|| Sri Sainath ||

New asymptotic conservation laws for electromagnetism

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Recent developments in S-matrix theory.

Contents

Soft theorems

- When energy of one of the scattering particles is taken to 0, the amplitude factorises into lower point amplitude times a universal 'soft factor'.

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- Leading soft photon theorem :

$$\lim_{\omega\to 0} \mathrm{Amp}_{n+1}(p_i, k) = \Big[\frac{S_0}{\omega} + ...\Big]\mathrm{Amp}_n(p_i).$$

$k = \omega(1, \vec{q})$ is the soft momentum.
n is number of hard particles in the scattering process.
$(n+1)^{th}$ particle is the soft photon.

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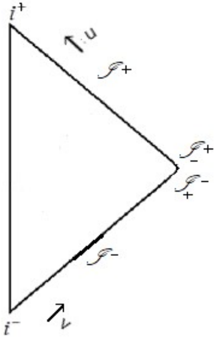
- Soft factors are universal.

$$S_0 = \sum_{i=1}^{n} e_i \frac{\varepsilon . p_i}{p_i . q}.$$

Asymptotic Conservation laws



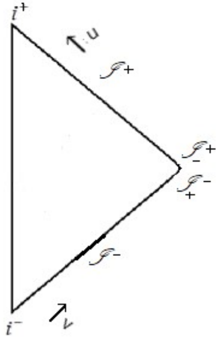
Asymptotic Conservation laws

Asymptotic conservation laws :

$$Q^+[\epsilon^+] \mid_{\mathcal{I}^+_-} \quad = \quad Q^-[\epsilon^-] \mid_{\mathcal{I}^-_+}.$$

$$\epsilon^+(\hat{x}) = \epsilon^-(-\hat{x}).$$

$\mathcal{I}^+_-$ is the $u \to -\infty$ sphere of $\mathcal{I}^+$.
$\mathcal{I}^-_+$ is the $v \to \infty$ sphere of $\mathcal{I}^-$.



Asymptotic Conservation law

- Classical conservation law :

$$Q_0^+[\epsilon^+] \mid_{\mathcal{I}^+_-} \quad = \quad Q_0^-[\epsilon^-] \mid_{\mathcal{I}^-_+}.$$

Q_0 is defined in terms of radial component of electric field.

- At quantum level, S-matrix has to satisfy the Ward identity :

$$< \text{out}| \ Q_0^+ \ S \ - \ S \ Q_0^- \ |\text{in} > = \ 0.$$

Asymptotic Conservation law

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- These charges form a subgroup of U(1) and are called large gauge transformations.

Subleading term for loop amplitudes

- Sahoo and Sen (arxiv:1808.03288) showed that the subleading term in the soft expansion of loop amplitudes is given by :

$$\lim_{\omega \to 0} \mathrm{Amp}_{n+1}(p_i, k) = \left[\frac{S_0}{\omega} + S_{\mathrm{log}} \log \omega + \ldots\right] \mathrm{Amp}_n(p_i).$$

- S_{log} is universal.
- S_{log} is 1-loop exact.

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Is this soft theorem related to an asymptotic symmetry?

$\rightarrow$This study was initiated by Campiglia and Laddha (arxiv:1903.09133).

$\rightarrow$ We build upon their work and dicuss the asymptotic conservation law underying this soft theorem.

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- $\log \omega$ term in the soft expansion is exclusive to 4 spacetime dimensions. It is related to the long range forces present in 4 spacetime dimensions.

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- Forces $\sim \frac{1}{r^2}$ are called long range forces. Non trivial effect at $r, t \to \infty$.
  Asymptotic trajectory of a point particle : $\vec{x} \sim \vec{p}t + \vec{c} \log t$.
- So, asymptotically particles are not free. They radiate and and this produces the $\log \omega$ term.

Asymptotic conservation laws

- We will incorporate the effect of long range forces on asymptotic dynamics.
- This leads to new asymptotic conservation laws.

Plan

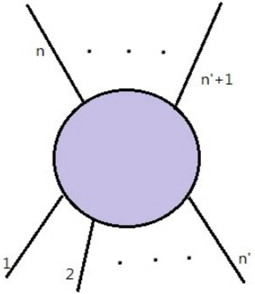
- Part 1 : Classical theory
  - Asymtotic dynamics in presence of long range forces
  - New Q_m-conservation laws (arxiv:2007.03627)

- Part 2 : Quantum theory
  - Q_1-Ward identity $\leftrightarrow$ $\log\omega$ soft theorem (arxiv:1912.10229)
  - Expectation at m-loop order

Part 1

Asymptotic conservation laws for classical scattering.

