogonal group of F

$$A = \langle S_1, S_2, S_3, S_4 \rangle, \quad S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$A \leq O_F(\mathbb{Z})$, A IS THIN!

ORBIT OF A ON $F(x) = 0$ YIELDS AN INTEGRAL APOLLONIAN PACKING

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{etc}$$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WHICH NUMBERS ARE CURVATURES?
ARE THERE INFINITELY MANY "PRIMES"?
TWIN " .

THEOREM (S 07): THERE ARE INFINITELY MANY PAIRS OF TANGENT CIRCLES WHOSE CURVATURES ARE PRIME.

LOCAL TO GLOBAL CONJECTURE:
[GRAHAM, LAGARIAS, MELLONS, WILK, YAN]
[FUCHS - SANDEN]:

$a(C)$ = curvature of $C \equiv 0, 4, 12, 13, 16, 21 \pmod{24}$

THERE IS $N_0 < \infty$ S.T. FOR $n > N_0$ AND n satisfying the congruence there is a C with $a(C) = n$.

THEOREM (BOURGAIN - KONTOROVICH):
THE SET n of exceptions to the local to global conjecture has zero density in $\mathbb{N}$.

(2) GROUP THEORETIC APPLICATIONS OF SIEVING

THEOREM (LUBOTZKY-MEIRI):

Let Γ be a finitely generated subgroup of $GL_n(\mathbb{C})$ (i.e. linear group) which is not virtually solvable. Then the powers

$$P = \bigcup_{m=2}^{\infty} \Gamma^m$$

is an exponentially small subset of Γ in the sense that a (full) random walk on Γ has exponentially small probability of hitting P.

In particular finitely many translates of P can never cover Γ.

(3) GONALITY AND HEEGARD GENUS

X a Riemann surface (curve / $\mathbb{C}$) of genus g.

X can be realized as a degree h cover of $\mathbb{P}^1$ with $h \leq \frac{g}{2} + 1$ (Riemann-Roch).

Let $d(X)$ be the least such h, it is a subtle invariant called the gonality.

If $X = \Gamma \backslash \mathbb{H}$ a hyperbolic surface then [Yang-Yau] $\Rightarrow$

$$d(X) \geqslant \frac{\lambda_1(X)(g-1)}{4\pi}.$$

So if $\lambda_1 \geqslant \varepsilon_0 > 0$ then $d(X)$ is linear in the genus!

eg for $X(N) = \Gamma(N) \backslash \mathbb{H}$ this is true by Selberg.

Controling the gonality is often a key side condition in applying Faltings's big Theorem (1990)

Eg: $X(N)/\mathbb{Q}$ has only finitely many rational points if $g(X(N)) \geqslant 2$ (Mazur for these, Faltings all curves)

- Fix D, if $N \geqslant 230 D$ then the set of points from all number fields of degree $d \leq D$ on $X(N)/\mathbb{Q}$ is finite! (gonality + Faltings observ. by FREY).

- There are similar diophantine problems on towers of curves for which the gonality growth is proven using the Fundamental expansion Theorem (Ellenberg-Hall-Kowalski 2011).

Hyperbolic 3-manifolds:

X a hyperbolic 3-manifold of finite volume. The minimal genus of a surface S in X such that X is realized as two handle bodies meeting ~~as~~ in S, is the Heegard genus of X, $g(X)$ _ A subtle topological invariant.

Lackenby:
$$g(X) \geq \frac{\lambda_1(X)\,Vol(X)}{32\pi}$$

(always $g(X) \leq 100\, Vol(X)$).

So for congruence hyperbolic 3 manifolds The Heegard genus is linear in the volume (by property tau).

. Similar application to towers of any 3-manifold using expansion for thin groups (Long-Lubotzky-Reid 2010).

Ubiquity of thin groups

As we will see they arise in many contexts;
However in many situations we don't know how to tell if the group is thin or not!

(Mihailova 1958) There is no finite procedure to decide whether a (general) set of say 7 elements in $\Gamma = SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ is finite index in Γ or not.

