Tensor Models and Melonic Large N Limits

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Talk mostly based on

- IK, G. Tarnopolsky, “Uncolored Random Tensors, Melon Diagrams, and the SYK models,” arXiv:1611.08915
- These slides were prepared in collaboration with Grigory Tarnopolsky.

Spenta

- For over 30 years I have enjoyed a great commonality of research interests with Spenta, and I have been strongly influenced by his work.
- One example is the “double-trace” paper by Das, Dhar, Sengupta and Wadia.
- And, of course, the c=1 matrix model and the two-dimensional black hole.
- Spenta and I share a passion for large N theories.

Large N Limits

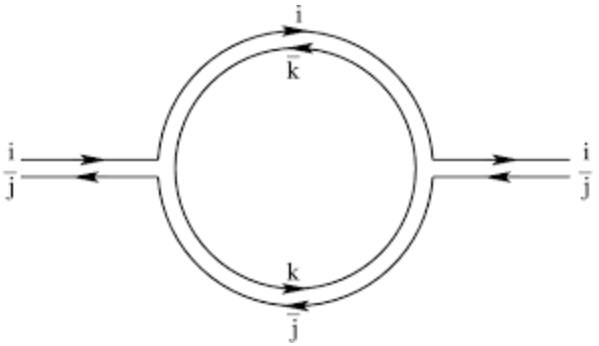
- An important theoretical tool is the study of certain models in the limit of a large number of degrees of freedom.
- One class of such large N limits is for theories where fields transform as vectors under O(N) symmetry with actions like

$$S_{\text{Wilson}-\text{Fisher}} = \int d^d x \left(\frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{g}{4} (\phi^i \phi^i)^2 \right)$$

- The O(N) theory is solvable in the limit where N is sent to infinity while keeping gN fixed.

‘t Hooft Limit and Planar Graphs

- Another famous large N limit is for theories of N x N matrices with single-trace interactions, where only the planar Feynman diagrams survive.
- This has been explored widely in the context of large N QCD: SU(N) gauge theory coupled to matter.
- The ‘t Hooft double line notation is very helpful:



Matrix Quantum Mechanics

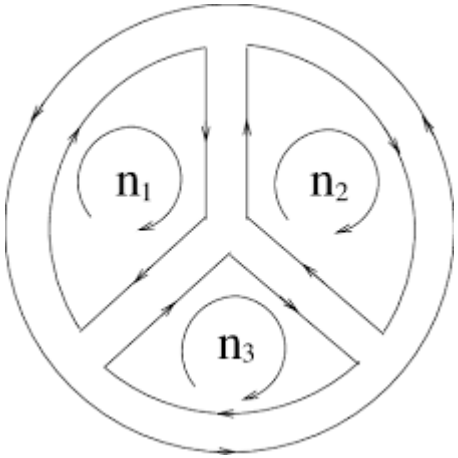
- A well-known solvable model is the QM of a hermitian NxN matrix with SU(N) symmetry

$$\Phi(t) \to V^\dagger \Phi(t) V$$

- The partition function

$$Z \sim \int D^{N^2}\Phi(x) \exp\left[-N \int_{-T/2}^{T/2} dx \ \mathrm{Tr}\left(\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2 + \frac{1}{2\alpha'}\Phi^2 - \frac{\kappa}{3!}\Phi^3\right)\right]$$

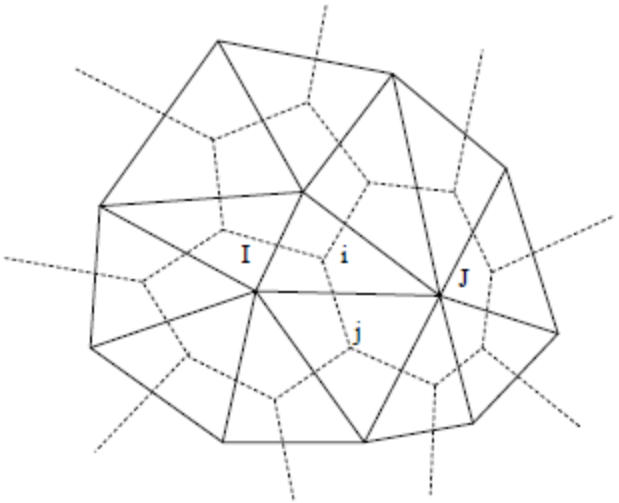
is dominated by planar graphs like



- Reviewed in hep-th/9108019, the 19th paper in arXiv.

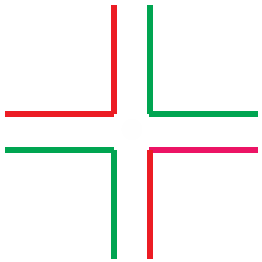
Discretized Random Surfaces

- The dual graphs are made of triangles. The limit where Feynman graphs become large describes two-dimensional quantum gravity coupled to a massless scalar field.
- But the model seems too simple to provide useful information about the two-dimensional black holes, which were discovered by Mandal, Sengupta and Wadia, and by Witten.

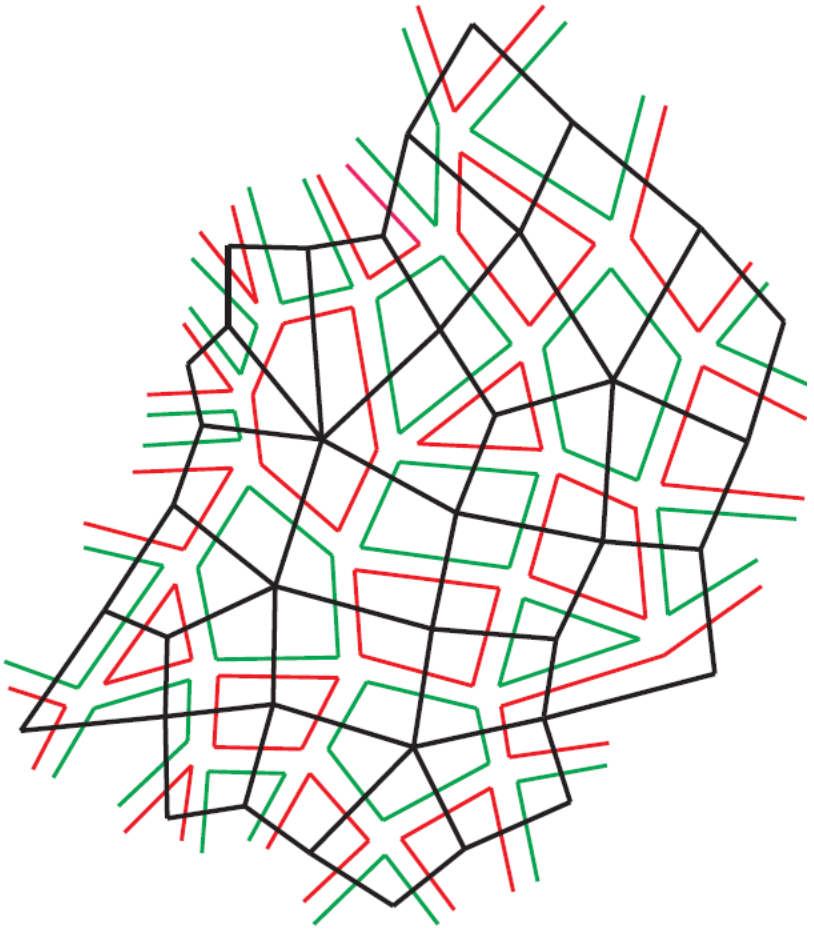


Product Groups

- Another class of matrix models: theories of real matrices M^{ab} with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a \times O(N)_b$ symmetry.
- The interaction must be quartic: g tr MM^TMM^T
- Propagators are represented by colored double lines, and the interaction vertex is



- In the large N limit where gN is held fixed we again find planar Feynman graphs, but now each index loop may be red or green.
- The dual graphs shown in black may be thought of as random surfaces tiled with squares whose vertices have alternating colors (red, green, red, green).