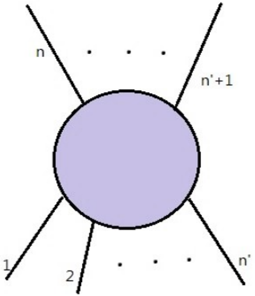
Scattering process



Let us consider scattering of charged particles where n' number of particles come in and interact in a finite region say a sphere of radius L around $r = 0$. At the end, they produce $(n - n')$ number of outgoing particles.

This interaction could be of any sort or of any strength.

Scattering process



For $r > L$, the particles are apart enough so that only possible interactions between them would be the long range forces.

Asymtotic dynamics at leading order

- We will calculate $F_{\mu\nu}$ perturbatively in e and $\frac{1}{\tau}$.

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- We will calculate $F_{\mu\nu}$ perturbatively in e and $\frac{1}{\tau}$.
- Let us first restrict ourselves to the leading order in coupling e, then we can ignore the effect of long range electromagetic interactions on the asymptotic trajectories.
- Hence an incoming particle has the trajectory :

$$x_i^\mu = [V_i^\mu \tau + d_i]\Theta(-T - \tau).$$

Similarly, an outgoing particle has the trajectory :

$$x_j^\mu = [V_j^\mu \tau + d_j]\Theta(\tau - T).$$

Asymtotic dynamics at leading order

- We have :

$$j_\sigma(x') = \int d\tau \Big[\sum_{i=n'+1}^{n} e_i V_{i\sigma} \; \delta^4(x' - x_i) \; \Theta(\tau - T) + \text{ in } \Big].$$

- Using $\Box A_\mu = -j_\mu$,

$$A_\sigma(x) = \frac{1}{2\pi} \int d\tau \; \delta(\; (x - x'(\tau))^2) \; j_\sigma(\tau).$$

$F_{\mu\nu}$ at $\mathcal{O}(e)$

- Let us find the field at $\mathcal{I}^+$: $r \to \infty$ for fixed $u = t - r, \hat{x}$.

- We get :

$$F_{\mu\nu}|_{\mathcal{I}^+_-} \sim \sum_{\substack{m,n \\ m<n}} \frac{u^m}{r^n} + \cdots \ .$$

'$\cdots$' represent terms that are atleast exponentially suppressed.

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- In particular, we have :

$$F^2_{rA}|_{\mathcal{I}^+_-} = u \ F^{(2,-1)}_{rA} + u^0 \ F^{(2,0)}_{rA} + \cdots \ .$$

A denotes S^2 co-ordinates.
The coefficients are a function of $\hat{x}$.

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- Long range forces lead to new logarithmic terms in field strength.

Asymptotic dynamics including subleading term

$$m_i \frac{\partial^2 x_i^\mu}{\partial \tau^2} = e_i \; F^{\mu\nu}(\tau) \; V_{i\nu}.$$

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Substitute $\mathcal{O}(e)$ solution of $F^{\mu\nu}$ in above equation to get

$$m_i \frac{\partial^2 x_i^\mu}{\partial \tau^2} = \mathcal{O}(\frac{e^2}{\tau^2}).$$

Hence we get :

$$x_i^\mu = V_i^\mu \; \tau + c_i^\mu \log \tau + d_i + \mathcal{O}(\frac{1}{\tau}).$$

Asymptotic dynamics including subleading term

We have :

$$x_i^\mu = V_i^\mu \, \tau + c_i^\mu \log \tau + d_i + \mathcal{O}(\frac{1}{\tau}).$$

$$c_i^\mu = -\frac{1}{4\pi} \sum_{\substack{j=n'+1, \\ j \neq i}}^{n} e_i e_j \frac{p_i.p_j \;\; m_j^2 p_i^\mu + m_i^2 m_j^2 p_j^\mu}{[(p_i.p_j)^2 - m_i^2 m_j^2]^{3/2}}.$$

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• Above expression represents effect of other particles j on the i^{th} particle.

• c_i's for an outgoing particle includes contribution only from outgoing particles.

