From Moduli Space Localization to AdS Amplitudes

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based on hep-th/1906.02099 (Springer, 2020)

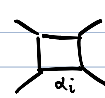
hep-th/1910.11852 (JHEP 02 159) with A. Pokraka

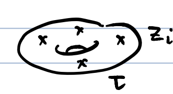
hep-th/1912.03397 (PRL 124 14)

hep-th/2007.06574 with L. Eberhardt and S. Komatsu

→ What do we mean by localization in this context?

$$\underbrace{\int D\phi \, e^{-S} \cdots}_{\substack{\text{path integral} \\ (\infty\text{-dim})}} \xmapsto{\text{free theories}} \underbrace{\int d\mu \cdots}_{\substack{\text{moduli space integral} \\ (\text{finite-dim})}} \xmapsto{\text{localization}} \underbrace{\sum_{i=1}^{N<\infty} \cdots}_{\substack{\text{finite sum} \\ (0\text{-dim})}}$$

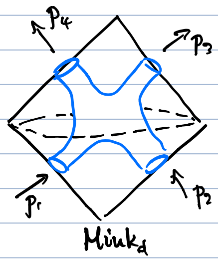




→ The goal of this talk is to review three cases where moduli space localization appears in the S-matrix theory.

→ Simplification: everything will be 4-pt, tree-level, and massless which illustrates all the main points.

→ Plan of the talk:

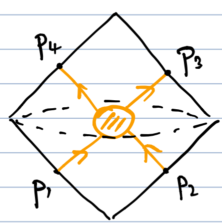


string perturbation theory in high-energy limit $\longmapsto$

[Fairlie-Roberts '72]
[Gross-Mende '87]

$$\sum_{j\neq i} \frac{p_i \cdot p_j}{z_i - z_j} = 0$$

different equations!

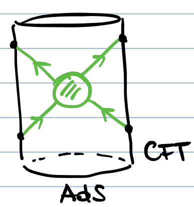


flat space S-matrix in QFT

[Cachazo-He-Yuan '13]

$$\sum_{j\neq i} \frac{p_i \cdot p_j}{z_i - z_j} = 0 .$$

Feynman diagrams



AdS amplitudes / CFT correlators in QFT

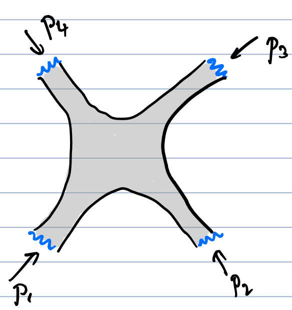
$$\sum_{j\neq i} \frac{D_i \cdot D_j}{z_i - z_j} = 0$$

Witten diagrams

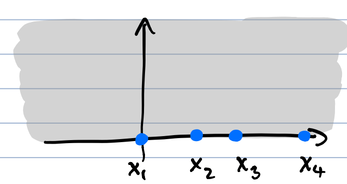
[with Eberhardt & Komatsu]
see also [Roehrig-Skinner '20]

→ Let's start with revisiting the result of Gross-Mende

[Appendix A of 1910.11852]



state-operator map



using $SL(2,\mathbb{R})$ freedom we fix

$$(x_1, x_3, x_4) = (0, 1, \infty)$$

momentum cons.

$$\sum_{i=1}^{4} p_i = 0$$

→ The remaining moduli space $\mathcal{M}_{0,4}(\mathbb{R})$ is parametrized by x_2

$$A_4 = \int_{x_2=0}^{1} \underbrace{e^{\alpha'(s\log|x_2| + t\log|1-x_2|)}}_{\text{Koba-Nielsen factor (universal)}} \underbrace{\varphi}_{\substack{dx_2\text{-form depends} \\ \text{on the matter content} \\ \text{(irrelevant for us)}}} .$$

