Celestial Amplitudes and Asymptotic Symmetries (I)

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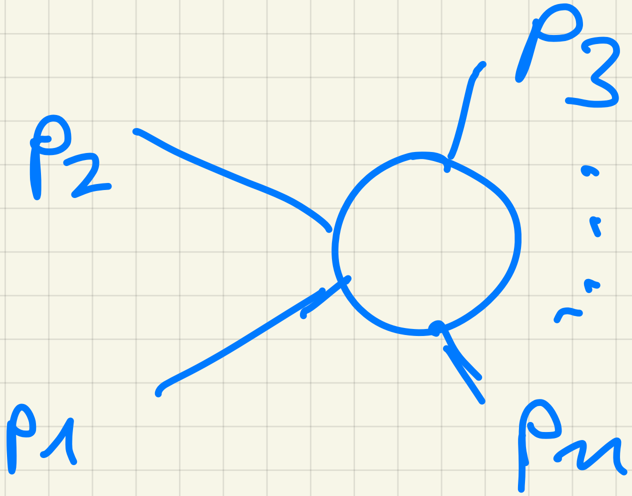
S-Matrix

p_K^μ ; $p_K^2 = m_K^2$; $K = 1, \dots, n$

traditional momentum space



scatter asymptotically

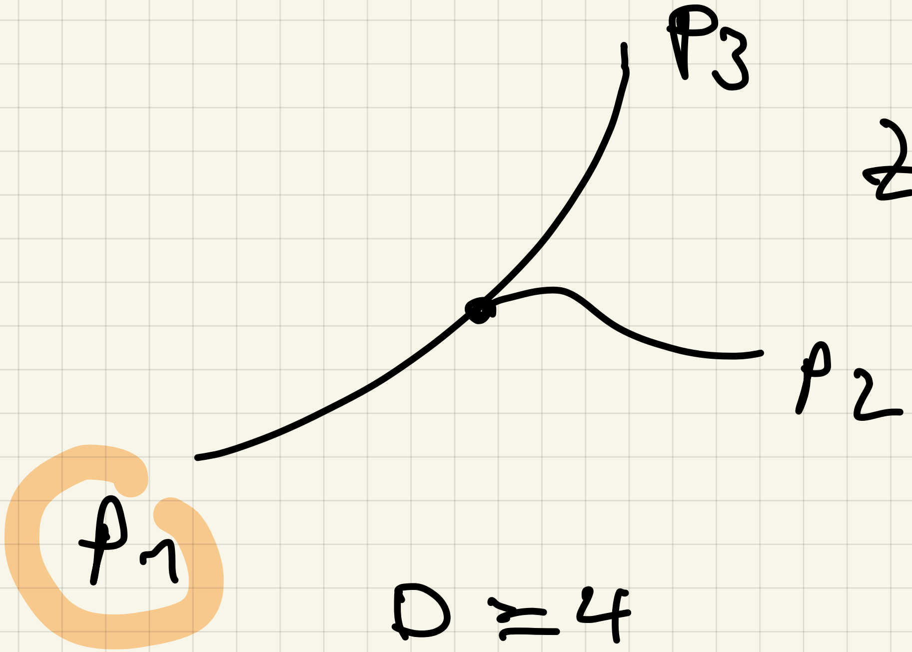
e^{ipx} $\longrightarrow$ Feynman

- amplitudes specified by <u>asymptotic wave functions</u>, which transform simply under space-time translations
- with manifest <u>translation symmetry</u>
- traditional amplitudes describe <u>transitions</u> between <u>momentum eigenstates</u>

$D=4$ Minkowski space probably not the right space to see <u>all</u> symmetries of scattering amplitudes