$\mathcal{O}(e^3)$ fall offs in the field strength

- Using $\Box A_\mu = -j_\mu$,

$$A_\sigma(x) = \frac{1}{2\pi}\int d\tau\ \delta(\ [x - x'(\tau)]^2)\ j_\sigma(\tau).$$

  Now j_σ includes $\mathcal{O}(e^3)$ terms.

- The fields admit new fall offs :

$$F^2_{rA}|_{\mathcal{I}^+_-} \quad = \quad u\ F^{(2,-1)}_{rA} + \log u\ F^{(2,\log)}_{rA} + u^0\ F^{(2,0)}_{rA} + \cdots\ .$$

$\mathcal{O}(e^3)$ fall offs in the field strength

- At future :

$$F_{rA}|_{u\to-\infty} = \frac{1}{r^2}\left[u\ F_{rA}^{(2,-1)} + \log u\ F_{rA}^{(2,\log)} + \ldots\right] + \mathcal{O}(\frac{1}{r^3})\ . \quad (1)$$

- We repeat the similar calculation at past null infinity (**??**).

$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2}\ v^0\ F_{rA}^{(\log,0)} + \mathcal{O}(\frac{1}{r^2})\ .$$

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$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2}\ v^0\ F_{rA}^{(\log,0)} + \mathcal{O}(\frac{1}{r^2})\ .$$

- We show that :

$$F_{rA}^{(2,\log)}(\hat{x})\ |_{\mathcal{I}^+_-} = F_{rA}^{(\log,0)}(-\hat{x})\ |_{\mathcal{I}^-_+}\ .$$

This law was suggested by Campiglia and Laddha. We proved it. This is the $m = 1$ conservation law.

Asymptotic dynamics including subsubleading term

- Asymptotically particles accelerate under long range force and radiate.
- This radiation backreacts on the particles and corrects the trajectory of matter particles.

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$$m_i \frac{\partial^2 x_i^\mu}{\partial \tau^2} = e_i \; F^{\mu\nu}(\tau) \; V_{i\nu}^{\text{cor}}(\tau).$$

Substituting $F_{\mu\nu} \sim \mathcal{O}(e^3)$:

$$m_i \frac{\partial^2 x_i^\mu}{\partial \tau^2} \sim \frac{e^2}{\tau^2} + e^4 \frac{\log \tau}{\tau^3} + \cdots .$$

Asymptotic dynamics including subsubleading term

Thus, asymptotic trajectories of the particles are corrected to :

$$x_i^\mu = V_i^\mu \ \tau + c_i^\mu \log\tau + d_i + f_{i\sigma} \frac{\log\tau}{\tau},$$

where

$$f_i^\mu = -\sum_{\substack{i=n'+1,\\ i\neq j}}^{n} m_i m_j^2 \frac{Q_i Q_j}{2} \ \Big[3 m_i m_j p_j.c_i \ \frac{(p_i.p_j \ \ p_i^\mu + m_i^2 p_j^\mu)}{[(p_i.p_j)^2 - m_i^2 m_j^2]^{5/2}} + \ \frac{[p_i.p_j \ \ c_i^\mu - (p_i.p_j \ \ c_j^\mu - p_j.c_i \ \ p_j^\mu)]}{[(p_i.p_j)^2 - m_i^2 m_j^2]^{3/2}} \Big].$$

$\mathcal{O}(e^5)$ fall offs in the field strength

- Including the $\mathcal{O}(e^5)$ terms around future null infinity :

$$F^3_{rA}|_{u\to-\infty} = u^2\ F^{(3,-2)}_{rA}\ + u\log u\ F^{(3,-1)}_{rA} + (\log u)^2\ F^{(3,\log^2)}_{rA}\ + \dots\ .$$

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- Expansion around the past null infinity is given by :

$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2} \; v^0 \; F^{(\log,0)}_{rA} \; + \frac{(\log r)^2}{r^3} \; v^0 \; F^{(\log^2,0)}_{rA} \; + \mathcal{O}(\frac{1}{r^2}) \; .$$

$\mathcal{O}(e^5)$ fall offs in the field strength

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- We proved following $\mathcal{O}(e^5)$ conservation law :

$$F_{rA}^{(3,\log^2)}(\hat{x})|_{\mathcal{I}^+_-} = F_{rA}^{(\log^2,0)}(-\hat{x})|_{\mathcal{I}^-_+}.$$

This is the $m = 2$ conservation law.