From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

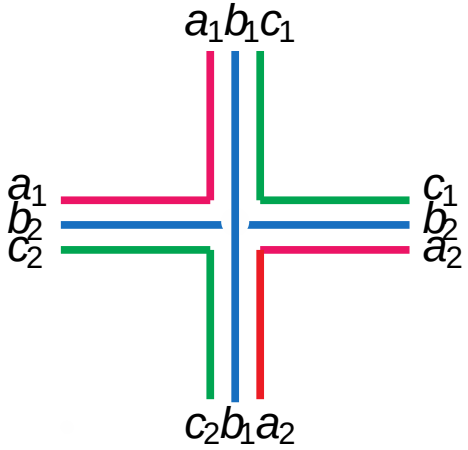
$$\left\langle \phi^{abc} \phi^{a'b'c'} \right\rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires

- An interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

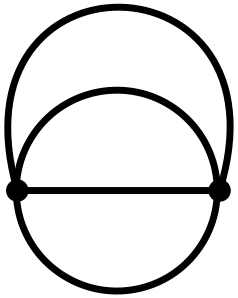
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$

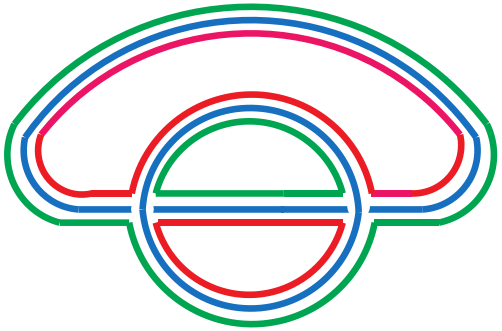
$$\lambda = g N^{3/2}$$



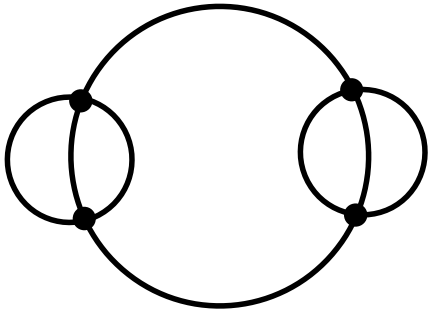
Cables and Wires

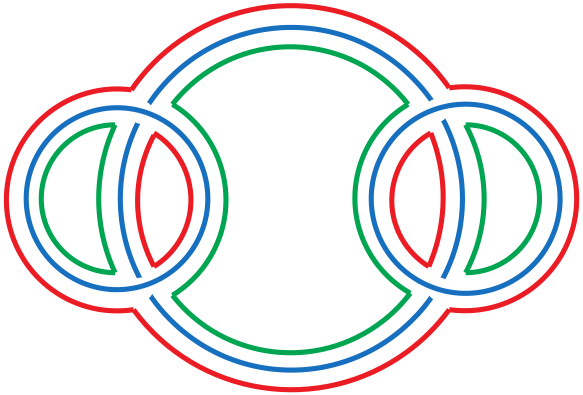
- The Feynman graphs of the quartic field theory may be resolved in terms of the colored wires (triple lines)





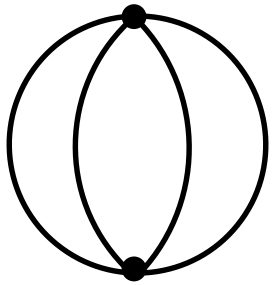
$$g^2 N^6 \sim N^3 \lambda^2$$

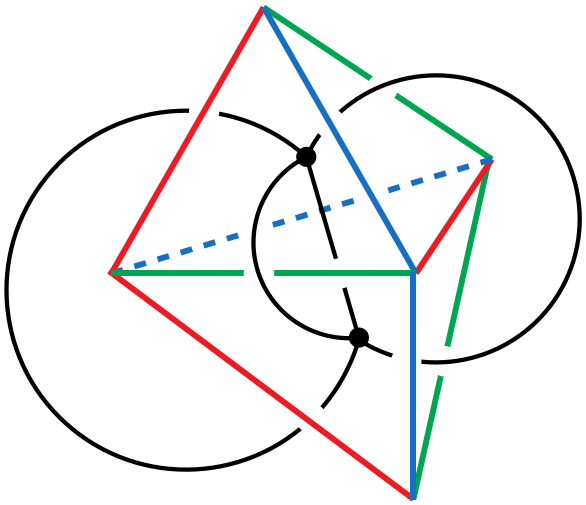


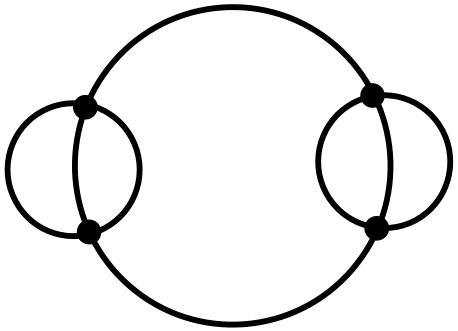


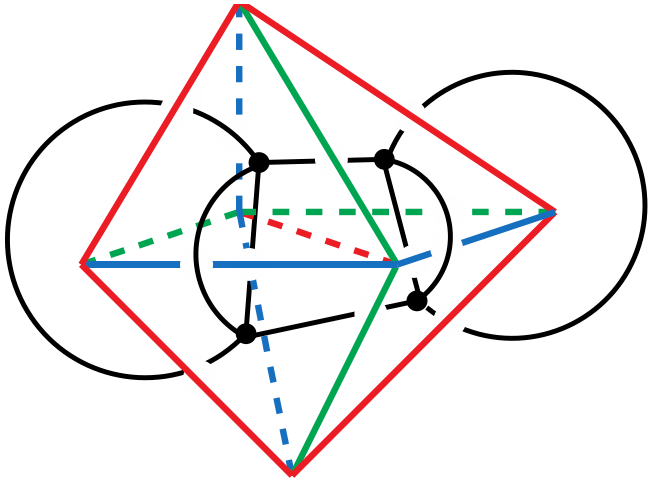
$$g^4 N^9 \sim N^3 \lambda^4$$

- The dual graphs may be represented by tetrahedra glued along the triangular faces. The sides of each triangle have different colors.