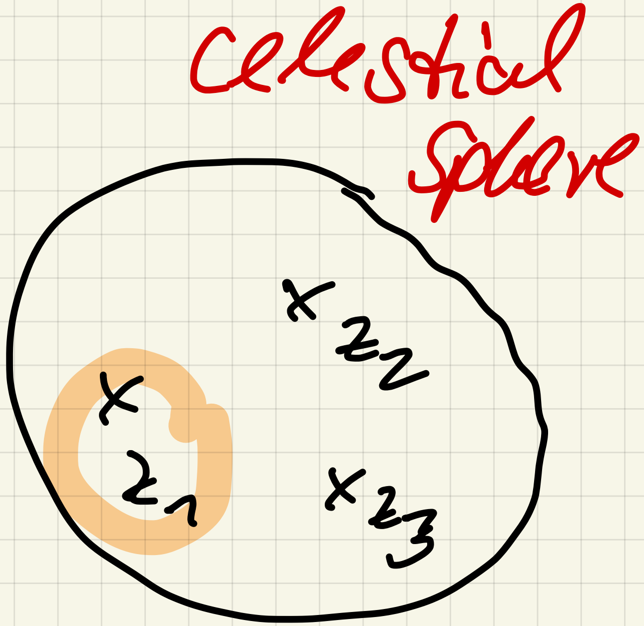
Scattering amplitudes in $D=4$ have interpretation as Euklidian $D=2$ CFT correlators

0. Basic Idea



$D=4$

$$z_k = \frac{p_k^1 + i p_k^2}{p_k^0 + p_k^3}$$



$D=2$

$$\sim g \, |z_{12}|^{h_1+h_2-h_3} \, |z_{23}|^{h_2+h_3-h_1} \, |z_{13}|^{h_1+h_3-h_2}$$

D = 4 space-time
QFT correlators

D=2 Euklidian
CFT correlators

Lorentz symmetry $\longrightarrow$ global conformal symmetry on $\mathcal{CS}^2$

$SO(1,3) \cong SL(2,\mathbb{C})$

$$z_i \to \frac{a z_i + b}{c z_i + d}$$

$$ad - bc = 1$$

$\to$ local 2D conformal

Why ?

- Constrain S-matrix and understand amplitude relations

From studying scattering amplitudes:
deep connections between
gravity and gauge interactions
e.g.: *KLT, BCJ, EYM (double-copy-construction)*

- scattering amplitudes in both gauge and gravity theories suggest a deeper connection

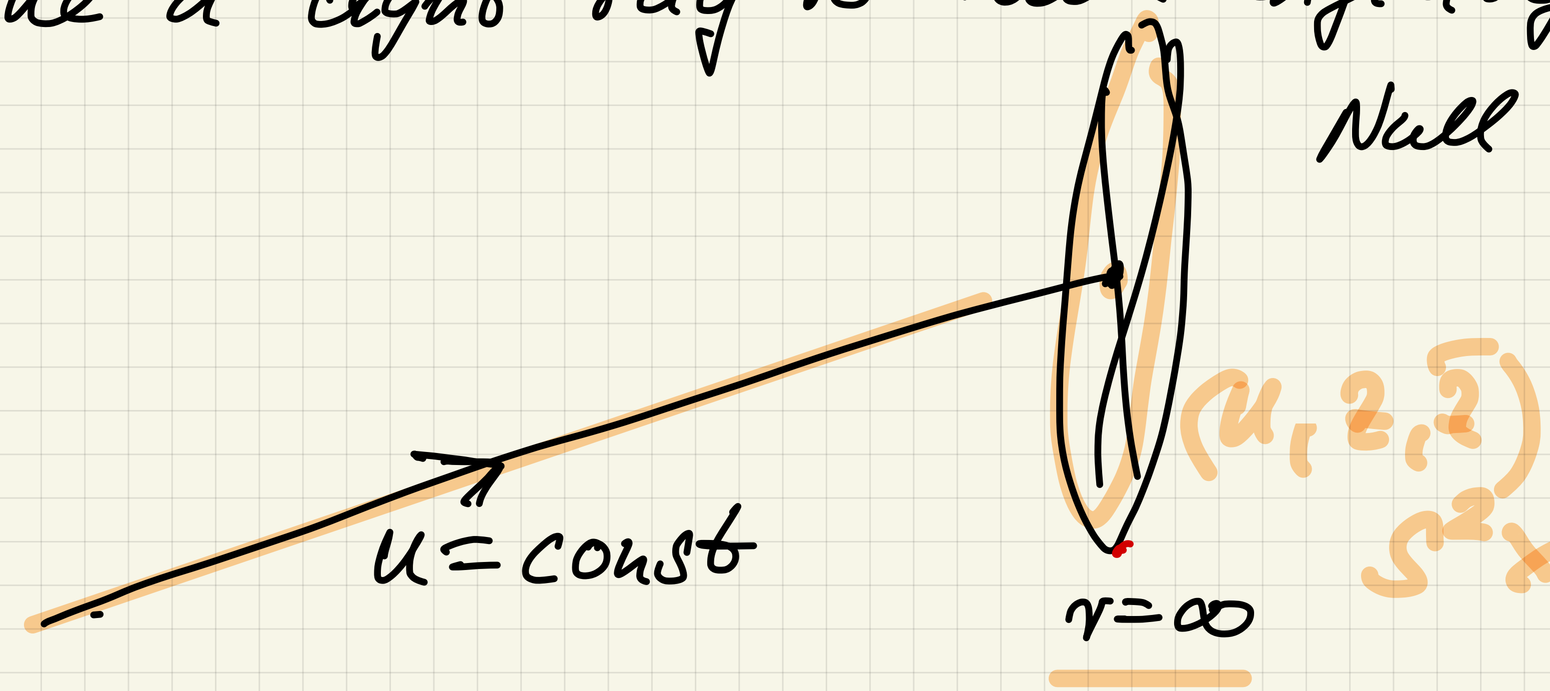
- indication for the existence of some gauge structure in quantum gravity