Summary so far

We have established conservation laws for following modes of F_{rA} :

$$\mathcal{O}(e^3): \quad \frac{\log u}{r^2} \text{ and } \frac{\log r}{r^2}.$$

$$\mathcal{O}(e^5): \quad \frac{(\log u)^2}{r^3} \text{ and } \frac{(\log r)^2}{r^3}.$$

m^{th} order asymptotic conservation laws

- We expect these conservation laws to exist for all m-modes of F_{rA} :

$$\mathcal{O}(e^{2m+1}) : \quad \frac{(\log u)^m}{r^{m+1}} \text{ and } \frac{(\log r)^m}{r^{m+1}}.$$

Proved for $m = 1, 2, 3$.

Concluding Part 1

- Classical theory admits a set of conservation laws $(m = 1, 2, 3)$:

$$Q^+_m[Y^+_m] \mid_{\mathcal{I}^+_-} \quad = \quad Q^-_m[Y^-_m] \mid_{\mathcal{I}^-_+} .$$

- Next we will discuss the implications of these Q_m charges in the quantum theory.

Part 2

- Strominger and his colloborators have established a correspondence between asymptotic conservation laws and soft theorems.

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- Strominger and his colloborators have established a correspondence between asymptotic conservation laws and soft theorems.
- Q_1-conservation law is equivalent to the Sahoo-Sen $\log \omega$ soft theorem.
- We expect higher Q_m's also to be related to soft theorems.

Soft theorems

Let us go back to the quantum theory.

$$\lim_{\omega \to 0} \mathrm{Amp}_{n+1}(p_i, k) = \left[\frac{S_0}{\omega} + S_{\log} \log \omega + \ldots \right] \mathrm{Amp}_n(p_i).$$

Soft theorems

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$$\lim_{\omega\to 0} \mathrm{Amp}_{n+1}(p_i, k) = \left[\frac{S_0}{\omega} + S_{\log}\,\log\omega + ...\right]\mathrm{Amp}_n(p_i).$$

- The subleading term can be isolated as follows :

$$\lim_{\omega\to 0} \omega\;\partial_\omega^2\;\omega\;\mathrm{Amp}_{n+1} = S_{\log}\;\mathrm{Amp}_n.$$

Our next goal is to reproduce this soft theorem from Q_1 conservation law for massless scalar QED.

Massless scalar QED

$$\lim_{\omega \to 0} \omega \, \partial_\omega^2 \, \omega \; \mathrm{Amp}_{n+1} = S_{\log} \; \mathrm{Amp}_n.$$

We have for massless hard particles :

$$S_{\log} \; = -\frac{1}{4\pi^2} \sum_{\substack{i,j \\ i \neq j}} e_i^2 e_j \frac{\varepsilon_\mu k_\rho}{p_i . k} \; p_{i[\mu} \partial_{i\rho]} \; \log[p_i . p_j].$$

• Let us construct the asymptotic charge Q_1 for massless scalar QED.

• The charge gets contribution from the dressing of massless scalar field under electromagnetic force.

Asymptotics of free massless scalar

- Free equation of motion of the scalar is :

$$\Box\phi(x) \; = 0.$$

- Around future null infinity :

$$\phi(u, r, \hat{x}) = \frac{1}{r}\ \phi^1(u, \hat{x}) + \frac{1}{r^2}\ \phi^2(u, \hat{x}) + \dots\ .$$

- $\phi^1(u, \hat{x}) = \frac{-i}{8\pi^2} \int d\omega\ [b(\omega, \hat{x})\ e^{-i\omega u} - d^\dagger(\omega, \hat{x})\ e^{i\omega u}\ ].$

Long range force on massless scalar field

The dominant effect of electromagnetic coupling is given by :

$$\Box\phi = -2ie\frac{A_r^1(\hat{x})}{r}\partial_u\phi.$$

$$A_r = \frac{A_r^1}{r} + \ldots\ .$$

The solution of above equation is given by :

$$\phi(x) = \frac{-i}{8\pi^2 r}e^{iA_r^1(\hat{x})\log r}\int d\omega\ [b\ e^{-i\omega u} - d^\dagger\ e^{i\omega u}\ ].$$

Corrections to U(1) current

- Currents get corrections due to the dressing :

$$j_A = ie\phi D_A\phi^* - ie\phi^* D_A\phi,$$
$$= \frac{j_A^2}{r^2} + \overset{\log}{j_A} \frac{\log r}{r^2} + \ldots$$
$$\overset{\log}{j_A} \sim e^2 \, \partial_A A_r^1 \, (bb^\dagger + dd^\dagger).$$

- The logarithmic mode contributes to the charge.