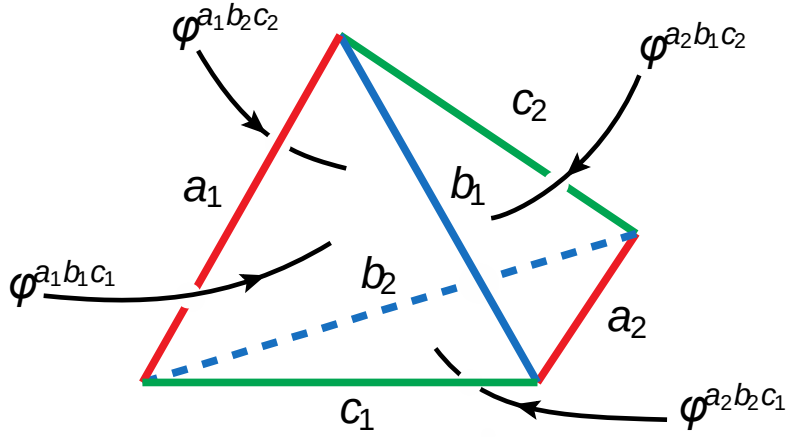




Discretized 3-Geometries

- The study of similar Random Tensor Models was initiated long ago with the goal of generating a class of discretized Euclidean 3-dimensional geometries. Ambjorn, Durhuus, Jonsson; N. Sasakura; M. Gross
- The original models involved 3-index tensors transforming under a single O(N) group. Their large N limit is hard to analyze.
- Since 2009 major progress was achieved by Razvan Gurau and others, who found models with multiple O(N) symmetries which possess a new “melonic” large N limit. Gurau, Rivasseau, Bonzom, Ryan, Tanasa, Carrozza, …

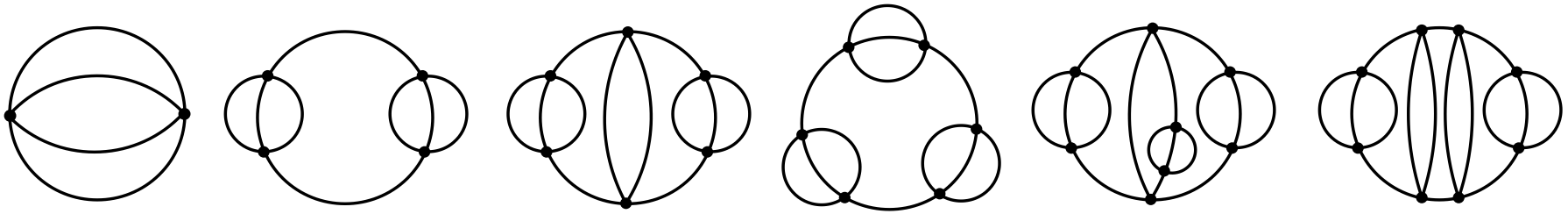
- The 3-geometry interpretation emerges directly is we associate each 3-index tensor with a face of a tetrahedron



- Wick contractions glue a pair of triangles in a special orientation: red to red, blue to blue, green to green.

Melonic Graphs

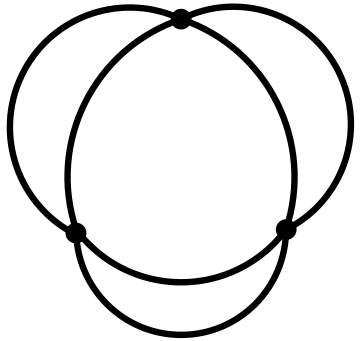
- In some models with multiple O(N) or U(N) symmetries only melon graphs survive in the large N limit where λ is held fixed.

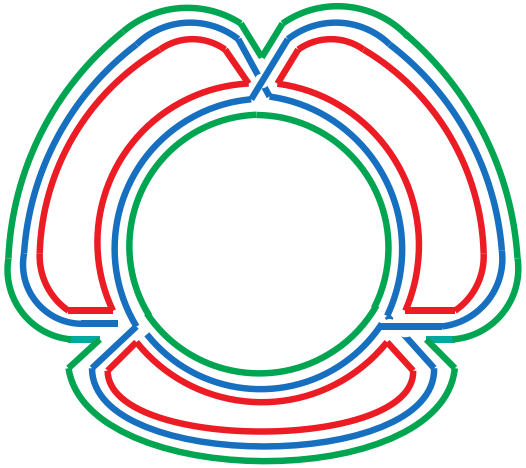


- Remarkably, these graphs may be summed explicitly, so the "melonic" large N limit is exactly solvable!
- The dual structure of glued tetrahedra is dominated by the branched polymers, which is only a tiny subclass of 3-geometries.

Non-Melonic Graphs

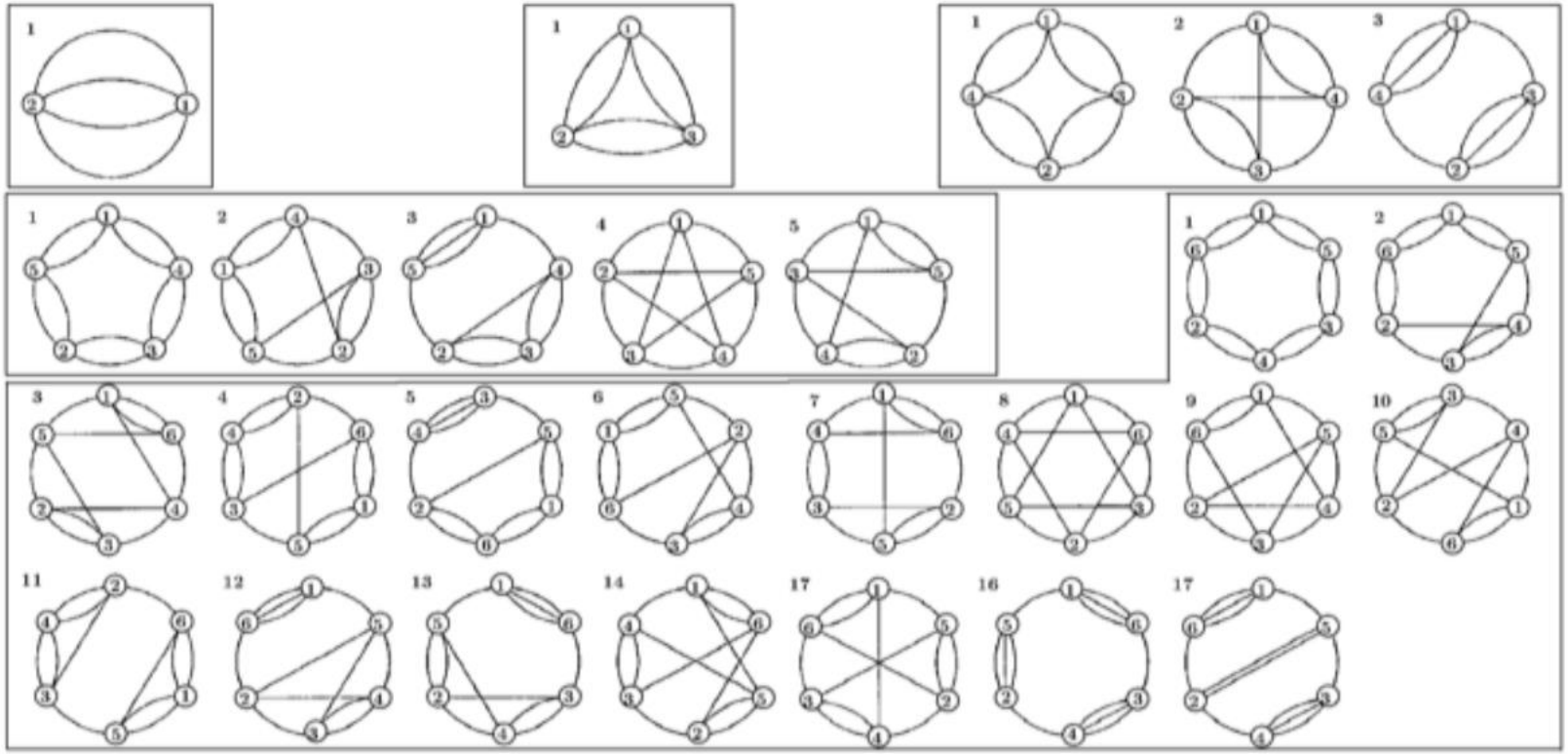
- Most Feynman graphs in the quartic field theory are not melonic are therefore subdominant in the new large N limit, e.g.