- New way of looking at quantum field theory and quantum gravity

- flat space-time holography

1. What is the celestial sphere?

Take a light-ray to Null infinity



natural arena for in and out states

retarded coordinate $u = x^0 - r$

Minkowski metric $ds^2 = -dt^2 + d\vec{x}^2$

$t = x^0$

$$x^0 = u + r$$

$$x^1 = r\,\frac{(z+\bar z)}{1+|z|^2}$$

$$x^2 = -i\,\frac{r(z-\bar z)}{1+|z|^2}$$

$$x^3 = \frac{r(1-|z|^2)}{1+|z|^2}$$

$$r^2 = \vec{x}^2$$

retarded or Bondi

Coordinates

$(u, r, z, \bar z)$

$$ds^2 = \underline{-du^2 - 2\,du\,dr} + \underbrace{\frac{4r^2}{(1+|z|^2)^2}}_{CS^2}\,dz\,d\bar z$$

2a Massless particle on celestial sphere

described by
- point $z \in CS^2$ at which it enters or exits the celestial sphere
- $SL(2,\mathbb{C})$ Lorentz quantum numbers $(h,\bar{h})$

at large r $\quad u = \text{const.}$

$$x^0 = u + r$$

large time x^0

lets take $u=0$:
$$\frac{x^\mu}{r} = \underbrace{\frac{1}{1+|z|^2}\left(1+|z|^2, z+\bar{z}, -i(z-\bar{z}), 1-|z|^2\right)}_{q^\mu}$$

$$=: \underline{\frac{1}{1+|z|^2}\, q^\mu}$$

$$\Rightarrow \quad q_\mu q^\mu = 0$$

$\hookrightarrow$ use it to define p^μ

$$\boxed{p^\mu := \frac{\omega}{1+|z|^2}\, q^\mu}$$

$$\omega = E$$

$$p^0 = E$$

$$\underbrace{p^\mu}_{\text{3 real parameters}} \longrightarrow \underbrace{(\omega, z, \bar{z})}_{\text{3 real parameters}}$$

2b) Particles $\Leftrightarrow$ operators

$$\phi_{h,\bar h}\left(\frac{az+b}{cz+d},\frac{\bar a\bar z+\bar b}{\bar c\bar z+\bar d}\right)=(cz+d)^{2h}(\bar c\bar z+\bar d)^{2\bar h}\times\phi_{h,h}(z,\bar z)$$

$h+\bar h=\Delta$ dimension

$h-\bar h=j$ spin

$$\Rightarrow (h,\bar h)=1/2\,(\Delta+j,\Delta-j)$$

plane wave in Minkowski $\exp\{\pm i p_\mu x^\mu\}$

$\downarrow$

boost eigenstates $\exp\{\pm i E u\}$

"State operator correspondence"