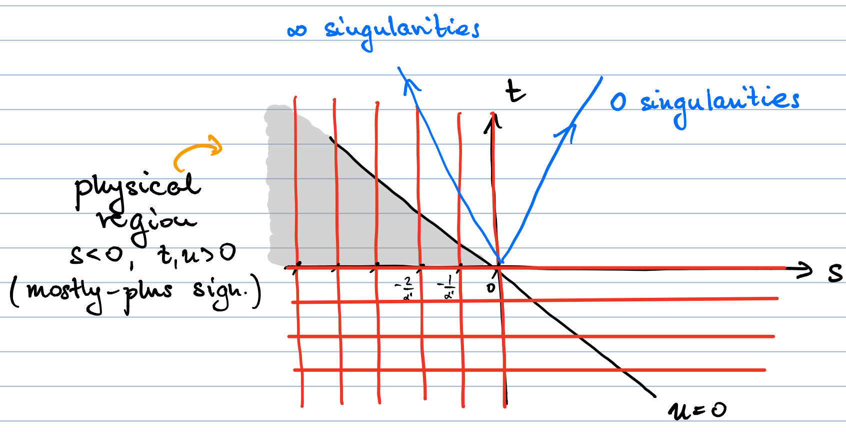
e.g. Veneziano $\varphi = \dfrac{dx_2}{x_2(1-x_2)}$ $\Rightarrow$ $A_4 = \dfrac{\Gamma(\alpha' s)\,\Gamma(\alpha' t)}{\Gamma(\alpha' s + \alpha' t)}$

→ Recall: $s = (p_1+p_2)^2$, $t = (p_2+p_3)^2$, $u = (p_1+p_3)^2$

$$s+t+u = \sum_i m_i^2 = 0 .$$

scattering angle $\sin^2 \phi/2 = \dfrac{\alpha' t}{\alpha' s}$.

→ Let's consider fixed-angle high-energy limit: $\alpha' \to \infty$.



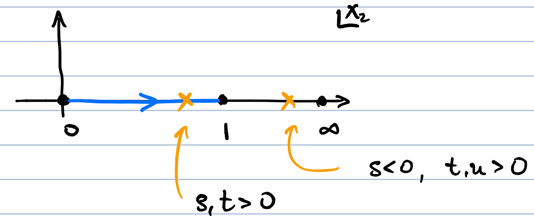
This feature is called the Stokes phenomenon. We want to understand it from a worldsheet perspective.

→ Saddle-point approximation, naively:

$$\frac{\partial}{\partial x_2}\left(s\log|x_2| + t\log|1-x_2|\right) = \frac{s}{x_2} + \frac{t}{x_2-1} = 0 .$$

$$\Rightarrow \quad x_2^* = \frac{s}{s+t} = 1+\frac{t}{u}$$

Integration in $\mathcal{M}_{0,4}$:



→ What's going on? To analyze asymptotics correctly, we first need a holomorphic integrand:

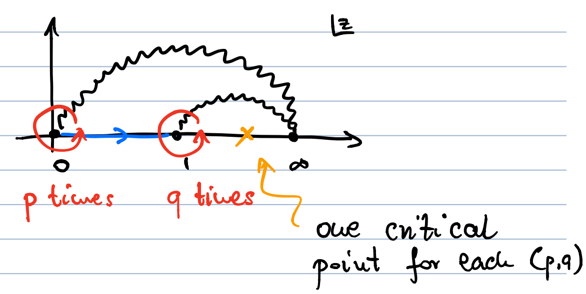
$$\log|x_2| \longmapsto \log z$$

but a consequence of doing so is that the integrand now has infinite branches

$$A_4 = \int_0^1 e^{\alpha'(s\log z + t\log(1-z))}\,\underset{\text{dz-form}}{\varphi}$$

This multi-branched surface is $\tilde{\mathcal{M}}_{0,4}$, the maximal Abelian cover of $\mathcal{M}_{0,4}$

Lattice $(p,q)\in\mathbb{Z}^2$



→ The way to see it more clearly is via the change of variables:

$$z = e^u, \qquad 1-z = e^v, \qquad e^u + e^v = 1 .$$

$$A_4 = \alpha' \int_\Gamma e^{\alpha'\left(\overbrace{su + tv + 2\pi i\, w\,(e^u + e^v - 1)}^{W}\right)}\, du\, dv\, dw$$