- Scales as $g^3 N^6 \sim N^3 \lambda^3 N^{-3/2}$
- None of the graphs with an odd number of vertices are melonic.

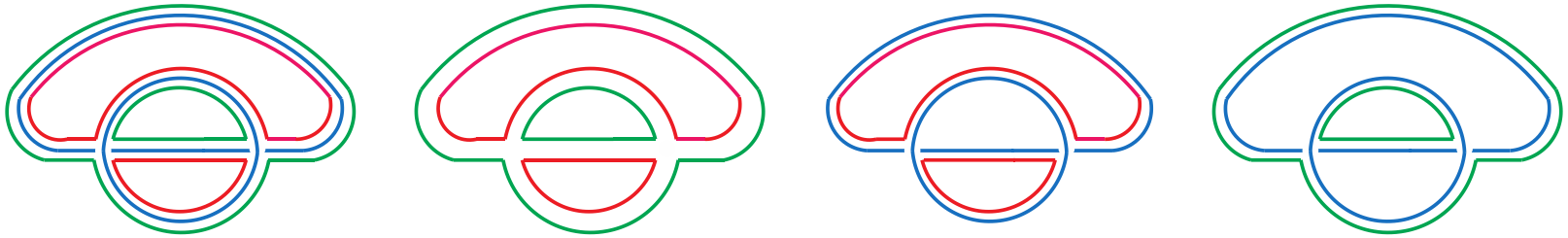
- Here is the list of vacuum graphs up to 6 vertices Kleinert, Schulte-Frohlinde



- Only 4 out of these 27 graphs are melonic.

Large N Scaling

- "Forgetting" one color we get a double-line graph.



- The number of loops in a double-line graph is $f = \chi + e - v$ where χ is the Euler characteristic, e is the number of edges, and v is the number of vertices, $e = 2v$

- If we erase the blue lines we get $f_{rg} = \chi_{rg} + v$

- Adding up such formulas we find

$$f_{bg} + f_{rg} + f_{br} = 2(f_b + f_g + f_r) = \chi_{bg} + \chi_{br} + \chi_{rg} + 3v$$

- The total number of index loops is

$$f_{\text{total}} = f_b + f_g + f_r = \frac{3v}{2} + 3 - g_{bg} - g_{br} - g_{rg}$$

- The genus of a graph is $g = 1 - \chi/2$
- Since $g \geqslant 0$, for a "maximal graph" which dominates at large N all its subgraphs must have genus zero: $f_{\text{total}} = 3 + 3v/2$
- Scales as $N^3(gN^{3/2})^v$
- In the 3-tensor models $\lambda = gN^{3/2}$ must be held fixed in the large N limit.

Digression: M5-Branes

- The number of degrees of freedom on N coincident M5-branes scales as N^3, just as in the 3-tensor models. IK, Tseytlin
- It was speculated that the 3-index objects are M2-branes with 3 boundaries attached to 3 different M5-branes.

The Sachdev-Ye-Kitaev Model

- Recently, another connection of the random tensor models was noted by Witten: the Sachdev-Ye-Kitaev model is also dominated by melon graphs!
- Hamiltonian $H = \sum_{iklm} j_{iklm} \psi_i \psi_k \psi_l \psi_m$
- Random couplings j_{iklm} have a Gaussian distribution with zero mean and a width $\mathcal{J}/N^{3/2}$
- The model flows to strong coupling and becomes nearly conformal. Georges, Parcollet, Sachdev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; …

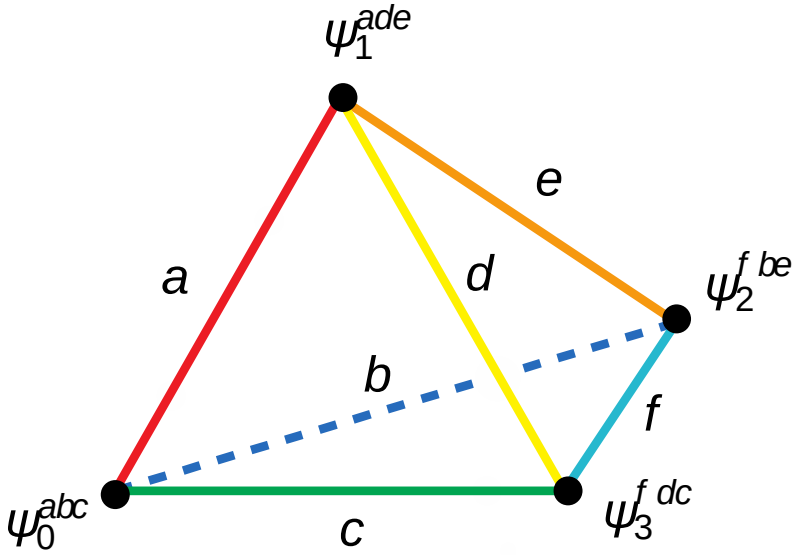
The Gurau-Witten Model

- This model is called “colored” in the random tensor literature because the anticommuting 3-tensor fields ψ_A^{abc} carry a color label A=0,1,2,3.

$$S_{\text{Gurau-Witten}} = \int dt \left(\frac{i}{2} \psi_A^{abc} \partial_t \psi_A^{abc} + g \psi_0^{abc} \psi_1^{ade} \psi_2^{fbe} \psi_3^{fdc} \right)$$

- The model has $O(N)^6$ symmetry with each tensor in a tri-fundamental under a different subset of the six symmetry groups.

- The 4 different fields may be associated with 4 vertices of a tetrahedron, and the 6 edges correspond to the different symmetry groups:



- As stressed by Witten, gauging the symmetry removes the non-singlet states in the QM, removing a crucial conceptual obstacle in the search for a gravity dual.

The Uncolored Model

- Remove the "color label" so that there are N^3 anticommuting components with action

$$S = \int dt \left(\frac{i}{2} \psi^{abc} \partial_t \psi^{abc} + \frac{1}{4} g \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} \right)$$

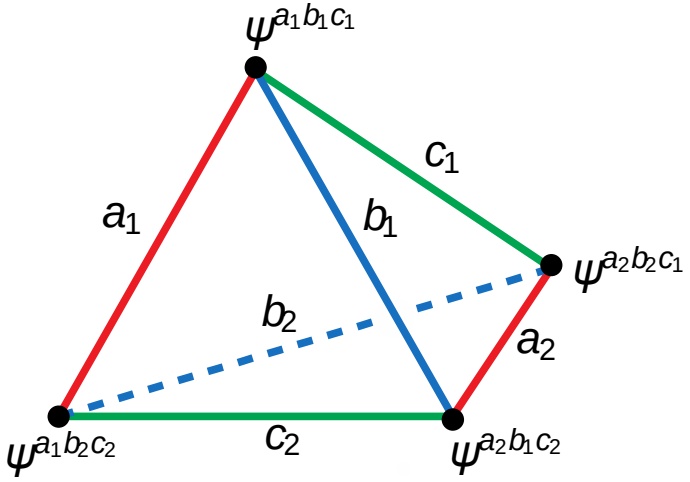
- Has $O(N)^3$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \quad M_1, M_2, M_3 \in O(N)$$