Asymptotic charge

The Q_1-conservation law :

$$\overset{2,\log}{F}_{rA}(\hat{x}) \mid_{\mathcal{I}^+_-} \quad = \quad \overset{\log,0}{F}_{rA}(-\hat{x}) \mid_{\mathcal{I}^-_+} .$$

Asymptotic charge

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$$\overset{2,\log}{F}_{rA}(\hat{x}) \mid_{\mathcal{I}^+_-} \quad = \quad \overset{\log,0}{F}_{rA}(-\hat{x}) \mid_{\mathcal{I}^-_+} .$$

- The charge is defined as follows :

$$Q_+[Y] = -\int d^2z \; Y^A \overset{2,\log}{F}_{rA} \mid_{u\to-\infty}$$

$$= \; -\int_{\mathcal{I}^+} du \; d^2z \; Y^A \partial_u \, [u^2 \partial_u^2 \overset{2}{F}_{rA}] \; - \; \int d^2z \; Y^A \overset{2,\log}{F}_{rA} \mid_{u\to\infty}$$

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- $Q_+^{\text{soft}}[Y] = \; -\int du \; d^2z \; Y^A \partial_u \, [u^2 \partial_u^2 \overset{2}{F}_{rA}]$

- $Q_+^{\text{hard}}[Y] = -\int d^2z \; Y^A \overset{2,\log}{F}_{rA} \mid_{u\to\infty}$

Soft charge

$$Q_{+}^{\text{soft}} = -\int du' \; d^2z' \; Y^A(\hat{x}, \hat{x}')\partial_{u'} \; [u'^2 \partial_{u'}^2 F_{rA}^2(x')]$$

Soft charge

$$Q_+^{\text{soft}} = -\int du' \; d^2z' \; Y^A(\hat{x}, \hat{x}')\partial_{u'} \; [u'^2 \partial_{u'}^2 F_{rA}^2(x')]$$

- Let us choose

$$Y^z(\hat{x}, \hat{x}') = \sqrt{2}(1 + z'\bar{z}')\frac{z' - z}{\bar{z}' - \bar{z}} \; , \quad Y^{\bar{z}} = 0.$$

Using Maxwell's equations we get,

$$Q_+^{\text{soft}} = -i \lim_{\omega \to 0} \omega \; \partial_\omega^2 \; \omega \; a_-(\omega, \hat{x}).$$

- Above operator picks out coefficient of $\log \omega$.

Ward identity

- Ward identity takes following form :

$$\Big[\ Q_1\ ,\ S\ \Big] = 0$$

$$\Big[\ Q^{\text{soft}}\ ,\ S\ \Big] = -\Big[\ Q^{\text{hard}}\ ,\ S\ \Big]$$

- Q^{soft} inserts soft photon.
- Q^{hard} acts on the states and produces the soft factor S_{log}.

Hard charge

- Using Maxwell's equations, we get :

$$Q_+^{\text{hard}} = - \int du' d^2 z' \, \frac{q^\nu \ \epsilon_-^\mu}{q.q'} \, q'_{[\mu} \ D'^A q'_{\nu]} \ j_A^{\log},$$
$$= \sum_i \ e_i^2 \ \frac{q^\nu \ \epsilon_-^\mu}{q.q_i} \ q_{i[\mu} \partial_{i\nu]} \ A_r^1(x_i).$$

- We recall that A_r^1 is the dressing of massless scalar field under electromagnetic force.

Hard charge

- Using Maxwell's equations, we get :

$$Q_+^{\text{hard}} = - \int du' d^2 z' \, \frac{q^\nu \ \epsilon_-^\mu}{q.q'} \, q'_{[\mu} \, D'^A q'_{\nu]} \, j_A^{\log},$$
$$= \sum_i e_i^2 \, \frac{q^\nu \ \epsilon_-^\mu}{q.q_i} \, q_{i[\mu}\partial_{i\nu]} \, A_r^1(x_i).$$

- We recall that A_r^1 is the dressing of massless scalar field under electromagnetic force.
- A sidepoint : when we incorporate gravity then above expression gets corrected according to $A_r^1 \to A_r^1 + h_{rr}^1$. h_{rr}^1 is the dressing of massless scalar field under gravitational force.
- It is interesting to note that the hard charge is directly related to the dressing of fields under long range forces.