Γ: $\mathbb{R}_-^2 \times \mathbb{R}$ $\qquad$ w: Lagrange multiplier

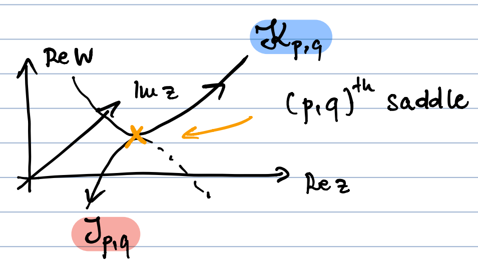
Saddle point equations:

$$\begin{cases} s + 2\pi i\, w e^u = 0 \\ t + 2\pi i\, w e^v = 0 \\ 2\pi i\,(e^u + e^v - 1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} u^* = \log\left(\frac{s}{s+t}\right) + 2\pi i p \\ v^* = \log\left(\frac{t}{s+t}\right) + 2\pi i q \\ w^* = \frac{i}{2\pi}(s+t) \end{cases}$$

$(p,q) \in \mathbb{Z}^2$ dependence disappears after $(u,v,w) \longmapsto z$

→ Asymptotics is governed by Lefschetz thimbles: paths of steepest descent $\mathcal{J}_{p,q}$ and ascent $\mathcal{K}_{p,q}$ of $\mathrm{Re}\, W$ extending from each saddle point.



$$\to \quad A_4 = \alpha' \sum_{(p,q)\in\mathbb{Z}^2} \underbrace{\langle \Gamma \,|\, \mathcal{K}_{p,q}\rangle}_{\substack{\text{intersection} \\ \text{numbers } \in \mathbb{Z}}} \int_{\mathcal{J}_{p,q}} e^{\alpha' W}\, du\, dv\, dw$$

$$\xrightarrow[\alpha' \to \infty]{} \underbrace{\left(\sum_{(p,q)\in\mathbb{Z}^2} \langle \Gamma \,|\, \mathcal{K}_{p,q}\rangle\, e^{2\pi i \alpha' (ps+qt)} \right)}_{= f(s,t) \text{ depends on the direction}} \underbrace{\frac{e^{\alpha'\left(s \log\left(\frac{s}{s+t}\right) + t \log\left(\frac{t}{s+t}\right)\right)}}{\sqrt{\frac{\alpha' s t}{2\pi (s+t)}}}}_{\text{universal suppression}}$$

→ Computing $f(s,t)$ in the physical region gives a sum over an infinite number of saddles, yielding

$$f(s,t) = \frac{e^{-2\pi i\alpha' t} - e^{2\pi i \alpha' s}}{1 - e^{2\pi i \alpha' s}} \qquad s<0, \; t,u>0.$$

resonances at $s = 0, -\frac{1}{\alpha'}, -\frac{2}{\alpha'}, \dots$

→ The way such results can be obtained easily is using twisted homology theory, which packages together the information of all branches at once.

→ Let's change the topics. Veneziano amplitude was given by

$$\int_0^1 e^{\alpha' W} \varphi = \int_0^1 e^{\alpha' W} \varphi + d(e^{\alpha' W} \xi)$$

$$W = \alpha'(s \log z + t \log(1-z)) \qquad = \int_0^1 e^{\alpha' W} \big(\varphi + \underbrace{d\xi + \alpha' dW \wedge \xi}_{\nabla_{dW}\xi \text{ with } \nabla_{dW} = d + \alpha' dW \wedge} \big)$$

→ Hence $\varphi \sim \varphi + \nabla_{dW}\xi$ defines a cohomology class.

One can ask a (mathematical) question about what kind of invariants can be computed with φ:

I $\int_\Gamma e^{\alpha' W} \varphi$: open-string amplitudes

II $\int_{M_{0,4}} |e^{\alpha' W}|^2 \varphi_L \wedge \overline{\varphi}_R$: closed-string amplitudes

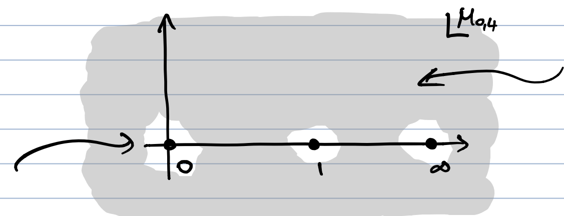
Recently we've learned about a third option,
intersection numbers:

III $\quad \langle \varphi_- | \varphi_+ \rangle = \int_{\mathcal{M}_{0,4}} \varphi_- \wedge \varphi_+ \; :$ QFT amplitudes

$$\varphi_\pm \sim \varphi_\pm + \nabla_{\pm dW} \xi$$ holomorphic

Currently no consistent worldsheet formulation, but a lot of progress by Siegel, Jusinskas, Azevedo, Casali, Tourkine, ... under names chiral/sectorized/twisted strings.

