Celestial Amplitudes and Asymptotic Symmetries (II)

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Recap

in momentum basis: plane waves with momentum $p = \omega q(z)$

in conformal basis: conformal primary wave functions $\Delta = h + \bar{h} \in \mathbf{C}$

construct complete set of on-shell wave functions in D=4:

solves D=4 wave (Maxwell) equations and transforms as SL(2,Z) conformal primaries

Pasterski, Shao arXiv:1705.01027

bases	plane waves	conformal primary wavefunctions
vector fields	$A_{\mu\ell}(x;p) = \epsilon_{\mu\ell}(p)\ exp\{\mp i p_\mu x^\mu\}$	$V^{\Delta\pm}_{\mu J}(x;z,\bar{z}) = (\partial_J q_\mu)\ (-q_\mu x^\mu \mp i\epsilon)^{-\Delta}$
3 continuous parameters	p^μ $p^2 = 0, \quad p > 0$	$\Delta = 1 + i\lambda\ (\lambda \in \mathbf{R})$ $z \in CS^2$
2 discrete parameters	$4d$ helicity $\ell = \pm 1$ incoming vs. outgoing	$2d$ spin $J = \pm 1$ incoming vs. outgoing

in the massless case the change of basis is furnished by **Mellin transform** of plane wave (or plus a shadow transform):

$$\boxed{\begin{aligned} V^{\Delta\pm}_{\mu J}(x^\mu; z, \bar{z}) &= \frac{\partial_J q_\mu}{\sqrt{2}} \int_0^\infty d\omega\ \omega^{\Delta-1} e^{\pm i\omega q\cdot x - \epsilon\omega} \\ &= (\mp i)^\Delta \frac{\Gamma(\Delta)}{\sqrt{2}} \frac{\partial_J q_\mu}{(-q_\mu x^\mu \mp i\epsilon)^\Delta} \end{aligned}}$$

specified by x and conformal dimension

$\Delta = 1 + i\lambda, \quad \lambda \in \mathbf{R}$

no dependence on D=4 momentum p^μ

$$\partial_J q^\mu = \begin{cases} \partial_z q^\mu = \sqrt{2}\epsilon^\mu_+(q) = (\bar{z}, 1, -i, -\bar{z}) \\ \partial_{\bar{z}} q^\mu = \sqrt{2}\epsilon^\mu_-(q) = (z, 1, +i, -z) \end{cases}$$

Pasterski, Shao, Strominger, 2017

n-point amplitude on celestial sphere

$$\mathcal{A}(\{p_i, \epsilon_j\}) = i(2\pi)^4\ \delta^{(4)}\left(p_1 + p_2 - \sum_{k=3}^{n} p_k\right) A(\{p_i, \epsilon_j\})$$

Celestial amplitudes $\tilde{\mathcal{A}}$ of massless particles are obtained
from momentum-space amplitudes $\mathcal{A}$
by Mellin transforms w.r.t. particle energies $\Delta_j = 1 + i\lambda_j$

$$\boxed{\tilde{\mathcal{A}}_{\{\Delta_l\}}(z_l, \bar{z}_l) = \left(\prod_{l=1}^{n}\int_0^\infty \omega_l^{\Delta_l - 1}\ d\omega_l\right)\ \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^{N}\omega_k q_k) \times A(\omega_n, z_n, \bar{z}_n)}$$

D=2 CFT correlators involve conformal wave packets

Three-point Amplitudes

(i) Mostly-plus three-gluon amplitude

$$\mathcal{M}(-,-,+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} = \frac{\omega_1\omega_2}{\omega_3}\frac{z_{12}^3}{z_{13}z_{23}}$$

$$\boxed{\tilde{\mathcal{A}}(-,-,+) = \frac{8\pi}{z_{21}}\left(\frac{z_{23}}{z_{21}}\right)^{\Delta_1-2}\left(\frac{z_{31}}{z_{21}}\right)^{\Delta_2-2}\delta(\bar{z}_{13})\,\delta(\bar{z}_{23}) \times\delta(\Delta_1+\Delta_2+\Delta_3-3)}$$