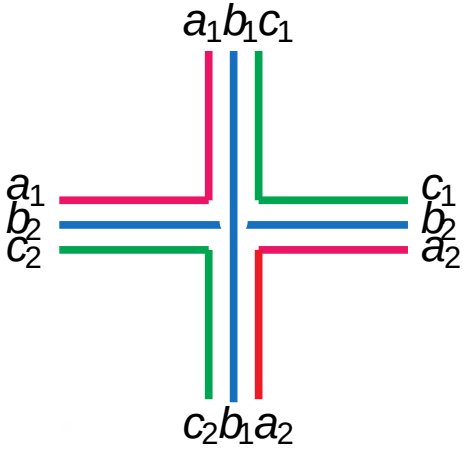
- May be gauged by replacing

$$\partial_t \psi^{abc} \to (D_t \psi)^{abc} = \partial_t \psi^{abc} + A_1^{aa'} \psi^{a'bc} + A_2^{bb'} \psi^{ab'c} + A_3^{cc'} \psi^{abc'}$$

- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.

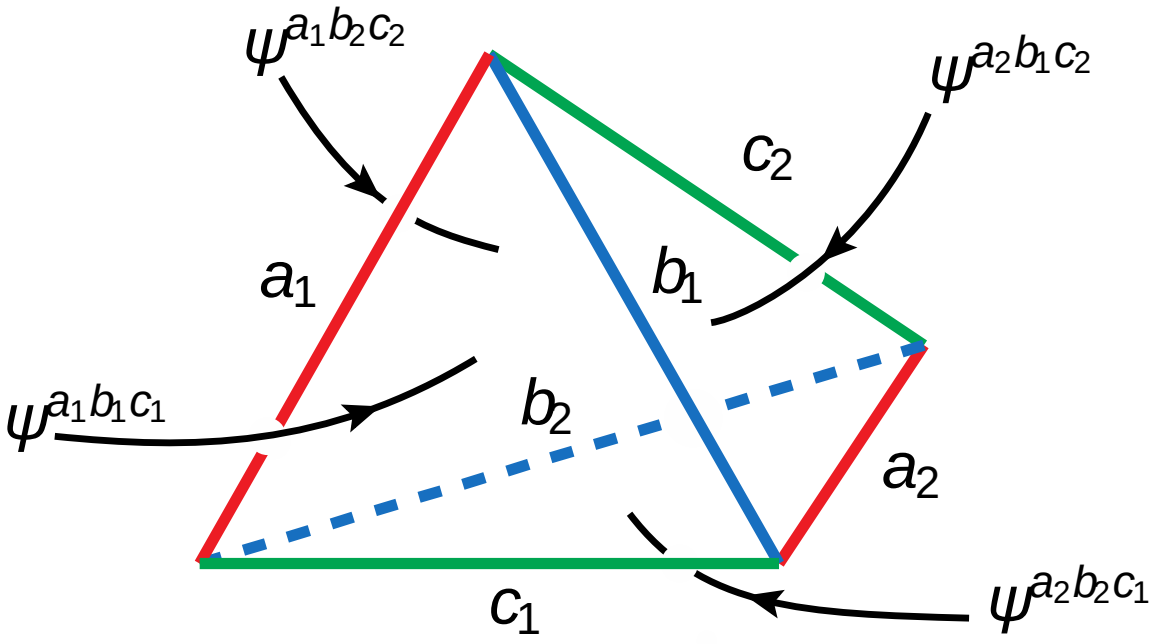


- This is equivalent to



- The 3-line Feynman graphs are produced using the propagator

- Alternatively, can associate the 3-tensors with the faces. This generates the dual graphs made by gluing the tetrahedra along the faces.

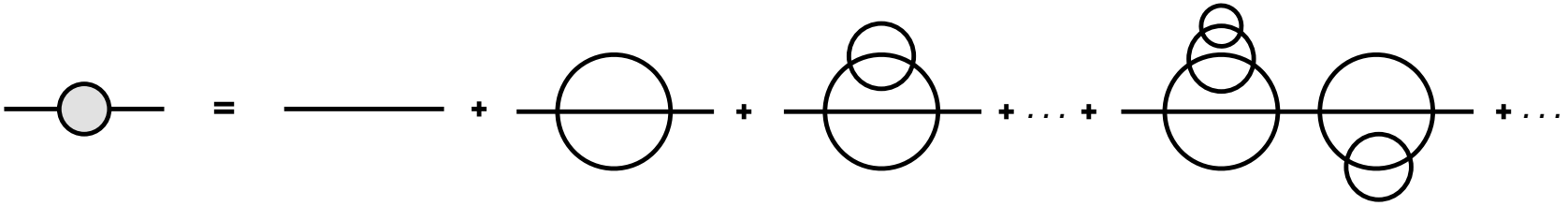


Gauge Invariant Operators

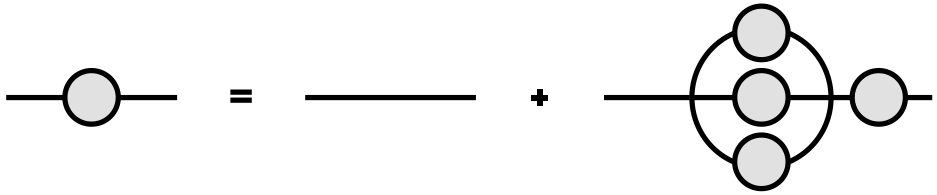
- Two-particle operators, which are analogous to a “single Regge trajectory” $O_2^n = \psi^{abc}(D_t^n \psi)^{abc}$
- n must be odd
- There is also a multitude of gauge invariant multi-particle operators, which may be visualized as glued tetrahedra.
- They should be dual to many additional fields coupled to the dual two-dimensional gravity.

Schwinger-Dyson Equations

- The two-point function obeys the Schwinger-Dyson equation like in SYK model Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; Gross, Rosenhaus



$$G(t_1 - t_2) = G_0(t_1 - t_2) + g^2 N^3 \int dt dt' G_0(t_1 - t) G(t - t')^3 G(t' - t_2)$$

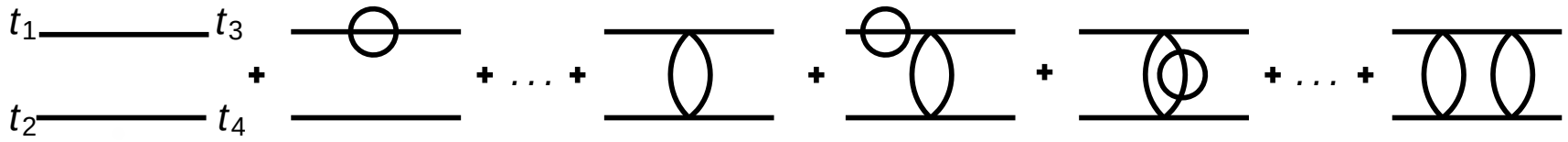


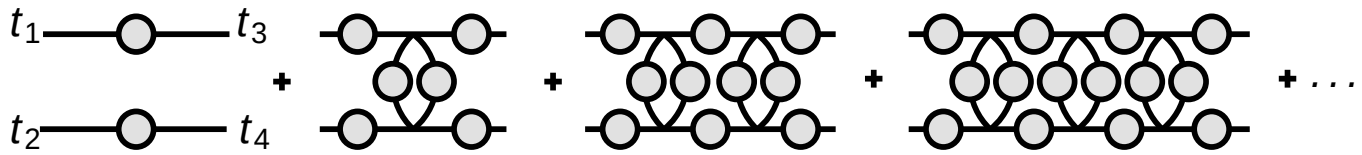
- Neglecting the bare term in IR we find

$$G(t_1 - t_2) = -\left(\frac{1}{4\pi g^2 N^3}\right)^{1/4} \frac{\mathrm{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2}}$$