1) Random Groups (via generators)
"Ping-Pong"

Take $X_1, X_2, \ldots, X_\ell$ ($\ell \geq 2$ fixed) elements at random in $SL_n(\mathbb{Z})$ ($n \geq 2$) by either

(i) running a full random walk on $SL_n(\mathbb{Z})$ ~~and~~ for a long time and taking X_1 — then repeat for $X_2, \ldots X_\ell$

(ii) Choose $X_1, \ldots, X_\ell$ so that X_j, X_j^{-1} lie in a big Euclidean ball in $M_n(\mathbb{R})$ independently and with uniform measure, then $\Pi = \langle X_1, \ldots, X_\ell \rangle$ will be Zariski dense, free and thin with prob $\to 1$.

(Aoun 2010 ; Fuchs 2012)

2) Non-arithmetic groups

If $\Gamma \leq G$ is a nonarithmetic lattice in a noncompact simple Lie group, then by local rigidity ($G \neq SL_2(\mathbb{R})$) Γ is thin. It can be conjugated to have entries in a nd field K and the certificate of thin is being discrete in a factor.

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(3) Hyperbolic Reflection Groups

$f(x_1, \ldots, x_n)$ integral quadratic form.

$n \geq 3$.

O_f its orthogonal group

assume that $O_f(\mathbb{R})$ has signature $(n-1,1)$ so $O_f(\mathbb{R})/K \cong \mathbb{H}^{n-1}$.

$O_f(\mathbb{Z})$ is an arithmetic lattice.

Let $R_f(\mathbb{Z})$ be the subgroup of $O_F(\mathbb{Z})$ which is generated by proper reflections in $\mathbb{H}^{n-1}$.

- (Vinberg - Nikulin) Except for finitely many f's (and $n \geq 3$ varying) $R_f(\mathbb{Z})$ is infinite index in $O_f(\mathbb{Z})$, hence "thin".

(4). Monodromy Groups:

Perhaps the most interesting way to get in $GL_n(\mathbb{Z})$ which are given as generated by some elements - is as monodromy of families of varieties or differential equations

Already the simplest hypergeometric equation is mysterious:

$${}_nF_{n-1}(z, \dots)$$

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n, \quad 0 \le \alpha_j < 1$$

$$\beta = (\beta_1, \dots \beta_n) \in \mathbb{Q}^n, \quad 0 \le \beta_j < 1$$

$$\alpha_j \neq \beta_k$$

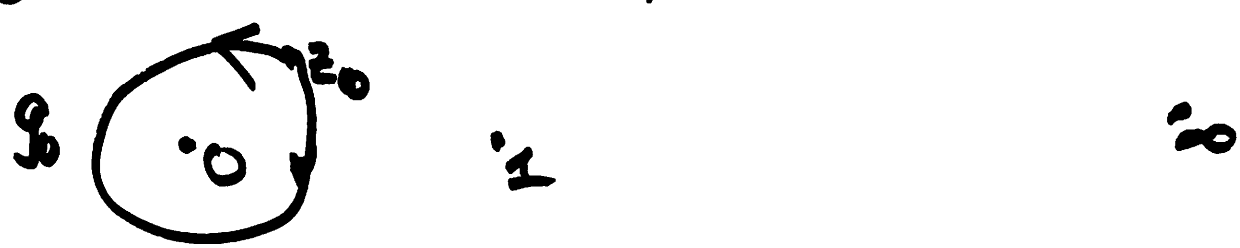
$$D = (\theta + \beta_1 - 1)(\theta + \beta_2 - 1) \cdots (\theta + \beta_n - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_n),$$

$$\theta = z \frac{d}{dz}$$

solutions of $\quad Du = 0 \quad$ n-th order linear o.d.e.

singular at $\{0, 1, \infty\}$

$$S = \mathbb{P} \setminus \{0, 1, \infty\},$$



analytic continuation of solutions gives a representation $\rho : \pi_1(S) \to$ solution near z_0.

The image of ρ is denoted $H(\alpha,\beta)$ and is the hypergeometric monodromy. We assume that it can be conjugated to lie in $GL_n(\mathbb{Z})$.

$H(\alpha,\beta)$ is generated by $\rho(g_0), \rho(g_\infty)$.

