SPECTRAL SUM RULES FOR

CONFORMAL FIELD THEORIES

IN GENERAL DIMENSIONS

with Subham Dutta Choudhury and Shiroman Prakash

arXiv: 1612.00609

I was a student of Spenta (1994-1999) during the time of the 2nd string revolution.

It was an very exciting time for most of us here.

But for me, these times were very confusing and even at times quite depressing.

The seminars at TIFR were all a buzz with:

- p-branes
- D-branes
- M-Theory
- F-Theory
- Matrix-theory (there were 2 of them !)
- AdS/ (slash ?) CFT

Together with this ‘nomenclature’ we were supposed to digest the fact that all strings theories are related.

We were also supposed to be like the ‘blind’ given access to only parts of the ‘elephant’.

All these developments led to a huge amount of literature which appeared on the xxx.lanl listing of abstracts everyday.

To an average graduate student this is a horrible night mare.

It was in this situation working with Spenta helped me overcome the crisis of studying superflous literature.

Spenta's advice was simple, easy to follow and has helped me ever since.

1. Focus on first principles and a basic understanding.
2. The problem should be physically relevant.
3. Find a good working model, for your question.
4. Extract the essential physics from the working model.

This advice sounds like any reasonable advisor would give to his student.
However Spenta did not give this explicitly to me.

He used to give me snippets about:

- Yukawa postulating the existence of a ‘particle’ (material) which mediates the nuclear force.
- About the Sakata model (precursor to the quark model)
- About how there is an emphasis on the ‘material’ in the method of discovering laws of nature.
- And all this being tied to Marxist philosophy.

Now, if Spenta had given the advice directly,
I would not have cared very much.

(Very recently, I realized that there is a philosphy of science, the three-stage methodology of science developed by Sakata and Taketani which emphasizes 'materialistic dialectics'.)

Since the 'advice' was cryptic to follow
it made me curious,
I kept looking for guidelines in Spenta's work.

I learnt the importance of obtaining a working model (material) from the following paper.

ELSEVIER

7 November 1996

Physics Letters B 388 (1996) 51–59

PHYSICS LETTERS B

Absorption versus decay of black holes in string theory and T-symmetry

Avinash Dhar [1], Gautam Mandal [2], Spenta R. Wadia [3]

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

Received 9 June 1996
Editor: L. Alvarez-Gaumé

Abstract

Classically a black hole can absorb but not emit energy. We discuss how this T-asymmetric property of black holes arises in the recently proposed (T-symmetric) microscopic models of black holes based on bound states of D-branes. In these string theory based models, the nonvanishing classical absorption is made possible essentially by the exponentially increasing degeneracy of quantum states with mass of the black hole. The classical limit of the absorption cross section computed in the microscopic model agrees with the result obtained from a classical analysis of a wave propagating in the background metric of the corresponding black hole (upto a numerical factor).

The microscopic model (material) is the D1-D5 system.
(The word model/models occurs 26 times in this paper.)

The description is valid a weak coupling.

However even at weak coupling the paper demonstrates that the microscopic model shows the features of a black hole.

That is they are ‘black’ , with the absorption cross section proportional to that obtained from the black hole description valid at strong coupling.

The detailed study of the properties of the D1/D5 system in light of the AdS/CFT correspondence essentially formed my thesis.
(We fixed the normalisation constant in the absorption cross section by appealing to AdS/CFT)

This model continued/continues to play an important role in many of things I do even till now.

It is still a very rich model (material) for both black hole physics as well as a important example of AdS/CFT with various unsolved problems.

Working with Spenta, through those exciting times taught me how crucial it is choose a good model to address the physical question at hand.

Focussing on the physical question and the model,

eliminates the need to rely too much on literature and

enables independent investigation of the physical phenomenon.

I am going talk about how conformal invariance can be used to constrain spectral densities of CFT's at finite temperature.

It is appropriate at this stage to introduce this by another work of Spenta which I know he cherishes from my student years with him.