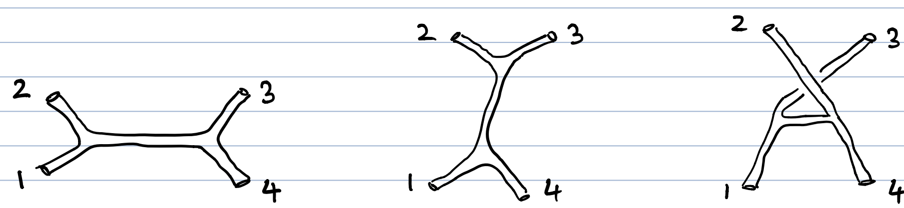
→ How is $\langle \varphi_- | \varphi_+ \rangle$ even well-defined? $\varphi_\pm \propto dz$, $dz \wedge dz = 0$.



in the bulk $\int_{\text{bulk}} \varphi_- \wedge \varphi_+ = 0$.

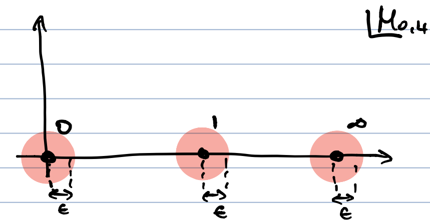
close to bdry "0/0" problem $\int_{\text{bdry}} \varphi_- \wedge \varphi_+$

→ This is the first sign of localization: once we regularize the "0/0" problem, the integral will only receive contributions from three points: $\partial \mathcal{M}_{0,4} = \{0, 1, \infty\}$.



$z \to 0$ | $z \to 1$ | $z \to \infty$
s-channel | t-channel | u-channel

→ The idea for regularization is extremely simple: we'll exploit the fact that $\varphi_+ \sim \varphi_+ + \nabla_{dW}\xi$ to cook up ξ making the integrand vanish near boundaries



$$\varphi_+^c = \varphi_+ - \nabla_{dW}\Big(\sum_{p=0,1,\infty} \Theta(|z-p|^2-\epsilon^2)\, \nabla_{dW}^{-1}\varphi_+ \Big)$$

step function

$$= \varphi_+\Big(1-\sum_p \Theta(|z-p|^2-\epsilon^2)\Big) - \sum_p \underbrace{\delta(|z-p|^2-\epsilon^2)}_{\text{non-holomorphic}}\, \nabla_{dW}^{-1}\varphi_+$$

vanishes inside the ϵ-neighborhoods of each boundary.

→ Let's go ahead and compute:

$$\langle \varphi_- | \varphi_+ \rangle = \frac{-1}{2\pi i}\int_{\mathcal{M}_{0,4}} \varphi_- \wedge \varphi_+^c$$

$$= \frac{1}{2\pi i}\sum_{p=0,1,\infty} \oint_{|z-p|=\epsilon} \varphi_- \nabla_{dW}^{-1}\varphi_+ .$$

$$= \sum_{p=0,1,\infty} \operatorname*{Res}_{z=p}\big(\varphi_- \nabla_{dW}^{-1}\varphi_+\big).$$

localization!

→ To get some intuition let's work out a specific example.

Ex. $\varphi_\pm = \frac{dz}{z(1-z)}$. Need to compute $\nabla_{dw}^{-1}\varphi_+ = \psi_p$,

i.e., locally solve the differential equation:

$$\varphi_+ = \nabla_{dw}\psi_{p=0} = \left(d + \alpha'\left(\frac{s}{z}+\frac{t}{z-1}\right)dz\right)\left(\psi_0^{(0)} + z\,\psi_0^{(1)} + \dots\right)$$

$$= \frac{dz}{z} + \dots$$

$\Rightarrow$ $\psi_0^{(0)} = \frac{1}{\alpha' s}$, similarly $\psi_1^{(0)} = -\frac{1}{\alpha' t}$, $\psi_\infty^{(0)} = 0$.