Beukers-Heckman determine the Zariski closure of $H(\alpha,\beta)$ and in particular when it is finite.

When is it Thin?

$H(\alpha,\beta) \subset GL_n(\mathbb{Z})$ and infinite

$$\Rightarrow \quad Zcl(H(\alpha,\beta)) = \begin{cases} Sp(n) \\ O(n). \end{cases}$$

Based on numerical experiments (FUCHS-RIVIN) it appears that all q these ~~are~~ except a finite no' are thin!

But it is difficult to show (prove) that they are.

<u>EXAMPLES:</u> DWORK / CALABI-YAU FAMILIES

n even

$\alpha = (0, 0, \ldots 0)$, $\beta = \left(\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}\right)$

$H = Sp(n)$.

(These we used by Taylor et al in their proof q Sato-Tate - the key point being used being strong approximation) for $n=2$, H is commensurable with $\overline{SL_2(\mathbb{Z})}$.

$n=4$ (also considered by Candelas et al ...)

$H(\alpha,\beta)$ is generated by

$$\begin{bmatrix} -9 & -3 & 5 & 3 \\ 0 & 1 & 0 & -1 \\ -20 & -5 & 11 & 5 \\ -15 & 5 & 8 & -4 \end{bmatrix}, \quad \begin{bmatrix} 51 & 90 & -25 & 0 \\ 0 & 1 & 0 & 0 \\ 100 & 175 & -49 & 0 \\ -75 & -125 & 35 & 1 \end{bmatrix}$$

$H(\alpha,\beta) \leq Sp(4,\mathbb{Z})$, is H thin? seems hard!

$n \geq 6$?

THERE IS AN INFINITE FAMILY OF ARITHMETIC SYMPLECTIC HYPERGEOMETRICS

VENKATARAMANA: $n \geq 4$, n even

$\alpha = \left(\frac{1}{2}+\frac{1}{n+1}, \frac{1}{2}+\frac{2}{n+1}, \dots, \frac{1}{2}+\frac{n}{n+1}\right)$, $\beta = \left(0, \frac{1}{2}+\frac{1}{n}, \frac{1}{2}+\frac{2}{n}, \dots, \frac{1}{2}+\frac{n-1}{n}\right)$

$H(\alpha,\beta) \subset Sp(n,\mathbb{Z})$ and H is arithmetic.

For hyperbolic hypergeometrics, i.e. $G = \mathrm{Zcl}(H) = O(n)$ and over $\mathbb{R}$ G has signature $(n-1,1)$ then it seems only finitely many are arithmetic.

One such family is, n odd

$\alpha = \left(0, \frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)}, \dots, \frac{n}{n+1}\right)$

$\beta = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{1}{2}\right)$

THEOREM (FUCHS, MEIRI, S): FOR $n \geq 5$, $H(\alpha,\beta)$ IS THIN.

Some references:

0). A. Lubotzky "Expander graphs in pure and applied Math"

1). S. Hoory - N. Linial - A. Wigderson "Expanders" BAMS 43 (2006) 439-561

2). P. Sarnak "Integral Apollonian Packing" Amer. Math. Monthly 118, (2011) 291-307

3). J. Bourgain - A. Gamburd - P. Sarnak Invent. Math. 179 (2010) 559-644

4). H. Helfgott Ann. Math. 167 (2008) 601-623

5). L. Pyber and E. Szabo "Growth in finite simple groups of Lie type" ArX. 1001.4556

6). E. Breuillard - B. Green - T. Tao "Linear approximate groups" ArX. 1006.3365

7). P. Varju "Expansion in $SL_d(\mathcal{O}/I)$" ArX. 1001.3664

8). A. Salehi and P. Varju "Expansion in perfect groups" preprint 2011

9) J. Bourgain and A. Kontorovich, ArX. 1103.0422.

only game in town, is something you can play if you can choose the players but it is hard to play if someone chooses them for you.

REFERENCES

[Gr] B. Green, "What is an approximate group" to appear in Notices of AMS.