PHYSICAL REVIEW D VOLUME 32, NUMBER 10 15 NOVEMBER 1985

Conformal invariance and string theory in compact space: Bosons

Sanjay Jain, R. Shankar, and Spenta R. Wadia
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay-400005, India
(Received 19 February 1985)

The area law of the Nambu-Goto string is generalized to include a solid-angle-type term, which is purely topological in nature. Such a term exists and is unique provided the manifold M in which the string lives satisfies certain topological conditions. This generalization may be useful to maintain conformal invariance in case M is compact. Using methods of Polyakov and Friedan we identify the conformal anomaly coefficient with the central charge of the Virasoro algebra of this string theory. As an illustration we choose M to be a compact Lie group and compute the anomaly coefficient following the work of Knizhnik and Zamolodchikov.

I. INTRODUCTION

Superstring theory in 10 dimensions offers an attractive possibility of unifying all known interactions including gravity.[1] For special gauge groups like SO(32) and $E_8 \times E_8$ the theory is free of gauge and gravitational anomalies.[2] This theory is based on a supersymmetric generalization of the Nambu-Goto string where the action is proportional to the area swept by the string.

However, two questions which are not yet well understood relate to the nature and mechanism of the compactification of string theory,[3] and to its short-distance properties. In this paper we consider only Bose strings and present a natural generalization of the area law of string theory. This generalization may be of significance in a discussion of the above two questions. Our work has been

If we write the monopole term in polar coordinates r,θ,ϕ, it becomes $i(k/2)\int_\Sigma \sin\theta\, d\theta\, d\phi$. The integral over Σ is the solid angle subtended by Σ at the origin. This immediately explains why $S(C)$ is independent of local deformations of Σ. Also, since the total solid angle of a closed surface enclosing the origin is 4π, it is clear that if $e^{-S(C)}$ is to be totally independent of Σ, k must be an integer.

For a relativistic particle the first term in (1) is just the length of C. By adding the monopole term we are adding another "geometrical" term, the solid angle subtended by C at the origin. This term also has the obvious property that it is not sensitive *per se* to the length or spatial extent of the trajectory C. It picks out only the "compact dimensions," for it measures only the angular spread of the trajectory.

This paper discusses the role of a topological solid-angle term to preserve conformal invariance of the world-sheet propagating on compact manifolds.

This, as far as I know is the first paper on

String theory:

from India in the

past

and also emphasises the role of conformal invariance which seems to be in fashion even in the

present.

INTRODUCTION

The sum rules of our interest are of the form

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \rho(\omega) = \langle 1\mathrm{pt\ functions} \rangle$$

where

$$\rho(\omega) = \mathrm{Im}\, G_R(\omega)$$

$G_R(\omega)$ is the retarded Greens function at finite temperature.

• The sum rule relates 2-point functions and 1-point functions. We will see they will have information of the 3-point functions.

• Sum rules result from analyticity of the 2 point functions in the complex ω-plane.

• The sum rule puts constraints on the 2 point function even though one does not know much about it .

This is useful in lattice calculations where one has good estimates of one point functions, but it is difficult to obtain the retarded real time correlator.

• Sum rules involve an integral over the full frequency domain.

The RHS of the sum rule results from contributions at high frequency as well as low frequency hydrodynamic behaviour of the theory.

• We focus on sum rules in conformal field theories in $d > 2$.

An operator universal to CFT's is the stress tensor $T_{\mu\nu}$.

We will look at the sum rule for the retarded correlator

$$G_R(t, \vec{x}) = i\theta(t)\langle[T_{xy}(t, \vec{x}), T_{xy}(0, 0)]\rangle$$

evaluated at finite temperature.