- Four point function

$$\langle \psi^{a_1 b_1 c_1}(t_1)\psi^{a_1 b_1 c_1}(t_2)\psi^{a_2 b_2 c_2}(t_3)\psi^{a_2 b_2 c_2}(t_4)\rangle = N^6 G(t_{12})G(t_{34}) + \Gamma(t_1,\ldots,t_4)$$





- If we denote by Γ_n the ladder with n rungs

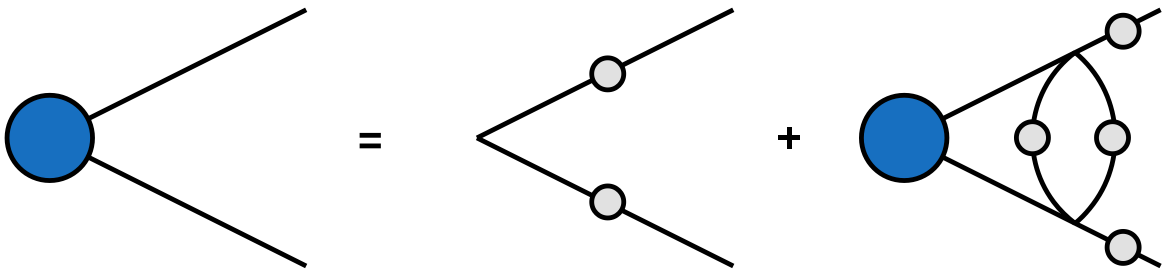
$$\Gamma = \sum_n \Gamma_n$$

$$\Gamma_{n+1}(t_1,\ldots,t_4) = \int dt dt' K(t_1,t_2;t,t')\Gamma_n(t,t',t_3,t_4)$$

$$K(t_1,t_2;t_3,t_4) = -3g^2 N^3 G(t_{13})G(t_{24})G(t_{34})^2$$

Spectrum of two-particle operators

- S-D equation for the three-point function Gross, Rosenhaus



$$v(t_0,t_1,t_2) = g(h)\int dt_3 dt_4 K(t_1,t_2;t_3,t_4)v(t_0,t_3,t_4)$$

$$v(t_0,t_1,t_2) = \langle O_2^n(t_0)\psi^{abc}(t_1)\psi^{abc}(t_2)\rangle = \frac{\mathrm{sgn}(t_1-t_2)}{|t_0-t_1|^h|t_0-t_2|^h|t_1-t_2|^{1/2-h}}$$

- Scaling dimensions determined by $g(h) = 1$

- Can use SL(2) invariance to take t_0 to infinity and consider eigenfunctions of the form

$$v(t_1,t_2)=\frac{\mathrm{sgn}(t_1-t_2)}{|t_1-t_2|^{1/2-h}}$$

- Two basic integrals

$$\int_{-\infty}^{+\infty}du\frac{\mathrm{sgn}(u-t_1)\,\mathrm{sgn}(u-t_2)}{|u-t_1|^a|u-t_2|^b}=l^+_{a,b}\frac{1}{|t_1-t_2|^{a+b-1}},$$

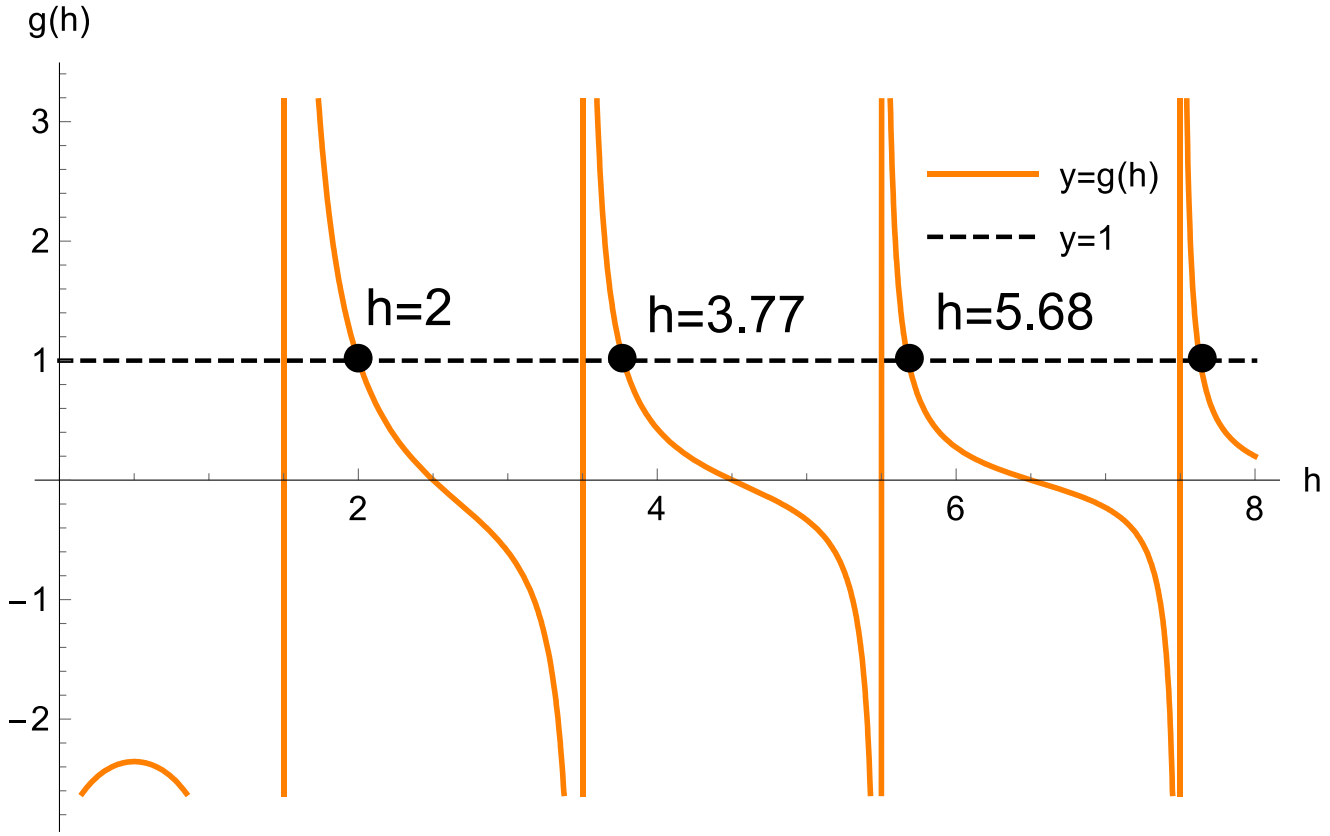
$$\int_{-\infty}^{+\infty}du\frac{\mathrm{sgn}(u-t_2)}{|u-t_1|^a|u-t_2|^b}=l^-_{a,b}\frac{\mathrm{sgn}(t_1-t_2)}{|t_1-t_2|^{a+b-1}},$$

$$l^\pm_{a,b}=\beta(1-a,a+b-1)\pm(\beta(1-b,a+b-1)-\beta(1-a,1-b))$$

- Find the result

$$g(h)=-\frac{3}{4\pi}l^+_{\frac{3}{2}-h,\frac{1}{2}}l^-_{1-h,\frac{1}{2}}=-\frac{3}{2}\frac{\tan(\frac{\pi}{2}(h-\frac{1}{2}))}{h-1/2}$$

- The first solution is h=2; dual to gravity.