For massless particles the map between momentum-space and conformal basis is a Mellin transform:

e.g.: plane wave $\quad e^{\pm i p_\mu X^\mu}$

$$p^\mu, \quad p^2 = 0, \quad p^0 > 0$$

$$\varphi^{\pm}_{\Delta}(X, z, \bar z) = \int_0^\infty d\omega \; \omega^{\Delta - 1} \; e^{\pm \overbrace{\omega \underset{\uparrow z}{q_\mu}}^{p^\mu} X^\mu}$$

$$= \{ x^\mu q_\mu(z,\bar z) \}^{-\Delta}$$

Satisfies D=4 Klein-Gordon Equation

$y = 0$

$\Delta = 1 + i\lambda$ $\quad \lambda \in \mathbb{R}$

completeness

(Pasterski & Shao)

$\hookrightarrow$ principal conformal series

$\Rightarrow$ $h = \bar h = \Delta/2$

for vectors, spin 2 : polarization $\varepsilon_\pm^\mu$

functions on $z, \bar z$

3. Celestial Amplitudes

celestial amplitudes $\tilde{\mathcal{A}}$ of massless particles are obtained from momentum-space amplitudes $\mathcal{A}$ by <u>Mellin transforms</u> w.r.t. particle energies $\Delta_j = 1 + i\lambda_j$

n-point ampl.

$$\tilde{\mathcal{A}}_{\{\Delta_j\}}(\{z_j, \bar{z}_j\}) = \left(\prod_{j=1}^{\infty} \int_0^{\infty} d\omega_j \, \omega_j^{\Delta_j - 1}\right)$$

$$\times \delta^{(4)}\left(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=2}^{n} \omega_k q_k\right) \mathcal{A}(\{\omega_j, z_j, \bar{z}_j\})$$

↳ D=2 CFT correlators involve conformal wave packets

4. Preliminaries

$$p^\mu = \omega q^\mu \qquad q^\mu = \left(1+|z|^2, z+\bar{z}, -i(z-\bar{z}), 1-|z|^2\right)$$

> spinor helicity formalism

$$p^{\dot{\alpha}\beta} = p^\mu \bar{\sigma}_\mu{}^{\dot{\alpha}\alpha} = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix} = \tilde{\lambda}^{\dot{\alpha}} \lambda^{\beta}$$

$$\lambda^{\alpha} = \begin{pmatrix} p^0+p^3 \\ p^1-ip^2 \end{pmatrix} \frac{1}{\sqrt{p^0+p^3}}$$

$$\tilde{\lambda}^{\dot{\alpha}} = \begin{pmatrix} p^0+p^3 \\ p^1+ip^2 \end{pmatrix} \frac{1}{\sqrt{p^0+p^3}}$$

$$\overset{(*)}{=} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix} w$$

$$\Rightarrow \quad \tilde{\lambda}^{\dot{\alpha}} \equiv p]^{\dot{\alpha}} = w^{1/2} \begin{pmatrix} 1 \\ z \end{pmatrix}$$

$$\lambda^{\alpha} = \langle p^{\alpha} = \omega^{1/2} \begin{pmatrix} 1 \\ z \end{pmatrix}$$

$\Rightarrow$

$$\langle \lambda_i \lambda_j \rangle \equiv \varepsilon_{\alpha\beta} \lambda_i^{\alpha} \lambda_j^{\beta} = (\omega_i \omega_j)^{1/2} z_{ij}$$

$$z_{ij} := z_i - z_j$$

$$[\tilde{\lambda}_i \tilde{\lambda}_j] \equiv \varepsilon_{\dot\alpha\dot\beta} \tilde{\lambda}_i^{\dot\beta} \tilde{\lambda}_j^{\dot\alpha} = -(\omega_i \omega_j)^{1/2} \bar{z}_{ij}$$

$$s_{ij} = (p_i + p_j)^2 \underset{\uparrow}{=} 2 p_i p_j = \langle ij \rangle [ji]$$

massless

$$\Rightarrow \boxed{s_{ij} = \omega_i \omega_j \, |z_{ij}|^2}$$

conformal transformation

$$ad - bc = 1$$

$$z \to \frac{az + b}{cz + d} \qquad (\hat{=} \text{ Lorentz transformation})$$

$SL(2, \mathbb{C})$

6 generators

$\hookrightarrow \; L_0, L_{-1}, L_{+1} \; + hc$

$$z_i \to \frac{a z_i + b}{c z_i + d} \qquad z_j \to \frac{a z_j + b}{c z_i + d}$$