$$\left.\begin{aligned} h_1 &= \frac{i}{2}\lambda_1, & \bar{h}_1 &= 1+\frac{i}{2}\lambda_1,\\ h_2 &= \frac{i}{2}\lambda_2, & \bar{h}_2 &= 1+\frac{i}{2}\lambda_2,\\ h_3 &= 1+\frac{i}{2}\lambda_3, & \bar{h}_3 &= \frac{i}{2}\lambda_3, \end{aligned}\right\}$$

$$\Delta_n = 1 + i\lambda_n$$

$$J_1 = J_2 = -1, \quad J_3 = +1$$

(ii) Mostly-plus three-graviton amplitude

$$\mathcal{M}(--,--,++) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2} = \frac{\omega_1^2\omega_2^2}{\omega_3^2}\frac{z_{12}^6}{z_{13}^2 z_{23}^2}$$

$$\boxed{\begin{aligned}\tilde{\mathcal{A}}(--,--,++) = 8\pi\left(\frac{z_{23}}{z_{21}}\right)^{\Delta_1-2}\left(\frac{z_{31}}{z_{21}}\right)^{\Delta_2-2}\delta(\bar{z}_{13})\ \delta(\bar{z}_{23})\\ \times\,\delta(\Delta_1+\Delta_2+\Delta_3-2)\end{aligned}}$$

$$\left.\begin{aligned} h_1 &= -\frac{1}{2}+\frac{i}{2}\lambda_1, & \bar{h}_1 &= \frac{3}{2}+\frac{i}{2}\lambda_1,\\ h_2 &= -\frac{1}{2}+\frac{i}{2}\lambda_2, & \bar{h}_2 &= \frac{3}{2}+\frac{i}{2}\lambda_2,\\ h_3 &= \frac{3}{2}+\frac{i}{2}\lambda_3, & \bar{h}_3 &= -\frac{1}{2}+\frac{i}{2}\lambda_3 \end{aligned}\right\}$$

$$\Delta_n = 1 + i\lambda_n$$

$$J_1 = J_2 = -2,$$

$$J_3 = +2$$

(iii) Mostly-plus EYM (one graviton, two gluon) amplitude

$$\mathscr{M}(\,-\,-\,,-\,,+\,) = \frac{\langle 12\rangle^4}{\langle 23\rangle^2} = \frac{\omega_1^2\omega_2}{\omega_3}\frac{z_{12}^4}{z_{23}^2}$$

$$\boxed{\begin{gathered}\tilde{\mathscr{A}}(\,-\,-\,,-\,,+\,) = 8\pi\left(\frac{z_{23}}{z_{21}}\right)^{\Delta_1-2}\left(\frac{z_{31}}{z_{21}}\right)^{\Delta_2-1}\delta(\bar{z}_{13})\ \delta(\bar{z}_{23})\\ \times\,\delta(\Delta_1+\Delta_2+\Delta_3-2)\end{gathered}}$$

Four-gluon celestial amplitude

$$P_i = \varepsilon_i \omega_i q_i \qquad i = 1, \dots, 4$$

$$\varepsilon_i = \pm 1$$

$$\delta^{(4)}\left(\sum_{i=1}^{4} \varepsilon_i \omega_i q_i\right) = \frac{1}{4\omega_3} \times$$

$$\times\, \delta\left(\omega_1 - \omega_3 \frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23}\bar{z}_{34}}{z_{12}\bar{z}_{14}}\right)$$

$$\times\, \delta\left(\omega_2 - \omega_3 \, {}^{\varepsilon_3}/_{\varepsilon_2} \frac{z_{13}\bar{z}_{34}}{z_{12}\bar{z}_{42}}\right)$$

$$\times\ \delta\left(\omega_4 - \omega_3 \frac{\varepsilon_3}{\varepsilon_4} \frac{\bar{z}_{13} z_{23}}{\bar{z}_{14} z_{42}}\right)$$