- The higher scaling dimensions are

$h \approx 3.77,\ 5.68,\ 7.63,\ 9.60$ approaching $h_n \to n + \frac{1}{2}$

Model with a Complex Fermion

- The action

$$S = \int dt \left(i\bar{\psi}^{abc}\partial_t \psi^{abc} + \frac{1}{4} g \psi^{a_1 b_1 c_1} \bar{\psi}^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \bar{\psi}^{a_2 b_2 c_1} \right)$$

has enhanced $U(N) \times O(N) \times U(N)$ symmetry

- Gauge invariant two-particle operators

$$\mathcal{O}_2^n = \bar{\psi}^{abc} (D_t^n \psi)^{abc} \qquad n = 0, 1, \ldots$$

including $\bar{\psi}^{abc}\psi^{abc}$

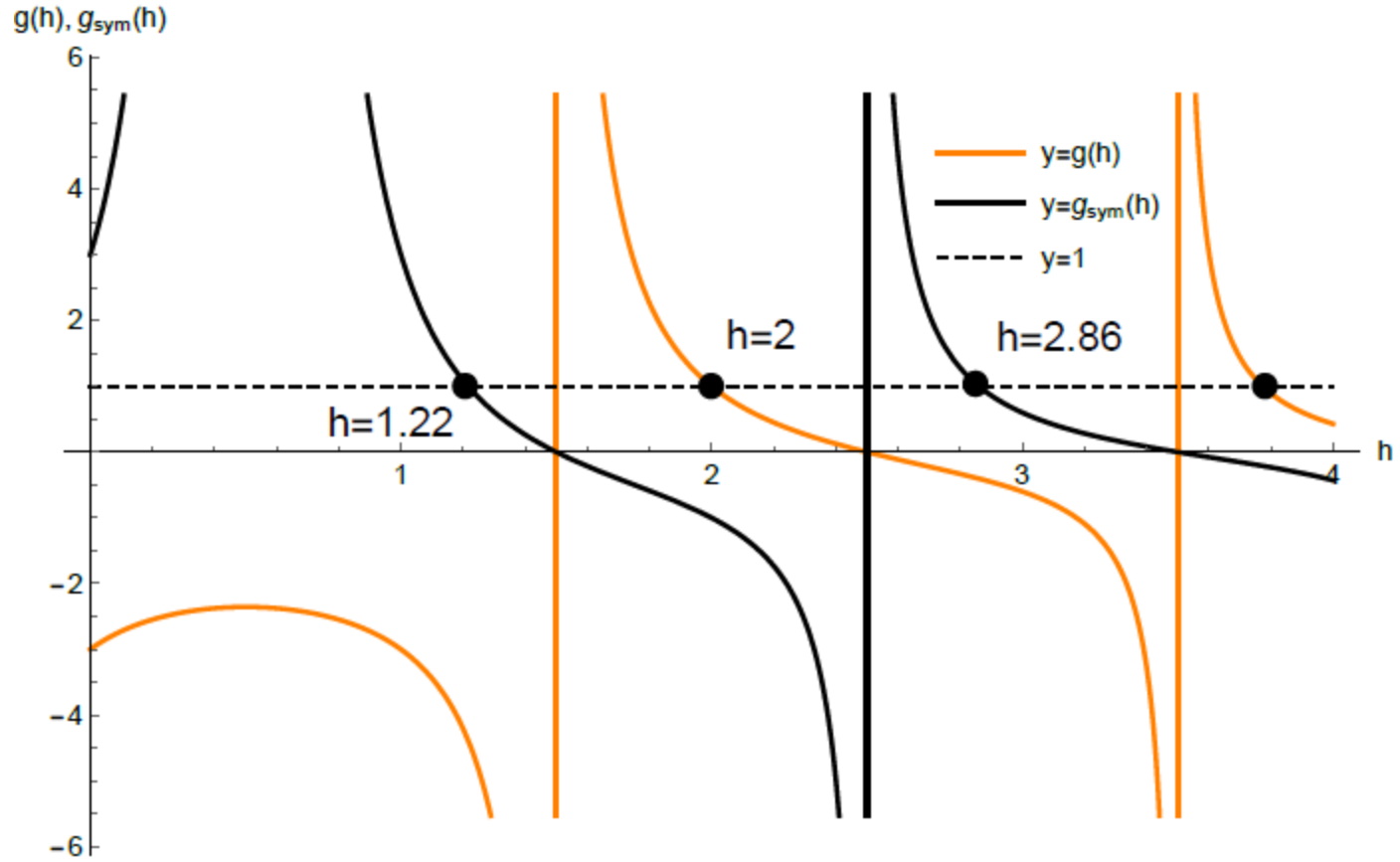
Spectrum of two-particle operators

- The same integral equation, but it also admits symmetric solutions

$$v(t_1, t_2) = \frac{1}{|t_1 - t_2|^{1/2-h}}$$

- Calculating the integrals we get

$$g_{\text{sym}}(h) = -\frac{3}{4\pi} l^{-}_{\frac{3}{2}-h,\frac{1}{2}} l^{+}_{1-h,\frac{1}{2}} = -\frac{3}{2} \frac{\tan\left(\frac{\pi}{2}\left(h+\frac{1}{2}\right)\right)}{h-1/2}$$



- The additional anomalous dimensions are

$$h \approx 1.22,\ \ 2.86,\ \ 4.72,\ \ 6.65,\ \ 8.62$$

approach $$h_n = n + \frac{1}{2} + \frac{3}{\pi n} + \mathcal{O}(n^{-3})$$

Bosonic Tensor Model in general d

- Action with a potential that is not positive definite

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc} + \frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1} \right)$$

- Schwinger-Dyson equation for 2pt function

$$G^{-1}(p) = -\lambda^2 \int \frac{d^d k d^d q}{(2\pi)^{2d}} G(q) G(k) G(p+q+k)$$

- Has solution

$$G(p) = \lambda^{-1/2} \left(\frac{(4\pi)^d d \Gamma(\frac{3d}{4})}{4\Gamma(1-\frac{d}{4})} \right)^{1/4} \frac{1}{(p^2)^{\frac{d}{4}}}$$

Spectrum of two-particle spin zero operators

- Schwinger-Dyson equation

$$\int d^d x_3 d^d x_4 K(x_1, x_2; x_3, x_4) v_h(x_3, x_4) = g(h) v_h(x_1, x_2)$$

$$K(x_1, x_2; x_3, x_4) = 3\lambda^2 G(x_{13}) G(x_{24}) G(x_{34})^2$$

$$v_h(x_1, x_2) = \frac{1}{\left[(x_1 - x_2)^2\right]^{\frac{1}{2}(\frac{d}{2} - h)}}$$

$$g_{\text{bos}}(h) = -\frac{3\Gamma\left(\frac{3d}{4}\right)\Gamma\left(\frac{d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{h}{2} - \frac{d}{4}\right)}{\Gamma\left(-\frac{d}{4}\right)\Gamma\left(\frac{3d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{d}{4} + \frac{h}{2}\right)}$$