Compute the amplitude:

$$\left\langle \frac{dz}{z(1-z)} \,\middle|\, \frac{dz}{z(1-z)} \right\rangle = \operatorname{Res}_{z=0}\left(\left(\frac{dz}{z}+\dots\right)\left(\frac{1}{\alpha' s}+\dots\right)\right) + \operatorname{Res}_{z=1}\left(-\frac{dz}{z-1}\,\frac{-1}{\alpha' t}\right)$$

$$= \frac{1}{\alpha' s} + \frac{1}{\alpha' t}. \quad \rightsquigarrow \text{planar } \phi^3 \text{ amplitude}$$

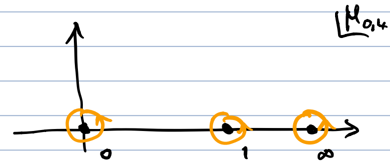
$\rightarrow$ In general one can show that

$$\langle \varphi_- | \varphi_+ \rangle = \frac{\overbrace{\left(\operatorname{Res}_{z=0}\varphi_-\right)}^{c_s}\overbrace{\left(\operatorname{Res}_{z=0}\varphi_+\right)}^{n_s}}{\alpha' s} + \frac{\overbrace{\left(\operatorname{Res}_{z=1}\varphi_-\right)}^{c_t}\overbrace{\left(\operatorname{Res}_{z=1}\varphi_+\right)}^{n_t}}{\alpha' t}$$

$$+ \frac{\overbrace{\left(\operatorname{Res}_{z=\infty}\varphi_-\right)}^{c_u}\overbrace{\left(\operatorname{Res}_{z=\infty}\varphi_+\right)}^{n_u}}{\alpha' u} + \mathcal{O}(\alpha'^0)$$

absent in massless amplitudes

Manifestation of color-kinematics duality:



$$\Rightarrow \begin{cases} c_s + c_t + c_u = 0 \\ n_s + n_t + n_u = 0 \end{cases}$$

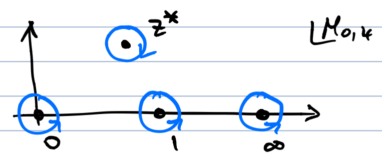
[Bern-Carrasco-Johansson '08]

[1912.03397]

→ Finally, for massless theories (where α' goes outside) we might as well take the opposite limit, $\alpha' \to \infty$.

→ Naively, $\nabla_{dW}^{-1} = \left(d + \alpha' \left(\frac{s}{z} + \frac{t}{z-1} \right) dz \wedge \right)^{-1} \xrightarrow[\alpha' \to \infty]{} \frac{1}{\alpha' \left(\frac{s}{z} + \frac{t}{z-1} \right) dz}$

$$\langle \varphi_- | \varphi_+ \rangle = \frac{1}{\alpha'} \sum_{p=0,1,\infty} \underset{z=p}{\mathrm{Res}} \underbrace{\left(\frac{\varphi_- \varphi_+}{\left(\frac{s}{z} + \frac{t}{z-1} \right) dz} \right)}_{\text{independent of } p} + \mathcal{O}(\alpha'^{-2})$$



where $\frac{s}{z^*} + \frac{t}{z^*-1} = 0$.

This gives yet another example of localization:

$$\langle \varphi_- | \varphi_+ \rangle = \frac{1}{\alpha'} \underset{z=z^*}{\mathrm{Res}} \left(\frac{\varphi_- \varphi_+}{\left(\frac{s}{z} + \frac{t}{z-1} \right) dz} \right) + \ldots$$

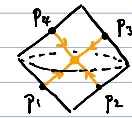
scattering equations

$$F_i = \sum_{j \neq i} \frac{p_i \cdot p_j}{z_i - z_j} = 0.$$

[Cachazo-He-Yuan '13]

→ Let's move on to AdS. But before that, it's useful to review the n-pt scattering equations formula:

$$A_n = \oint_{F_i=0} \frac{\varphi_- \, \varphi_+}{\prod_{i=2}^{n-2} F_i \, dz_i} \; \underbrace{\delta^{d+1}\Big(\sum_{i=1}^{n} p_i^{\mu}\Big)}_{\int_{\mathrm{Mink}_{d+1}} d^{d+1}x \, \prod_{i=1}^{n} e^{i x \cdot p_i}}$$