Ward identity

- So, we have following expressions :

$$Q_{+}^{\text{soft}} = -i \lim_{\omega \to 0} \omega\ \partial_{\omega}^{2}\ \omega\ a_{-}(\omega, \hat{x}),$$

$$Q^{\text{hard}} = \sum_{i} e_i^2 \frac{q^{\nu}\ \epsilon_{-}^{\mu}}{q.q_i}\ q_{i[\mu}\partial_{i\nu]}\ A_r^1(x_i). \qquad (2)$$

- The Ward identity (**??**) :

$$\Big[\ Q^{\text{soft}}\ ,\ S\ \Big] = -\Big[\ Q^{\text{hard}}\ ,\ S\ \Big].$$

Ward identity

$$\lim_{\omega\to 0}\omega\ \partial_\omega^2\ \omega\ \text{Amp}_{n+1}(p_i,k)$$

$$= -\frac{1}{4\pi^2}\sum_i\ e_i^2\,\frac{q^\nu\ \ \epsilon^\mu}{q.p_i}\ p_{i[\mu}\partial_{i\nu]}\sum_{j,j\neq i} e_j \log p_i.p_j\ \text{Amp}_n.$$

- This reproduces the Sahoo-Sen soft theorem.

- Q_1-conservation law $\leftrightarrow$ $\log\omega$ soft photon theorem.

Outlook - Strominger's triangle

- 1-loop exact $\log \omega$ soft theorem [Sahoo, Sen].
- Tail memory effect [Laddha, Sen].
- Asymptotic conservation law for Q_1.
(The underlying symmetry not clear yet.)

For $m > 1$

- Q_m charges which are $\mathcal{O}(e^{2m+1})$ are expected to be related to m-loop soft factors.
- These soft theorems have not been studied for $m > 1$.

For $m > 1$

- It can be argued that Q_m-conservation laws are related to following terms in soft expansion of loop amplitudes :

$$\lim_{\omega \to 0} \text{Amp}_{n+1}(p_i, k) = \left[\frac{S_0}{\omega} + \sum_m S_m \, \omega^{m-1} \log \omega^m + \ldots \right]. \quad (3)$$

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- These m^{th} level soft photon theorems for quantum amplitudes have not been explored for $m > 1$.
- In the paper, we have proved the classical version of soft theorems for $m = 2, 3, 4$.

(**??**)

Conclusion

- A new set of Q_m-conservation laws ($m = 1, 2, 3$). Charges are $\mathcal{O}(e^{2m+1})$ and tied to long range forces.
- Expected to be related to soft theorems for loop amplitudes. We demonstrated the equivalence for $m = 1$.

Conclusion

- A new set of Q_m-conservation laws ($m = 1, 2, 3$). Charges are $\mathcal{O}(e^{2m+1})$ and tied to long range forces.
- Expected to be related to soft theorems for loop amplitudes. We demonstrated the equivalence for $m = 1$.
- There is compelling evidence that this structure holds for all m's. So there are many interesting questions that need to be explored.

THANK YOU !

Comparison between future and past solutions (??)

- The retarded root of the delta function is

$$\tau_0 = -V_i.(x - d_i) - \left[\, (V_i.x - V_i.d_i)^2 + (x - d_i)^2 \, \right]^{1/2}. \qquad (4)$$

- Around future :

$$\tau_0|_{\mathcal{I}^+} = \frac{u}{(V_i^0 - \hat{x}.V_i)} + \mathcal{O}(\frac{1}{r}).$$

- Around past :

$$\tau_0|_{\mathcal{I}^-} = -2r \, (V_i^0 + \hat{x}.V_i) + \mathcal{O}(r^0).$$

A_r^1 - classical

- In Lorenz gauge, the mertric perturbations satisfy $\Box A_\mu = -j_\mu$. So,

$$A_\mu(x^\nu) = \frac{1}{\pi} \int d^4x' \; \delta\big(\, (x - x')^2 \, \big) \Theta(t - t') \; j_\mu(x').$$

- We get :

$$A_r^1(\hat{x}) = -\frac{1}{2\pi} \sum_j e_j. \tag{5}$$

A^1_r - quantum (**??**)