- Spectrum in d=1 again includes scaling dimension h=2, suggesting the existence of a gravity dual.
- However, the leading solution is complex, which suggests that the large N CFT is unstable Giombi, IK, Tarnopolsky (work in progress)
- Roughly, it corresponds to the operator

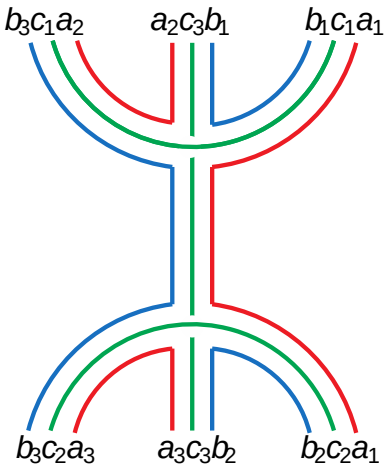
$$\phi^{abc}\phi^{abc}$$

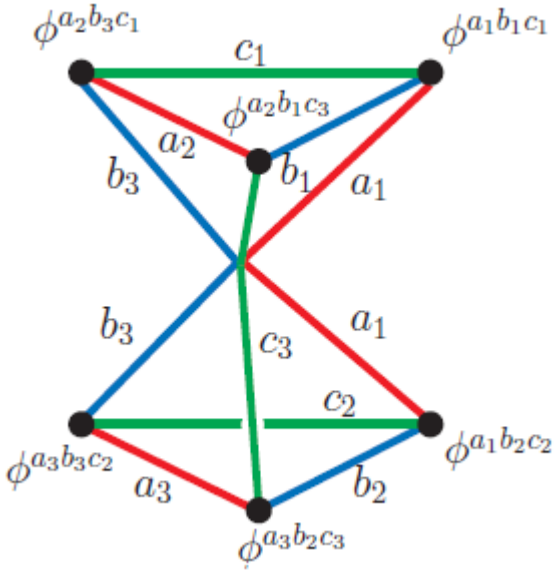
- The dual scalar field in AdS violates the Breitenlohner-Freedman bound.

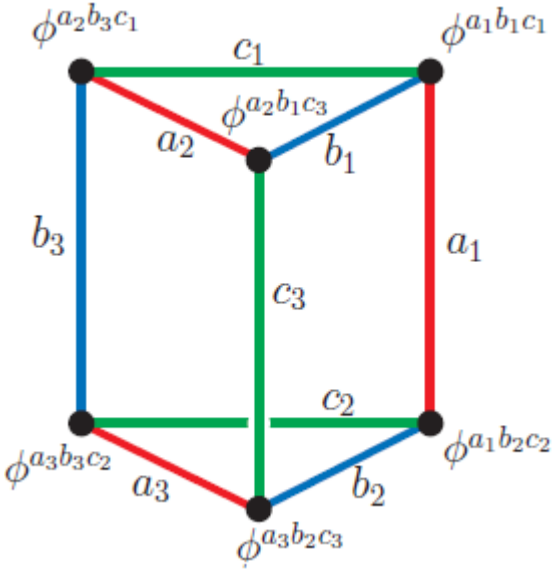
Gauge Invariant Operators

- Different representations of the sextic operator

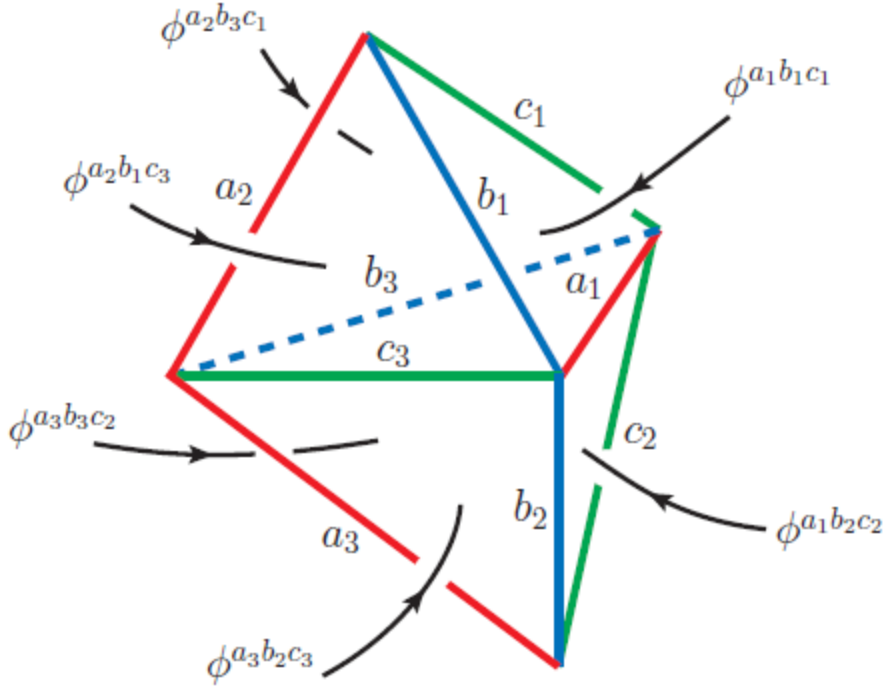
$$\phi^{a_1 b_1 c_1}\,\phi^{a_1 b_2 c_2}\,\phi^{a_2 b_1 c_3}\,\phi^{a_2 b_3 c_1}\,\phi^{a_3 b_2 c_3}\,\phi^{a_3 b_3 c_2}$$







- In the dual representation the operators are described by tetrahedra glued along the faces.
- For example, the sextic operator is the triangular bipyramid.
- Dual to the prism



Super Melons

- May consider a supersymmetric model with "tetrahedron superpotential"

$$W = \frac{1}{4} g \Phi^{a_1 b_1 c_1} \Phi^{a_1 b_2 c_2} \Phi^{a_2 b_1 c_2} \Phi^{a_2 b_2 c_1}$$

- For example, in SUSY Quantum Mechanics

$$\Phi^{a_1 b_1 c_1}(t, \theta, \bar{\theta}) = \phi^{a_1 b_1 c_1}(t) + i\theta \psi^{a_1 b_1 c_1}(t) - i\bar{\psi}^{a_1 b_1 c_1}(t)\bar{\theta} + \bar{\theta}\theta F^{a_1 b_1 c_1}(t)$$

$$S = \int dt d\theta d\bar{\theta} \left(\frac{1}{2} |D_\theta \Phi^{abc}|^2 + W \right)$$

- Includes a positive sextic scalar potential.

Conclusions

- Early exploration of the Random Tensor Models was to study discretized quantum geometries in d>2.
- For models with multiple symmetry groups, like $O(N)^6$ or $O(N)^3$ new solvable large N limits were later discovered, dominated by the melonic Feynman graphs.
- Very recently noted their connection with the SYK models, quantum mechanics with random coupling.

- Gauging the Random Tensor QM removes the non-singlet states.
- Leaves a rich spectrum of gauge invariant operators including the h=2 operator dual to gravity.
- The dual of the model, in the usual sense of gauge/gravity duality, should be similar to a black hole in two-dimensional dilaton gravity.
- The Random Tensor Quantum Mechanics will hopefully teach us a lot more about the quantum physics of black holes.