$$\times\ \delta\left(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}\right)$$

$$A_4(-,-,+,+) = \frac{z_{12}^{3}}{z_{23}\, z_{34}\, z_{41}} \quad \frac{\omega_1 \omega_2}{\omega_3 \omega_4}$$

$$\tilde{A}_4(-,-,+,+) = \left(\prod_{i=1}^{4} \int_0^\infty d\omega_i\, \omega_i^{\Delta_i - 1}\right) \frac{z_{12}^{3}}{z_{23} z_{34} z_{41}} \quad \frac{\omega_1 \omega_2}{\omega_3 \omega_4}$$

$$\times\ \delta^{(4)}(\ldots)$$

$$= \frac{1}{4} \left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23} \bar{z}_{34}}{z_{12} \bar{z}_{14}}\right)^{\Delta_1} \left(\frac{\varepsilon_3}{\varepsilon_2} \frac{z_{13} \bar{z}_{34}}{z_{12} \bar{z}_{42}}\right)^{\Delta_2}$$

$$\times \left(\frac{\varepsilon_3}{\varepsilon_4} \frac{\bar{z}_{13} z_{23}}{\bar{z}_{14} z_{42}} \right)^{\Delta_4 - 2} \qquad \delta\left(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34} \right)$$

$$\times \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \qquad \times \theta\left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23} \bar{z}_{34}}{z_{12} \bar{z}_{14}} \right) \theta\left(\frac{\varepsilon_3}{\varepsilon_2} \frac{z_{13} \bar{z}_{34}}{z_{12} \bar{z}_{42}} \right) \theta\left(\frac{\varepsilon_3}{\varepsilon_4} \frac{\bar{z}_{13} z_{23}}{\bar{z}_{14} z_{42}} \right)$$

$$\underbrace{\times \int_0^\infty d\omega_3 \, \omega_3^{\Delta_3 - 1 \underset{\substack{\uparrow \\ \delta^{(4)}}}{-1} \underset{\substack{\uparrow \\ \Delta_4}}{-1} + \Delta_1 + \Delta_2 + \Delta_4 - 2}}_{2\pi \, \delta\left(\underbrace{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - 4}_{\to \, \delta(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} \right)}$$

Gauge Amplitudes

four-gluon amplitude:

$$\tilde{\mathscr{A}}_4(-,-,+,+) = 8\pi\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$

$$\times\, r^{\frac{5}{3}}\,(r-1)^{\frac{2}{3}}\,\delta\left(-4+\sum_{i=1}^{4}\Delta_i\right)$$

$$r = \frac{z_{12}\,z_{34}}{z_{23}\,z_{41}}$$

conformal invariant
cross-ratio on CS^2

$$r^{-1} = \sin^2\left(\frac{\theta}{2}\right)$$

Pasterski, Shao, Strominger (2017)

$$h_1 = \frac{i}{2}\lambda_1,\;\; h_2 = \frac{i}{2}\lambda_2,\;\; h_3 = 1+\frac{i}{2}\lambda_3\;,h_4 = 1+\frac{i}{2}\lambda_4$$

$$\bar{h}_1 = 1+\frac{i}{2}\lambda_1,\;\; \bar{h}_2 = 1+\frac{i}{2}\lambda_2,\;\; \bar{h}_3 = \frac{i}{2}\lambda_3\;,\bar{h}_4 = \frac{i}{2}\lambda_4$$

higher-point: involve Gaussian hypergeometric functions like string amplitudes

Schreiber, Volovich, Zlotnikov (2017)

Graviton Amplitudes

four-graviton amplitude:

$$\tilde{\mathcal{A}}_4(--,--,++,++) = 2\pi\, \delta(r-\bar{r})\, \theta(r-1) \left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\, \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j} \right)$$

$$\times\, r^{\frac{11}{3}-\frac{\beta}{3}}\, (r-1)^{-\frac{1}{3}-\frac{\beta}{3}}\, \delta\left(-2 + \sum_{i=1}^{4} \Delta_i \right)$$