- There is a quantum mode in photon such that :

$$A^1_r(\hat{x}) = -\frac{1}{2\pi^2}\sum_j e_j \; \log q.p_j.$$

A_r^1 - quantum

A generic homogenous solution in Lorenz gauge can be written as :

$$A_\mu(x) = \int d^2q' \left[\varepsilon_\mu^- \ \dot{A}_z(u = x \cdot q', \hat{q}') + \varepsilon_\mu^+ \ \dot{A}_{\bar{z}}(u = x \cdot q', \hat{q}')\right] \tag{6}$$

A_r^1 - quantum

$$A_r(x) \;=\int d^2z'\Big[\varepsilon^-.q\;\dot{A}_{zz}(u=r\;q.q',\hat{q}')+...\;]\;.$$

Thus, $1/r$ term in A_r needs a $\log u$ term in A_B.

- But, classically we have

$$A_B^0 \;=\; \overset{0,0}{A_B}\;u^0\;+\overset{0,1}{A_B}\frac{1}{u}+...\;.$$

- This is consistent with the usual radial gauge choice.

$\log u$ term in photon field

$$A(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \tilde{A}(\omega) \; e^{-i\omega u}.$$

- If around $\omega = 0$,

$$\tilde{A}(\omega) \sim \frac{\tilde{A}^0_{\pm}}{\omega} + \dots \; .$$

then,

$$A(u)|_{u\to\infty} = \frac{1}{2\pi}[\; \tilde{A}^0_{+} - \tilde{A}^0_{-} \;] \; \log u^{-1} + \dots \; .$$

$\log u$ term in photon field

- Now, $[\tilde{A}]_+$ involves annihilation operator : $a(\omega, \hat{x})$
  $[\tilde{A}]_-$ involves creatuion operator : $a^\dagger(\omega, \hat{x})$
- So, we get :

$$\begin{aligned} A(u, \hat{x}) = & -\frac{1}{2\pi}\left[\tilde{A}^0_+(\hat{x}) - \tilde{A}^0_-(\hat{x})\right] \quad \log|u| + \dots , \\ = & -\frac{1}{2\pi} \lim_{\omega \to 0} \omega[a(\omega, \hat{x}) + a^\dagger(-\omega, \hat{x})] \quad \log|u| + \dots . \end{aligned}$$

Classical soft theorems

Classical soft theorems were studied in Prof Ashoke Sen's talk. These are statements about universal terms in soft limit of classical radiative field.

We calculate the soft limit of the radiative field generated in a general scattering process :

$$\lim_{\omega\to 0} \epsilon^{\mu} \tilde{A}_{\mu}(\omega)$$
$$= \left[\frac{S_0}{\omega} + S_1^{\text{class}} \log\omega + S_2^{\text{class}} \omega(\log\omega)^2 + S_3^{\text{class}} \omega^2(\log\omega)^3 + \ldots\right] \quad (7)$$

S_0 and S_1 terms are already known.

- S_m are $\mathcal{O}(e^{2m+1})$.
- S_m are related to Q_m charges. (**??**)

Log ω soft theorem

- $S_{\log}$ has 2 parts. A part that survives in the classical limit.

$$S_{\log}^{\text{class}} = \frac{i}{4\pi} \sum_{\substack{\eta_i\eta_j=1 \\ i\neq j}} e_i^2 e_j \frac{\epsilon^\mu q^\nu}{(q.p_i)} \; m_i^2 m_j^2 \; p_{i[\mu}\partial_{i\nu]} \; \frac{p_i.p_j}{\sqrt{(p_i.p_j)^2 - m_i^2 m_j^2}}$$

- This term appears in the soft radiation emitted in a classical scattering.
- Important to note that this term vanishes for massless particles.

Log ω soft theorem

- The other part is purely quantum and does not appear in classical physics.

$$S_{\log}^{\text{quan}} = -\frac{1}{8\pi^2}\sum_{i\neq j} e_i^2 e_j \frac{\epsilon^\mu q^\nu}{(q.p_i)}\; p_{i[\mu}\partial_{i\nu]}\; \frac{f(p_i,p_j)}{\sqrt{(p_i.p_j)^2 - m_i^2 m_j^2}}$$

- The exact form of this expression is not important for us. Interesting to note the relative factor of i between two terms.