$$z_{ij} \to \frac{z_{ij}}{(c z_i + d)(c z_j + d)}$$

$$S_{ij} \overset{!}{=} \text{inv} \qquad \text{(scalar under Lorentz)}$$

$$= w_i w_j \, |z_{ij}|^2$$

$$\boxed{w_i \to w_i \, (c z_i + d)(\bar{c} \bar{z}_i + \bar{d})}$$

The celestial amplitudes $\tilde{A}_n$ transform under $SL(2,\mathbb{C})$ like the correlation functions of n conformal primary fields with weights $(h_j, \bar{h}_j)$

$$\tilde{A}_n\Big(\{z_j', \bar{z}_j', \underbrace{\Delta_j, y_j}_{h_j, \bar{h}_j}\}\Big) = \prod_{i=1}^{n} (c z_i + d)^{\Delta_i + y_i} (\bar{c}\bar{z}_i + \bar{d})^{\Delta_i - y_i}$$

$$\times \tilde{A}_n\big(\{z_j, \bar{z}_j, \Delta_j, y_j\}\big)$$

Quick:

$$\phi(z'(z), \bar{z}'(\bar{z})) = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z})$$

$$z \to \frac{az+b}{cz+d} =: z'$$

$$(cz+d)^{+2h} \sim \Delta + J \qquad (c\bar{z}+d)^{2\bar{h}} \sim \Delta - J$$

$$\frac{\partial z'}{\partial z} = \frac{1}{(cz+d)^2}$$

Tedious:

(i) $$A_n(-,-,+,\dots,+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}$$

$$= \frac{z_{13}^{3}}{z_{23}\, z_{34} \cdots z_{n1}} \frac{\omega_1 \omega_2}{\omega_3 \cdots \omega_n}$$

$$\hookrightarrow (cz_1+d)^{1-2} (\bar{c}\bar{z}_1 + d)^{1} \cdots$$

$$\cdots (cz_n+d)^{-1+2} (\bar{c}\bar{z}_n + d)^{-1} \; A_n(-,-,+,\dots,+)$$

$$= \prod_{i=1}^{n} (cz_i+d)^{y_i} (\bar{c}\bar{z}_i + d)^{-y_i} A_n(\dots)$$

(ii) $\mathcal{A}_n = \underbrace{\delta^{(4)}(\ldots)}_{\text{inv.}} A_n$

$$\delta^{(4)}\Big(\underbrace{\omega_1 q_1}_{p^1} + \ldots + q_n \omega_n\Big) = \text{inv.}$$

$$p^1 = (z+\bar{z})\,\omega \longrightarrow \frac{z+\bar{z}}{|cz+d|^2}(cz+d)^2\omega$$

$$= \text{inv.}$$

(iii)

$$\tilde{\mathcal{A}}_n\left(\{z_j, \bar{z}_j, \Delta_j, y_j\}\right) = \underbrace{\int_0^\infty d\omega_j\, \omega_j^{\Delta_j - 1}}_{\text{}} \mathcal{A}_n\left(\{z_j, \bar{z}_j, \omega_j, y_j\}\right)$$

$$\times \left\{ \prod_{j=1}^{n} (c z_i + d)^{\Delta_i + y_i} (\bar{c} \bar{z}_i + \bar{d})^{\Delta_i - y_i} \right\}$$

5. Amplitude Examples

$$\delta^{(4)}\Big(\sum_{i=1}^{n} \varepsilon_i w_i q_i\Big) \xrightarrow{n=3} \delta^{(4)}\big(\underline{w_1 q_1 + w_2 q_2} - \underline{w_3 q_3}\big)$$