→ Let's review embedding space formalism in AdS_{d+1} first:

$$ds^2_{AdS} = \frac{dy^2 + dx^{\mu} dx_{\mu}}{y^2}, \qquad \mu = 1, 2, \dots, d$$

Embed in $\mathbb{R}^{2,d}$: $\qquad A = 0, 1, 2, \dots, d, d+1$

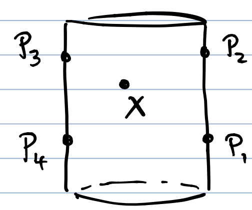
$$X^A = \frac{1}{y}\left(\frac{1 + y^2 + x_{\mu} x^{\mu}}{2}, \; x^{\mu}, \; \frac{1 - y^2 - x_{\mu} x^{\mu}}{2} \right),$$

which parametrizes the hypersurface $X \cdot X = -1$ in $\mathbb{R}^{2,d}$.

Boundary points can be represented by

$$P^A = \left(\frac{1 + x_{\mu} x^{\mu}}{2}, \; x^{\mu}, \; \frac{1 - x_{\mu} x^{\mu}}{2} \right).$$

with $P \cdot P = 0$ and $P^A \sim \lambda P^A$



→ The main advantage is that the conformal group $SO(2,d)$ acts linearly in these coordinates.

Generators: $$D_i^{AB} = P_i^A \frac{\partial}{\partial P_{iB}} - P_i^B \frac{\partial}{\partial P_{iA}}.$$

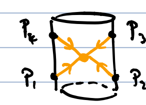
Casimir: $D_i^{AB} D_{i,AB} \sim \Delta(\Delta - d) = 0$ setting $\Delta = d$.

operator dimension

Bulk-to-bdry propagator: $\dfrac{1}{(-2X\cdot P_i)^{\Delta_i}}$

→ **We can now state a generalization of the flat-space formula:**

$$A_n = \oint_{E_i=0} \frac{\varphi_- \varphi_+}{\prod_{i=2}^{n-2} E_i \, dz_i} \underbrace{\int_{AdS_{d+1}} dX \prod_{i=1}^{n} \frac{1}{(-2X\cdot P_i)^d}}_{\substack{\text{contact diagram} \\ \mathcal{C} = \mathcal{C}(P_1, P_2, \dots, P_n)}}$$



with **operator-valued** scattering equations,

$$E_i = \sum_{j\neq i} \frac{D_i \cdot D_j}{z_i - z_j} .$$

→ **Let's give a plausibility argument why**

$$D_i\cdot D_j \int_{AdS_{d+1}} dX \prod_{i=1}^{n} \frac{1}{(-2X\cdot P_i)^{\Delta}} \xrightarrow{\text{flat space}} p_i\cdot p_j \int_{Mink_{d+1}} dx \prod_{i=1}^{n} e^{i x\cdot p_i}$$

in the limit $\Delta \to \infty$ [Komatsu, Paulos, van Rees, Zhao '20]

$$\mathcal{C} = \int_{AdS_{d+1}} dX \; e^{-S[X]} \qquad \text{with} \quad S[X] = \Delta \sum_i \log(-2X\cdot P_i) + \lambda(X^2+1)$$

Lagrange mult.

→ **Saddle-point approximation:**

$$\frac{\partial}{\partial X_A} S[X] = \Delta \sum_{i=1}^{n} \frac{P_i^A}{X\cdot P_i} + 2\lambda X^A = 0 .$$

contract with X_A : $\quad n\Delta - 2\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{n\Delta}{2}$.

$$\Rightarrow \quad \sum_{i=1}^{n} \underbrace{\frac{i\Delta}{R_{AdS}}\left(\frac{P_i^A}{X\cdot P_i} + X^A\right)}_{\tilde{p}_i \text{ "momentum"}} = 0.$$

where $\tilde{p}_i^{\,2} = -\left(\Delta/R_{AdS}\right)^2 \longrightarrow 0$ in the flat-space limit.