St.St., Taylor (2018)

$$h_1 = -\frac{1}{2} + \frac{i}{2}\lambda_1,\;\; h_2 = -\frac{1}{2} + \frac{i}{2}\lambda_2,\;\; h_3 = \frac{3}{2} + \frac{i}{2}\lambda_3\;, h_4 = \frac{3}{2} + \frac{i}{2}\lambda_4$$

$$\bar{h}_1 = \frac{3}{2} + \frac{i}{2}\lambda_1,\;\; \bar{h}_2 = \frac{3}{2} + \frac{i}{2}\lambda_2,\;\; \bar{h}_3 = -\frac{1}{2} + \frac{i}{2}\lambda_3\;, \bar{h}_4 = -\frac{1}{2} + \frac{i}{2}\lambda_4$$

$$\beta = 2 - \frac{1}{2}\sum_{i=1}^{4}\Delta_i$$

- *first calculation of graviton amplitude in the conformal basis*
- *important for the soft graviton theorems* $\Delta \to 1, 0, \ldots$ *in celestial basis*

no holomorphic factorization (due to supertranslation operator P)

6. Translation Symmetry

$$\tilde{\mathcal{A}} \sim \left(\prod_{j=1}^{n} \int_0^\infty \omega_j^{\overset{,,h_j + \bar{h}_j}{\Delta_j} - 1} d\omega_j \right) \delta^{(4)}(\ldots) A(\ldots)$$

$$P^0 = \frac{1}{2}(1+|z|^2)\,\omega$$

↑

$$e^{\frac{1}{2}(\partial_u + \partial_{\bar{u}})/2}$$

Note: $e^{c\partial_x} f(x) = f(x+c)$

$$e^{1/2\,(\partial_h+\partial_{\bar h})}$$

$$\begin{tikzcd} f(h,\bar h) \arrow[d] & = & f(h+1/2,\ \bar h+1/2) \arrow[d] \\ \omega^{h+\bar h-1} & & \omega^{h+\bar h-1}\cdot\omega \end{tikzcd}$$

$$p^\mu = q^\mu \cdot \omega$$

$$\times \quad p^0 = 1/2\,(1+|z|^2)\ e^{1/2\,(\partial_h+\partial_{\bar h})}$$

$$p^1 = 1/2\,(z+\bar z)\ e^{1/2\,(\partial_h+\partial_{\bar h})}$$

$$p^2 = -i/2\,(z-\bar z)\ e^{1/2\,(\partial_h+\partial_{\bar h})}$$

$$\times \quad p^3 = 1/2\,(1-|z|^2)\ e^{1/2\,(\partial_h+\partial_{\bar h})}$$

$$p^0 + p^3 = e^{1/2(\partial_u + \partial_{\bar{u}})}$$

$$\boxed{\begin{aligned}(p_i^0 + p_i^3)\, \tilde{\mathcal{A}}(\{z_i, \bar{z}_i, \Delta_i, J_i\}) \\ = \tilde{\mathcal{A}}(\{z_i, \bar{z}_i, \Delta_i + 1, J_i\})\end{aligned}}$$