Its Fourier transform is given by

$$G(\omega, 0) = \int d^d x e^{i\omega t} G_R(t, \vec{x})$$

SHEAR SUM RULE FOR CFT'S

• Sum rules result from analyticity of $G(\omega)$ in the ω-plane. They result from the following two properties

1. $G(\omega)$ is holomorphic in the upper half plane, including the real axis.
2. $\lim_{|\omega|\to\infty} G(\omega) = 0$ if $\mathrm{Im}(\omega) \geq 0$

Then using Cauchy's theorem:

$$G(0) = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho(\omega)}{\omega - i\epsilon}.$$

where $\rho(z) = \mathrm{Im}G(z)$.

Let us now consider CFT's at finite temperature.
For simplicity assume no operators of dimension $\Delta \leq d$ other than the stress tensor in the thermal vacuum.

From considering the OPE's of the stress tensor

$$\begin{aligned} \lim_{\omega \to \infty} G(i\omega) &\sim \omega^d f(\omega/\Lambda) + \omega^{(0)} \langle T \rangle + O(\omega^{(-1)}) \\ &\sim \omega^d f(\omega/\Lambda) + \mathcal{F} \end{aligned}$$

Therefore the second assumption to obtain the sum rule is violated.

There is a diverging term together with a finite term $\mathcal{F}$

The diverging term is identical to the one that is obtained for the retarded two point function if the theory is held at zero temperature.

So we consider

$$\hat{\delta}G(\omega) = G(\omega)|_{T\neq 0} - G(\omega)|_{T=0}$$

This removes the divergent term.

To remove the constant we consider

$$\delta G(\omega) = G(\omega)|_{T\neq 0} - G(\omega)|_{T=0} - \mathcal{F}$$

Apply Cauchy theorem and obtain the sum rule on $\delta G(\omega)$.

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \delta\rho(\omega) = \delta G(0)$$

and $\delta\rho(\omega)$ is the difference in the spectral densities

$$\begin{aligned}
\delta\rho(\omega) &= \mathrm{Im}(G(\omega)|_{T\neq 0} - G(\omega)|_{T=0}) \\
&= \rho(\omega)|_{T\neq 0} - \rho(\omega)|_{T=0} \\
\delta G(0) &= \lim_{\omega\to 0} G(\omega)|_{T\neq 0} - \mathcal{F}
\end{aligned}$$

Note the first term can be determined from hydrodynamics. The second term is obtained from the OPE, the high frequency behaviour.

The sum rule is a constraint on spectral density from the conformal invariance of the theory.

• To proceed further we need details of the OPE coefficients of the stress tensor:
Osborn, Petkos (1993)

The tensor structures in the OPE

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle \sim C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^{2d}} + \hat{A}_{\mu\nu\rho\sigma\alpha\beta}(x) \langle T_{\alpha\beta}(0) \rangle + \cdots$$

are known.

The coefficients are determined in terms of 3 parameters (a, b, c).

- To get a feel of these parameters.

Consider the 3 point function of stress tensor in $d = 4$.

The most general tensor structure occuring in the three point function can be written as

$$\langle TTT \rangle = n_s \langle TTT \rangle_{\text{free scalar}} + n_v \langle TTT \rangle_{free\ vectors} + n_f \langle TTT \rangle_{free\ fermions}$$

The tensor structures can be obtained by considering free theories and performing Wick contractions.

The basis of tensor structures used by Osborn and Petkos ($d = 4$) is related by

$$
\begin{aligned}
a &= \frac{1}{27\pi^6}(n_s - 54 n_v), \\
b &= -\frac{1}{54\pi^6}(8n_s + 27 n_f), \\
c &= \frac{1}{27\pi^6}[n_s + (27 n_f + 8 n_v)],
\end{aligned}
$$

Similarly C_T in $d = 4$ is given by

$$
C_T = \frac{\pi^2}{3}(14a - 2b - 5c)
$$

Performing the Fourier transform of the OPE coefficient $\hat{A}_{\mu\nu\rho\sigma\alpha\beta}(x)\langle T_{\alpha\beta}(0)\rangle$ we can obtain the finite term.

$$
\begin{aligned}
\mathcal{F} &= \mathcal{A} + \mathcal{B}, \\
\mathcal{A} &= \sum_{j=1}^{7} \hat{I}_j(\omega, 0), \\
&= \frac{(1-d)(a(d(d+4)-4)+d(2b-c))}{2\left(-a\left(d^2+d-6\right)+2b+cd+c\right)} P \\
\mathcal{B} &= P \frac{((2a+b)(-2+d)-cd)}{-2b-c(1+d)+a\left(-6+d+d^2\right)}
\end{aligned}
$$

$\mathcal{A}$ is the contribution from the tensor structures.