[Lu 1] A. Lubotzky BAMS 49 (2012), 113–162

[P-R] V. Platonov and A. Rapinchuk "Algebraic groups and Number Theory" AP 1991

[H-L-W] S. Hoory, N. Linial and A. Wigderson BAMS 43 (2006) 439–561

[Se] A. Selberg 1965 Proc. Symp Pure Math Vol VIII, 1–15

[B-S] M. Burger and P. Sarnak Inv. Math 106 (1991) 1–11.

[Cl] L. Clozel Inv. Math. 151 (2003) 297–328.

[Sa 1] P. Sarnak "Notes on the generalized Ramanujan Conjectures" publications ias.edu/sarnak

[Lu 2] A. Lubotzky "What is property tau" Not. A.M.S 52 (2005) 626–627

[M-V-W] C. Matthews, L. Vaserstein and B. Weisfeiler Proc. L.M.S 48 (1984) 514–532

[N 1] M. Nori Invent. Math. 88 (1987) 257–275.

[L-Pi] M. Larsen and Pink "Finite subgroups of algebraic groups" 1995

[S-X] P. Sarnak and Xue, DMJ 64 (1991) 207–227

[Ga] A. Gamburd ~~Israel~~ Math. Jnl 127 (2002) 157–200

[He] H. Helfgott Ann Math. 167 (2008) 601–623

[B-G] J. Bourgain and A. Gamburd, Ann Math 167 (2008) 625–642.

[B-G-S 1] J. Bourgain, A. Gamburd and P. Sarnak, Inv. Math. 179 (2010) 559–644

[P-Sz] L. Pyber, E. Szabo "Growth in finite simple groups" ArXiv 1001.4556

[B-G-T] E. Breuillard, B. Green and T. Tao, "Approximate subgroups of linear groups" ArXiv 1005.1881

[V] P. Varju "Expansion in $SL_d(\mathcal{O}_K/I)$, I-square free" ArX. 1001.3664

[S-V] A. Salehi and P. Varju "Expansion in perfect groups" ArXiv 1108.4900

[Ba-Sz] A. Balog, E. Szemeredi Combinatorica 14, 263–268, 1994

[Go] T. Gowers GEOM FUNCT. ANAL. 11 465–588 (2011)

[S-S] A. Salehi and P. Sarnak "Affine linear sieve" ArXiv 1109.6432

[Sa 2] P. Sarnak Am. Math. Month. 118 (2011) 291–306

[P] S. Patterson Acta Math 136 (1976) 241–273

[Su] D. Sullivan Pub IHES 50 (1979) 171–202.