7. Conformally soft theorem in gauge theory

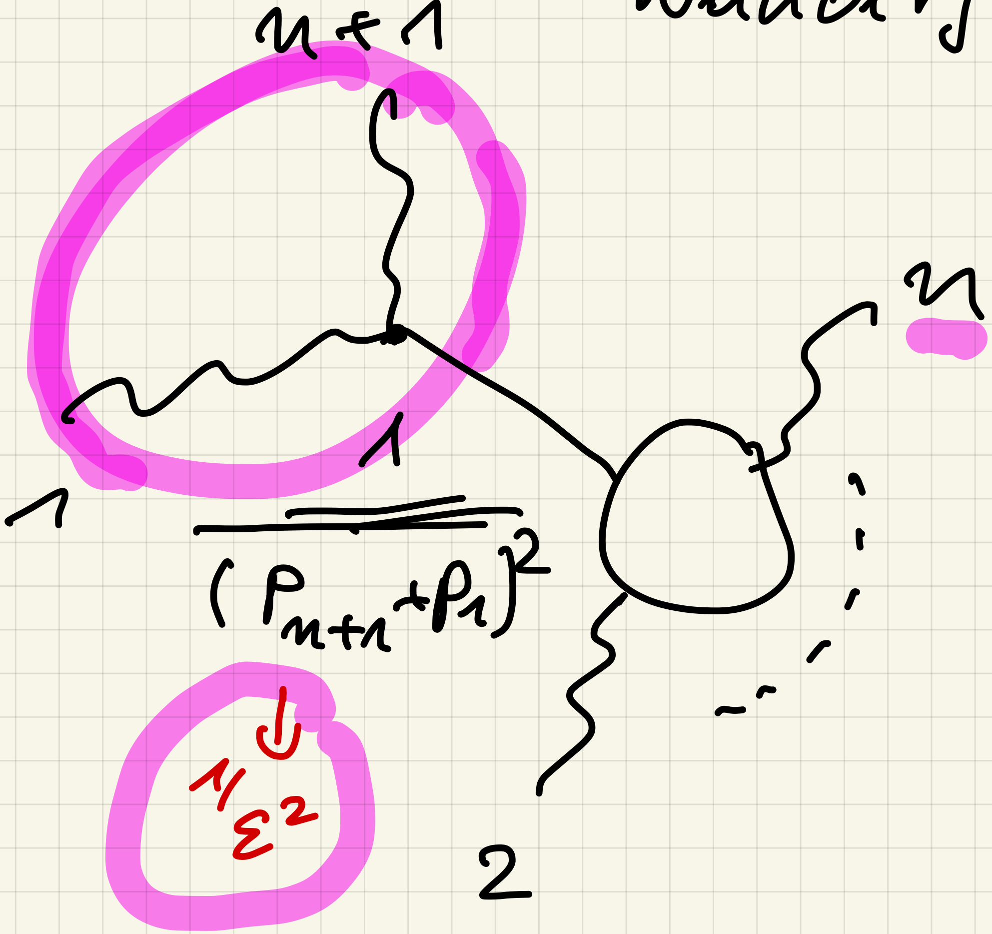
In usual QFT soft theorems E→0 play an important role in consistency and structure of amplitudes

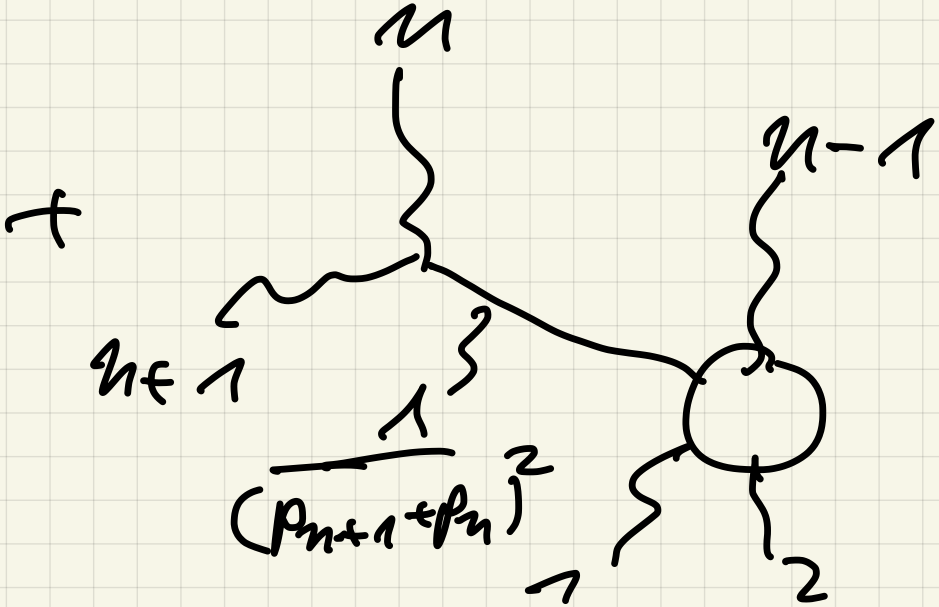
D=4:

$$A_{n+1} \xrightarrow{p_{n+1}\to 0} \left(\underline{\frac{1}{\varepsilon^2} \boxed{S_{YM}^{(0)}} + \boxed{\frac{1}{\varepsilon}}} \boxed{S_{YM}^{(1)}} + \dots \right) A_n$$

Weinberg

$p_{n+1} \to \varepsilon \to 0$





Soft theorems imply Ward identities for asymptotic symmetries

$\underline{D=2}$ on CS^2 Discuss 4-gluon

✓ $\Delta_4 \to 1 \quad \hat{=} \quad \lambda_4 \to 0$

$ \overset{\displaystyle \Delta_4}{} \; \| $

$1+i\lambda_4$

$$\lim_{\Delta_4 \to 1} \tilde{A}_4(-,-,+,+) \overset{(*)}{=}$$

$$\delta(x) = \lim_{\Delta_4 \to 1} \frac{\Delta_4 - 1}{2} |x|^{\Delta_4 - 2}$$

$$\underset{=}{(*)} \quad \frac{2}{\Delta_4 - 1}\left(\frac{2\pi}{4}\right) \delta(\Delta_1 + \Delta_2 + \Delta_3 - 3)$$

$$\delta\left(\frac{\varepsilon_3}{\varepsilon_4} \frac{\bar z_{13} z_{23}}{\bar z_{14} z_{42}}\right) \left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23} \bar z_{34}}{z_{12} \bar z_{14}}\right)^{\Delta_1} \left(\frac{\varepsilon_3}{\varepsilon_2} \frac{z_{13}}{z_{12}} \frac{\bar z_{34}}{\bar z_{42}}\right)^{\Delta_2}$$

$$\times \; \delta\left(z_{12} z_{34} \boxed{\bar z_{13}} \bar z_{24} - z_{13} z_{24} \bar z_{12} \bar z_{34}\right)$$

$$\times \; \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \quad \theta(\underline{\ldots}) \, \theta(\ldots) \, \theta(\ldots) \; \underset{=}{(*)}$$

now assuming z_i & $\bar{z}_i$ to be independent real variables

× with $z_{23} \neq 0$

$$\delta\left(\frac{\varepsilon_3}{a_4} \frac{\bar{z}_{13} z_{23}}{\bar{z}_{14} z_{42}} \right) = \mathrm{sgn}\,(\bar{z}_{14} z_{42} z_{23}) \frac{z_{42} \bar{z}_{14}}{z_{23}} \delta(\bar{z}_{13})$$

$$\delta\left(\qquad\qquad \right) \longrightarrow \delta\left(z_{13} z_{24} \bar{z}_{12} \bar{z}_{34} \right)$$

$$(*) = \frac{\pi}{\Delta_4 - 1} \operatorname{sgn}\left(\bar{z}_{14} z_{42} z_{23}\right) \frac{z_{42} \bar{z}_{14}}{z_{23}} \times$$

$$\times \frac{z_{12}^{3}}{z_{23} z_{34} z_{41}} \delta(\bar{z}_{13}) \, \delta\left(z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}\right)$$

$$\theta(0) \left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23}}{z_{12}}\right)^{\Delta_1} \left(\frac{\varepsilon_3}{\varepsilon_2} \frac{z_{13}}{z_{12}} \frac{\bar{z}_{34}}{\bar{z}_{42}}\right)^{\Delta_2}$$

$$\Theta\left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23}}{z_{12}}\right) \theta\left(\frac{\varepsilon_3}{\varepsilon_2} \frac{z_{13} \bar{z}_{34}}{z_{14} \bar{z}_{42}}\right) \quad (*) =$$

we further assume $z_{13} \neq 0$

we can furthermore $z_{24} \neq 0$

$\bar{z}_{34} \neq 0$

(no collinear singularity)

to obtain non-trivial 3-pt ampl.