$\mathcal{B}$ is the contribution from the contact term. needed in the OPE to ensure conformal Ward identities.

The coefficient $\mathcal{A}$ obtained from the high frequency limit with out the contact terms coincides with the Hofman-Maldacena coefficient.

$$\sum_{l=1}^{7} \hat{l}_l(\omega, 0) = 2(1-d)Pa_{T,0},$$

$$a_{T,0} = \frac{1}{4}\frac{a(d(d+4)-4)+d(2b-c)}{-a\,(d^2+d-6)+2b+cd+c}.$$

We are now in a position to state the sum rule. We look at the Greens function

$$\delta G(\omega) = G(\omega)|_{T\neq 0} - G(\omega)|_{T=0} - (\mathcal{A} + \mathcal{B})$$

The RHS of the sum rule is given by

$$\begin{aligned}\delta G(0) &= G(0)|_{T\neq 0} - (\mathcal{A} + \mathcal{B}) \\ &= \frac{\left(2c + d(c + 2bd - cd) + a\left(8 + d\left(-6 + d + d^2\right)\right)\right)P}{2(2b + c + cd) - 2a\left(-6 + d + d^2\right)}\end{aligned}$$

We have used

$$G(0)|_{T\neq 0} = P, \qquad G(0)|_{T=0} = 0$$

CHECK FROM HOLOGRAPHY

The coefficients a, b, c has been obtained in AdS_{d+1} by evaluating the 3 point function of the stress tensor holographically.

Arutyunov and Frolov (1999)

$$
\begin{aligned}
a &= -\frac{d^4 \pi^{-d} \Gamma[d]}{4(-1+d)^3} \\
b &= -\frac{d\left(1+(-3+d)d^2\right)\pi^{-d}\Gamma[1+d]}{4(-1+d)^3} \\
c &= \frac{d^3(1-2(-1+d)d)\pi^{-d}\Gamma[d]}{4(-1+d)^3}
\end{aligned}
$$

Evaluating $\mathcal{A}$ and $\mathcal{B}$ we obtain

$$
\mathcal{A} = -\frac{d(d-1)}{2(d+1)} P, \qquad \mathcal{B} = P
$$

Now evaluating

$$\delta G(0) = \epsilon \frac{d}{2(d+1)}$$

Thus the sum rule is given by

$$\int_{-\infty}^{\infty} \frac{\delta\rho(\omega)}{\omega} = \epsilon \frac{d}{2(d+1)}$$

Let us examine an alternate way of deriving the sum rule holographically.

It relies on evaluating the retarded Greens function directly in the AdS_{d+1} black hole background and obtaining the high frequency behaviour.

The retarded shear correlator is obtained by considering the minimially coupled scalar with in-going boundary conditions at the AdS_{d+1} black hole horizon.

From the equation obeyed by this scalar one can obtain a systematic expansion of the high frequency behaviour of the Green's function and evaluate $\delta G_R(0)$.

- Going through the analysis one evaluates

$$\delta G_R(0) = \frac{r_+^d}{2\kappa^2}\frac{d(d-1)}{2(d+1)}.$$

Using

$$P = \frac{r_+^d}{2\kappa^2}.$$

This results in

$$\delta G_R(0) = \frac{d(d-1)}{2(d+1)}P = \frac{d}{2(d+1)}\epsilon.$$

The evaluation of $\delta G_R(0)$ did not rely on the explicit expression given in given in terms of the parameters of the three point function a, b, c

and therefore provides a consistency check for the general sum rule.