$$\varepsilon_i = \pm 1$$

four delta functions lead to

$z_1 \bar{z}_1$ $\quad z_2 \bar{z}_2 \quad$ $z_3 \bar{z}_3$

$$
\begin{cases}
0 = \omega_1(1+|z_1|^2) + \omega_2(1+|z_2|^2) - \omega_3(1+|z_3|^2) \\
0 = \omega_1(z_1+\bar z_1) + \omega_2(z_2+\bar z_2) - \omega_3(z_3+\bar z_3) \\
0 = \omega_1(z_1-\bar z_1) + \omega_2(z_2-\bar z_2) - \omega_3(z_3-\bar z_3) \\
0 = \omega_1(1-z_1\bar z_1) + \omega_2(1-z_2\bar z_2) - \omega_3(1-z_3\bar z_3)
\end{cases}
$$

choice: $\bar z_{ij} = 0 \iff [ij] = 0 \quad \times$

$z_i \, , \bar z_i \in \mathbb{R} \rightarrow SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

$\Rightarrow \omega_1 + \omega_2 + \omega_3 = 0 \qquad ? \; | \; \omega_2 z_{21} - \omega_3 z_{31} = 0$

$$\omega_1 z_1 + \omega_2 z_2 - \omega_3 z_3 = 0 \quad \Big| \quad \omega_1 z_{12} - \omega_3 z_{32} = 0$$

$$\left.\begin{aligned} &\omega_1 \bar{z}_1 + \omega_2 \bar{z}_2 - \omega_3 \bar{z}_3 = 0 \\ &\omega_1 z_1 \bar{z}_1 + \omega_2 z_2 \bar{z}_2 - \omega_3 z_3 \bar{z}_3 = 0 \end{aligned}\right\} \text{nothing new for } \bar{z}_i = \bar{z}_j$$

Ansatz for $\delta^{(4)}(\ldots) \sim$

$$\sim \delta(\bar{z}_{13})\, \delta(\bar{z}_{23})\, \delta\!\left(\omega_1 - \omega_3 \frac{z_{32}}{z_{12}}\right)$$

$$\times\ \delta\!\left(\omega_2 - \omega_3 \frac{z_{31}}{z_{21}}\right)$$

$$\rightarrow (c z_2 + d)(\bar{c}\bar{z}_2 + \bar{d})$$

need factor: $\dfrac{4}{w_3^2 z_{23} z_{31}}$

$$\delta^{(4)}(\underline{\ldots\ldots}) = \frac{4}{w_3^2 z_{23} z_{31}} \, \delta\left(w_1 - w_3 \frac{z_{32}}{z_{12}}\right)$$

$$\times \; \delta\left(w_2 - w_3 \frac{z_{31}}{z_{21}}\right)$$

$$\times \; \delta(\bar{z}_{13}) \, \delta(\bar{z}_{23})$$

3-gluon $[ij]=0 \Leftrightarrow \bar{z}_{ij}=0$

$$A_3(-,-,+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} = \frac{\omega_1\omega_2}{\omega_3}\,\frac{z_{12}^3}{z_{13}z_{23}}$$

$$\mathcal{A}_3(-,-,+) = A_3(-,-,+)\cdot \delta^{(4)}(\ldots)$$

$$\tilde{\mathcal{A}}_3(-,-,+) = \left(\prod_{i=1}^{3}\int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1}\right)$$

$$\times\ \delta(\bar{z}_{13})\,\delta(\bar{z}_{23})\,\delta\!\left(\omega_1-\omega_3\frac{z_{32}}{z_{12}}\right)\delta\!\left(\omega_2-\omega_3\frac{z_{31}}{z_{21}}\right)$$

$$\times \frac{4}{\omega_3^2 z_{23} z_{31}} \times \frac{\omega_1 \omega_2}{\omega_3} \frac{z_{12}^3}{z_{12} z_{23}}$$

$$= 4\,\delta(\bar z_{13})\,\delta(\bar z_{23}) \int_0^\infty d\omega_3 \left(\omega_3 \frac{z_{32}}{z_{12}}\right)^{\Delta_1} \times$$

$$\times \left(\omega_3 \frac{z_{31}}{z_{21}}\right)^{\Delta_2} \omega_3^{\Delta_3 - 1 - 1}\, \omega_3^{-3} \frac{z_{12}^3}{z_{23}^2 z_{13}^2} (-1)$$

$$= 4\,\delta(\bar z_{13})\,\delta(\bar z_{23})\,\delta(\Delta_1 + \Delta_2 + \Delta_3 - 3)\,(2\pi)$$