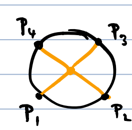
→ With this identification we have:

$$D_i\cdot D_j \int_{AdS_{d+1}} dX \prod_{k=1}^{n} \frac{1}{(-2X\cdot P_k)^{\Delta}} = -2\Delta^2\left(\frac{P_i\cdot P_j}{X\cdot P_i \;\, X\cdot P_j} + 1\right) e^{-S[X^*]}$$

$$= \frac{2}{R_{AdS}^2}\,\tilde{p}_i\cdot\tilde{p}_j\;\underbrace{e^{-S[X^*]}}_{\propto\, \delta\left(\sum_{i=1}^{n}\tilde{p}_i\right)}.$$

→ Great, so let's compute a 4-pt amplitude in AdS,
with $(z_1, z_2, z_3, z_4) = (0, z, 1, \infty)$ and $\varphi_\pm = \frac{dz}{z(1-z)}$:

$$A_4 = -\oint_{E_2=0} \frac{dz}{z^2(1-z)^2\underbrace{\left(\frac{D_1\cdot D_2}{z} + \frac{D_2\cdot D_3}{z-1}\right)}_{E_2}}$$



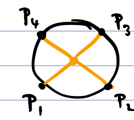
E_2 ← operator

→ Two options for computing the result:

I Decompose into eigenfunctions of E_2

II Contour deformation (this talk)

$$A_4 = \left(\oint_{z=0} + \oint_{z=1} + \oint_{z=\infty} \right) \frac{dz}{z^2(1-z)^2 \left(\frac{D_1 \cdot D_2}{z} + \frac{D_2 \cdot D_3}{z-1} \right)}$$



$$= \left(\frac{1}{D_1 \cdot D_2} + \frac{1}{D_2 \cdot D_3} \right) \int_{AdS_{d+1}} dX \prod_{i=1}^{n} \frac{1}{(-2X\cdot P_i)^\Delta}$$

→ To complete the derivation let's see how $\frac{1}{D_i \cdot D_j}$ can be converted to bulk-to-bulk propogators:

$$D_i^{AB} \frac{1}{(-2X\cdot P_i)^\Delta} = P_i^{[A} \frac{\partial}{\partial P_{i\,B]}} \frac{1}{(-2X\cdot P_i)^\Delta}$$

$$= 2\Delta \frac{P_i^{[A} X^{B]}}{(-2X\cdot P_i)^{\Delta+1}}$$

$$= -\nabla^{AB} \frac{1}{(-2X\cdot P_i)^\Delta},$$

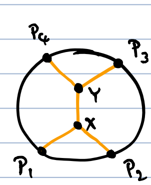
where $\nabla^{AB} = X^{[A} \frac{\partial}{\partial X_{B]}}$ such that $\nabla^2 = \nabla_{AB}\nabla^{AB}$ is the AdS Laplacian. Then

$$D_1 \cdot D_2 \frac{1}{(-2X\cdot P_1)^\Delta} \frac{1}{(-2X\cdot P_2)^\Delta} = \nabla^2 \frac{1}{(-2X\cdot P_1)^\Delta} \frac{1}{(-2X\cdot P_2)^\Delta},$$

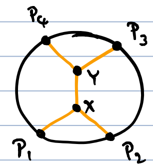
and hence the first term in A_4 is

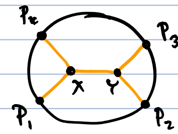
$$\frac{1}{D_1\cdot D_2} \int_{AdS_{d+1}} dX\, dY \frac{1}{(-2X\cdot P_1)^\Delta} \frac{1}{(-2X\cdot P_2)^\Delta} \delta^{d+1}(X-Y) \frac{1}{(-2Y\cdot P_3)^\Delta} \frac{1}{(-2Y\cdot P_4)^\Delta}$$

$$= -\int_{AdS_{d+1}} dX\, dY \frac{1}{(-2X\cdot P_1)^\Delta} \frac{1}{(-2X\cdot P_2)^\Delta} \underbrace{\frac{1}{\nabla^2} \delta^{d+1}(X-Y)}_{\text{bulk-to-bulk propagator}} \frac{1}{(-2Y\cdot P_3)^\Delta} \frac{1}{(-2Y\cdot P_4)^\Delta}$$



→ Therefore we found an expansion into Witten diagrams:

$A_4 =$ 

$+$ 

Higher-pt generalizations in [2007.06574 with Eberhardt & Komatsu]

see also [Roehrig-Skinner '20].

↑ Let's scroll up the slides for a recap ↑

Thanks!