Remarks: The shear sum rule was first studied in the context of holography in AdS_5
Romatschke, Son (2009)

The shear sum rule using the direct evaluation of the Greens function for AdS_{d+1} black hole was derived by Gulotta, Herzog, Kaminski (2010)

In the holographic dual one can show that the Green's function is analytic in the upper half ω-plane by studying the differential equations.

Such such rules were generalised in gravity for retarded Greens function of current correlators as well as situations with chemical potentials. Jain, Thakur, JRD (2011), Thakur JRD (2012)

SUM RULE AND

HOFMAN-MALDACENA VARIABLES

• Note that $\delta G_R(0)$ is a ratio of a linear functions of a, b, c. It should be possible to write this in terms of variables t_2, t_4 of Hofman-Maldacena.

$$
\begin{aligned}
t_2 &= \frac{2(1+d)(-d(c-3bd+2cd)+a(-1+d)(4+d(8+d)))}{d\left(-2b-c(1+d)+a\left(-6+d+d^2\right)\right)}, \\
t_4 &= \frac{(1+d)(2+d)\left(d(c-2bd+cd)+3a\left(1+d-2d^2\right)\right)}{d\left(-2b-c(1+d)+a\left(-6+d+d^2\right)\right)}.
\end{aligned}
$$

- In these variables the sum rule assumes the form

$$\delta G_R(0) = \left(\frac{(-1+d)d}{2(1+d)} + \frac{(3-d)t_2}{2(-1+d)} + \frac{\left(2+3d-d^2\right) t_4}{(-1+d)(1+d)^2} \right) P.$$

Einstein gravity in AdS_{d+1} lies at the origin in the $t_2 = t_4 = 0$.

However as expected for $d = 3$, we have only one condition $t_4 = 0$.

There are only 2 independent parity even constants which determine the three point function of the stress tensor in $d = 3$.

• In $d = 4$ the central charges $\hat{a}, \hat{c}$ are coefficients that occur in the trace anomaly of the theory.

A necessary condition, that the CFT admits an Einstein gravity dual is that $\hat{a} = \hat{c}$.

It can be seen that the condition

$$\delta G_R(0) = \frac{d}{2(d+1)}\epsilon$$

is a linearly independent necessary condition for the theory to admit an Einstein gravity dual.

• Implication of the causality bounds on the sum rule: In arbitrary dimensions, these bounds are stated by the inequalities Camanho and Edelstein (2009), Buchel et. al. (2009)

$$
\begin{aligned}
&1-\frac{1}{d-1}t_2-\frac{2}{(d+1)(d-1)}t_4 \geq 0 \\
&1-\frac{1}{d-1}t_2-\frac{2}{(d+1)(d-1)}t_4+\frac{1}{2}t_2 \geq 0 \\
&1-\frac{1}{d-1}t_2-\frac{2}{(d+1)(d-1)}t_4+\frac{d-2}{d-1}(t_2+t_4) \geq 0
\end{aligned}
$$

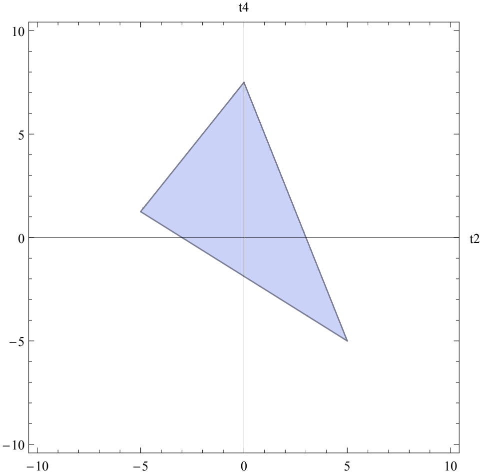

Figure: The allowed domain for t_2 and t_4 in $d = 4$ conformal field theories.

- The sum rule is a linear function of t_2, t_4.