$$x \quad z_{23}^{\Delta_1-2} \quad z_{31}^{\Delta_2-2} \quad z_{21}^{-\Delta_1-\Delta_2+3}$$

$$\int_0^\infty d\omega_3 \quad \omega_3^{\Delta_1+\Delta_2+\Delta_3-1-3}$$

$$= (2\pi)\ \delta(\Delta_1+\Delta_2+\Delta_3-3)$$

$$\int_0^\infty dx \quad x^{s-1} = \delta(s)$$

$$\langle \phi_1(z_1)\ \phi_2(z_2)\ \phi_3(z_3) \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\ z_{13}^{h_1+h_3-h_2}\ z_{23}^{h_2+h_3-h_1}}$$

$$\left.\begin{aligned} h_1 + h_2 - h_3 &= -i\lambda_3 - 1 \\ h_1 + h_3 - h_2 &= -i\lambda_2 + 1 \\ h_2 + h_3 - h_1 &= -i\lambda_1 + 1 \end{aligned}\right\} \Rightarrow \begin{aligned} h_1 &= + \frac{i\lambda_1}{2} \\ h_2 &= + \frac{i\lambda_2}{2} \\ h_3 &= 1 + \frac{i\lambda_3}{2} \end{aligned}$$

$$\Delta_i = 1 + i\lambda_i = h_i + \bar{h}_i$$

$$\bar{h}_1 = 1 + i/_2\, \lambda_1$$

$$\bar{h}_2 = 1 + i/_2\, \lambda_2$$

$$\bar{h}_3 = i/_2\, \lambda_3$$

$$\gamma_i = h_i - \bar{h}_i$$

3-graviton

MHV amplitude

$$M_3(--,--,++) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2 \langle 23\rangle^2} =$$

$$= \frac{\omega_1^2 \omega_2^2}{\omega_3^2} \frac{z_{12}^6}{z_{13}^2 z_{23}^2}$$

$$\tilde{\mathcal{M}}_3(--,--,++) = 4\,\delta(\bar{z}_{13})\,\delta(\bar{z}_{23})$$

$$\times \frac{1}{z_{23}\, z_{31}} \qquad \frac{z_{21}^{6}}{z_{31}^{2}\, z_{23}^{2}}$$

$$\times \int_0^\infty d\omega_3 \quad \omega_3^{\Delta_3 - 1} \left(\omega_3 \, \frac{z_{23}}{z_{21}} \right)^{\Delta_1 + 1} \left(\omega_3 \, \frac{z_{31}}{z_{21}} \right)^{\Delta_2 + 1}$$

$$\omega_3^{-2-2}$$

$$= 4\, \delta(\bar{z}_{13})\, \delta(\bar{z}_{23}) \quad \delta\left(\underline{\Delta_1 + \Delta_2 + \Delta_3 - 2}\right)$$

$$z_{23}^{\Delta_1 - 2} \quad z_{31}^{\Delta_2 - 2} \quad z_{21}^{-\Delta_1 - \Delta_2 + 4}$$

$$\Rightarrow \quad h_1 = -\tfrac{1}{2} + \frac{i\lambda_2}{2} \qquad \bar{h}_1 = \tfrac{3}{2} + \tfrac{i}{2}\lambda_1$$

$$h_2 = -\tfrac{1}{2} + \frac{i\lambda_2}{2} \qquad \bar{h}_2 = \tfrac{3}{2} + \tfrac{i}{2}\lambda_2$$

$$h_3 = +\tfrac{3}{2} + \frac{i\lambda_3}{2} \qquad \bar{h}_3 = -\tfrac{1}{2} + \tfrac{i}{2}\lambda_3$$

The main difference between gravitational and gauge amplitudes is the energy integral $d\omega_3$

EYM - amplitudes

$$M_3^{EYM}(-,-,+) = \frac{\langle 12\rangle^4}{\langle 23\rangle^2}$$

$$= \frac{\omega_1^2 \omega_2}{\omega_3} \frac{z_{12}^4}{z_{23}^2}$$

$$\hookrightarrow \tilde{\mathcal{M}}_3(-,-,+) = \dots \delta(\Delta_1+\Delta_2+\Delta_3-2)$$