[B-K-T] J. Bourgain, N. Katz, T. Tao GAFA 14 (2004) 27–57

[L-O] M. Lee and H. Oh, "Effective circle count for Apollonian Packings and closed horospheres" ArXiv 2012.
[L-P] P. Lax and R. Phillips J. Fun. Anal 46 (1982) 280-350
[K-O] A. Kontorovich and H. Oh JAMS 24 (2011) 603-648
[O-S] H. Oh and N. Shah "Equidistribution and counting for orbits of geometrically finite hyperbolic groups" preprint.
[B-G-S 2] J. Bourgain, A. Gamburd, P. Sarnak, ArXiv 0912.5021
[B-K 1] J. Bourgain and A. Kontorovich; in preparation.
[B-K 2] J. Bourgain and A. Kontorovich "On a conjecture of Zaremba" ArXiv 1107.3776
[G-L-M-W-Y] R. Graham, J. Lagarias, C. Mallows, R. Wilks and C. Yan, Jnl Numb. Th. 100 (2003) 1-45.
[F-S] E. Fuchs and K. Sanden Exp Math. Vol 20, 2011, 380-400
[B-F] J. Bourgain and E. Fuchs JAMS 24 (2011) 945-967.
[La] S. Lalley Acta Math 163, 1-55 (1989).
[Ri] I. Rivin D.M.J. 142 (2008) 353-379.
[Ko] E. Kowalski "The large sieve and its applications" CUP 2008
[L-M] A. Lubotzky and C. Meiri "Sieve methods in group theory powers in linear groups ARXIV. 1107-3666
[Z] P. Zograf J. Math. Sc. 36, 1, 106-114 (1994)
[A1] D. Abramovich IMRN 20 (1996) 1005-1011
[Y-Y] P. Yang and S. Yau Ann. Sc. Norm. Sup Pisa 7, 55-63 (1980)
[B-B] V. Blomer and F. Brumley Ann. Math. 174 (2011) 581-605
[A-B-S-W] I. Agol, M. Belolipetsky, P. Storm and K. Whyte, GROUPS GEOM DYN (2008) 481-498.
[L-Y] P. Li and S. Yau Inv. Math 69 (1982) 269-291
[L-M-R] D. Long, C. MacLachlan, A. Reid P.A.M.Q. Vol 2, 2, 1-31 (2006)
[Vi] E. Vinberg TRUDY MOSCOW MAT. OBS. 47 (1984) 68-102
[P] M. Prokhorov Izv. Akad Nauk SSSR 50 (1986) 413-424
[A2] D. Abramovich PHD. Thesis Harvard 1991.
[Fr] G. Frey Israel Jnl of Math (85) 1994, 79-83
[Fa] G. Faltings Ann of Math 133 (1991) 549-576.
[E-H-K] J. Ellenberg, C. Hall and E. Kowalski, "Expander graphs gonality and variations of Galois representations" arXiv. 1008.3675
[La] M. Lackenby, Inv. Math. 164 (2006), 317-359
[L-L-R] D. Long, A. Lubotzky and A. Reid, J. Top. 1 (2008) 152-158
[Mi] K. Mikhailova Dok. Akad Nauk SSSR 119 (1958) 1103-1105
[Rip] E. Rips BLMS 14 (1982) 45-47.
[B-M-S] G. Baumslag, C. Miller and H. Short, BLMS 26 (1994) 97-101

[Mc] C. McMullen, JAMS Vol 16, 4, 857-885 (2003)

[N-S] A. Nevo and P. Sarnak Acta Math 205 (2010) 361-402

[Ti] J. Tits J. Alg. 20: 250-270 (1972).

[B-Ge] E. Breuillard and T. Gelander J. Alg. 26, 448-467 (2003)

[A] R. Aoun "Random subgroups of linear groups are free" ArXiv 1005.3445

[Fu-Ri] E. Fuchs and I. Rivin, in preparation.

[D-M] P. Deligne and G. Mostow, PUBL. IHES 63 (1986) 5-89

[N2] M. Nori C. R. Acad. Sci. I Math. 302 (1986) 71-73

[Ni] V. Nikulin ICM Berkeley 1986 Vol 1, 654-671.

[Sa 3] P. Sarnak "Letter to Jim Davis 2005" publications.ias.edu/sarnak

[B-H] F. Beukers and G. Heckman Inv. Math 95 (1989) 325-354

[H-S-T] M. Harris, N. Shepherd-Baron, R. Taylor, Ann Math. 171 (2010) 779-813

[C-D-G-P] P. Candelas, X de la Ossa, P. Green, L. Parkes, Nuc. Phys B 359 (1991) 21-74

[C-Y-Y]. Y. Chen, Y. Yang and N. Yui J. Reine. Angew Math. 616 (2008) 167-203

[F-M-S] E. Fuchs, C. Meiri and P. Sarnak, in preparation.

Final Note: The gonality of a congruence arithmetic surface being linear in its genus and the Heegard genus of a congruence arithmetic hyperbolic 3-manifold being linear in its volume, as well as the proof that there are only finitely many maximal arithmetic reflection groups, all appeal to the uniform lower bounds for λ_1 for all such manifolds. This follows from what is known towards the Ramanujan Conjectures but it does not follow from the fundamental expansion theorem since the latter applies only to one tower at a time. As far as the general Ramanujan Conjectures some progress has been made since the report [Sa 1]. Namely in [The Endoscopic classification of representations of orthogonal and symplectic groups, J. Arthur 2011] a precise formulation of the Ramanujan Conjecture for these groups is given. Moreover it is shown (assuming the forms fundamental lemma which itself should be a theorem before too long) that these conjectures will follow, if one can prove them for GL_n.