On the allowed domain it is extremized at the vertices. This gives the bounds for $d > 3$

$$\frac{1}{2}P \leq \delta G_R(0) \leq \frac{d}{2}P.$$

For $d = 3$, the first inequality is an equality.

$$t_2 = 2 - \frac{t_4}{2}.$$

The remaining two inequalities reduce to

$$-4 \leq t_4 \leq 4.$$

Therefore

$$\frac{P}{2} \leq \delta G_R(0) \leq P.$$

APPLICATIONS

• For $d = 3$:

Free conformal field theories in $d = 3$ consist of n_s real bosons and n_f Dirac fermions. The contribution of these fields to the constants a, b, c are given by

$$a = \frac{27 n_s}{4096\pi^3}, \qquad b = -\frac{9(8n_f + 9n_s)}{4096\pi^3}, \qquad c = -\frac{9(16n_f + n_s)}{4096\pi^3}.$$

Consider the theory on a single M2-brane which consists of 8 real scalars and 8 real Majorana fermions for which

$$a = \frac{27}{8(4\pi)^3}, \qquad b = \frac{-117}{8(4\pi)^3}, \qquad c = \frac{-81}{8(4\pi)^3}.$$

Substituting this values in the sum rule for $d = 3$ we obtain

$$\delta G_R(0)|_{\mathrm{M2}} = \frac{3P}{4}.$$

This value for the sum rule agrees with that obtained in gravity for $d = 3$

• Similarly for the ABJM theory, the coefficient of the sum rule evaluated at weak coupling agrees with that of gravity.

• For large N Chern-Simons vector theories Maldacena, Zhiboedov (2012) ,
the values of a, b, c are known exactly at large N.
Let

$$\theta = \frac{\pi N}{k}$$

Then

$$\begin{aligned}
a &= \frac{27N\sin^2\frac{\theta}{2}\sin\theta}{32\theta}\frac{1}{(4\pi)^3} \\
b &= \frac{9N(-17+\cos\theta)\sin\theta}{64\theta}\frac{1}{(4\pi)^3} \\
c &= -\frac{9N(17+15\cos\theta)\sin\theta}{64\theta}\frac{1}{(4\pi)^3}
\end{aligned}$$

The sum rule reduces to

$$\delta G(0) \quad = \quad \frac{2P}{(3+\cos\theta)}$$

It is easy to check that $\delta G(0)$ is within the predicted bounds.

The bounds are saturated at $\theta = 0$, free fermions and $\theta = \pi$, free bosons.

$$\delta G(0)|_{\cos\theta=-1/3} = \frac{3P}{4}$$

which is the gravity result.

The theory at this point , effectively consists of n_f free fermions and n_s real scalars with $\frac{n_s}{n_f} = 2$.

In $d = 4$ for $\mathcal{N} = 4$ Yang-Mills the coefficients a, b, c are given by

$$a = -\frac{16}{9\pi^6}(N_c^2 - 1)$$
$$b = -\frac{17}{9\pi^6}(N_c^2 - 1)$$
$$c = -\frac{92}{9\pi^6}(N_c^2 - 1)$$

Substituting these values in the sum rule for $d = 4$ we obtain

$$\delta G_R(0) = \frac{6}{5}P$$

Again, the value at weak coupling agrees with that in gravity.

• For the M5-brane in $d = 6$ we have the $(2, 0)$ tensor multiplet, which is made up of a single self dual tensor, 2 Weyl fermions and 5 real scalars. Using this field content we obtain

$$a = -\frac{2592}{100\pi^9}, \qquad b = -\frac{7848}{100\pi^9}, \qquad c = -\frac{25488}{100\pi^9}.$$

$$\delta G_R(0) = \frac{15P}{7}.$$

Again, this agrees with the result from gravity for $d = 6$

OTHER CHANNELS

- The vector channel:

$$
\begin{aligned}
G_{R;V}(t,x) &= i\theta(t)\langle[T_{xt}, T_{xz}]\rangle, \\
G_{R;V}(\omega,p_z) &= \int d^d x e^{i\omega t - ip_z.z} i\theta(t)\langle[T_{xt}, T_{xz}]\rangle.
\end{aligned}
$$

The high frequency behaviour of this Greens can be obtained by studying the OPE of the stress tensors in these channels.

The Fourier transform of the OPE in this channel has the structure

$$\hat{A}_{x\tau xz}(\omega, p_z) \quad = \quad -\frac{p_z\omega}{(p_z^2+\omega^2)} G_2(\omega, p_z),$$

$$G_2(\omega, p_z) = -\delta G_R(0) + a_{T,1}\frac{8dp_z^2}{\omega^2+p_z^2}P.$$

$\delta G_R(0)$ is the RHS of the shear sum rule.

$a_{T,1}$ is the Hofman-Maldacena coefficient in the vector channel defined as

$$a_{T,1} = \frac{1}{8}\frac{b(2-3d)+2cd-a(-8+d(6+d))}{-(2b+c+cd)+a(-6+d+d^2)}.$$

- The sound channel:

$$
\begin{aligned}
G_{R;S}(t,x) &= i\theta(t)\langle[T_{tt},T_{tt}]\rangle \\
G_{R;S}(\omega,p_z) &= \int d^dx e^{i\omega t - ip_z z} i\theta(t)\langle[T_{tt},T_{tt}]\rangle
\end{aligned}
$$

The Fourier transform of the OPE in this channel admits the expansion

$$
\hat{A}_{tttt\alpha\beta}(\omega,p_z)\langle T_{\alpha\beta}\rangle = \frac{p_z^4}{(p_z^2+\omega^2)^2} G_3(\omega,p_z).
$$

$G_3(\omega, p_z)$ admits an expansion

$$G_3(\omega, p_z) \;=\; \left(F_1\left(\frac{\omega}{p_z}\right)^4 + F_2\left(\frac{\omega}{p_z}\right)^2 + F_3 + a_{T,2}\frac{64\left(\frac{p_z}{\omega}\right)^2}{1+\left(\frac{p_z}{\omega}\right)^2} \right) P.$$

F_1, F_2, F_3 are ratios of linear functions of the constants a, b, c.

Starting from the term $(\frac{p_z}{\omega})^2$ the entire expansion is determined by $a_{T,2}$, the Hofman-Maldacena coefficient in the tensor channel which is given by

$$a_{T,2} = -\frac{1}{32}\frac{(4a+2b-c)(-2+d)d}{(-2b-c(1+d)+a(-6+d+d^2))}.$$

CONCLUSIONS

We have derived the shear sum rule

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega} (\rho(\omega)_{T\neq 0} - \rho(\omega)_{T=0}) = \delta G(0)$$
$$= \left(\frac{(d-1)d}{2(1+d)} + \frac{(3-d)t_2}{2(d-1)} + \frac{(2+3d-d^2)\, t_4}{(d-1)(1+d)^2} \right) P$$

where ρ is the spectral density corresponding to the retarded shear correaltor in conformal field theories.

Assumption: no operator of dimension $\leq d$ acquires expectation value other than the stress tensor.

Causality constraints for $d > 3$ restrict the value $\delta G(0)$ to be within

$$\frac{1}{2}P \leq \delta G(0) \leq \frac{d}{2}P$$

For $d = 3$, the constraints are

$$\frac{1}{2}P \leq \delta G(0) \leq P$$

- It will be interesting to investigate the other channels in detail, especially holographically.

- Interesting and useful constraints on the spectral density can be obtained using conformal invariance and causality.

It will be rewarding to explore this direction further.

During the last 10 years, working for building the ICTS, Spenta complained to me, that he does not have time to think about the hard questions in physics that he loves.

- Conceptual questions regarding black hole physics.
- Understanding thermalization.

Perhaps, now he will have time to contribute towards these issues.

I wish him more momentous times with physics for the

future.