$$\delta(z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) = \mathrm{sgn}(z_{13} z_{24} \bar{z}_{34})\, \delta(\bar{z}_{12})$$

$$\times \frac{1}{z_{13} z_{24} z_{34}}$$

~~a~~)

$$= \frac{1}{2} \frac{\pi}{\Delta_4 - 1} \operatorname{sgn}(z_{31} z_{23}) \frac{\operatorname{sgn}(\bar z_{14} \bar z_{34})}{\delta(\bar z_{13})} = 1$$

$$\delta(\bar z_{12})\, \delta(\bar z_{13}) \frac{z_{12}^3}{z_{23}^2 z_{34} z_{31} z_{41}} \underset{1}{\underset{\shortparallel}{\frac{\bar z_{14}}{\bar z_{34}}}} \ \delta(\bar z_{13})$$

$$\times \left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23}}{z_{12}} \right)^{\Delta_1} \left(\frac{\varepsilon_3}{\varepsilon_2} \frac{z_{31}}{z_{12}} \right)^{\Delta_2}$$

$$\Theta\left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23}}{z_{12}} \right) \Theta\left(\varepsilon_3 / \varepsilon_2 \ \frac{z_{31}}{z_{12}} \right)$$

$$\times \ \delta(\Delta_1+\Delta_2+\Delta_3-3)$$

Recall:

$$\tilde{A}_3(-,-,+) = \left(\frac{\varepsilon_3\, z_{23}}{\varepsilon_1\, z_{12}}\right)^{\Delta_1-2} \left(\frac{\varepsilon_3\, z_{31}}{\varepsilon_2\, z_{12}}\right)^{\Delta_2-1}$$

$$\times \ \frac{1}{z_{12}}\, \delta(\bar{z}_{13})\, \delta(\bar{z}_{23})$$

$$\times \ \mathrm{sgn}(z_{31}\, z_{23})\, (-\pi)\, \delta(\Delta_1+\Delta_2+\Delta_3-3)$$

$$\theta\left(\frac{\varepsilon_3}{\varepsilon_1}\, \frac{z_{23}}{z_{12}}\right) \theta\left(\frac{\varepsilon_3}{\varepsilon_2}\, \frac{z_{31}}{z_{12}}\right)$$

$$\boxed{\Rightarrow \lim_{\Delta_4 \to 1} \tilde{A}_4(-,-,+,\underset{\uparrow}{+}) = -\frac{1}{2}\,\frac{1}{\Delta_4 - 1}\,\frac{z_{31}}{z_{34}\,z_{41}}\,\tilde{A}_3(-,-,+)}$$

$$\uparrow$$

$$S_{YM}^{(0)} = \frac{\langle \dot{3} 1 \rangle}{\langle 3 \underset{\cdot}{4} \rangle \langle 4 \underset{\cdot}{\underline{1}} \rangle}$$

$\Delta_4 \to 1$

Note: $$\int_0^\infty dw_4 \; w_4^{\Delta_4 - 1} \left\{ w_4^{-1} S_{YM}^{(0)} + \underline{w_4^{0}} S_{YM}^{(1)} + \dots \right\}$$

taking $\Delta_4 \to 1$

extract $\frac{1}{\Delta_4 - 1}$ $\quad w_4 \to 0$

$$\tilde{I}_{n+1} \longrightarrow \left(\underbrace{w_{n+1}^{-1} \; S_{YM}^{(0)}}_{\Delta_s \to 1} + \underbrace{w_{n+1}^{0} \; S_{YM}^{(1)}}_{\Delta_s \to 0} + \dots \right)$$

$\times \tilde{A}_n$

8. Operator Product Expansion

understand celestial conformal field theory (CCFT)

↪ OPEs for conformal primaries

$z_i \to z_j$ on CS^2

$$p_i^\mu = \omega_i \, q_i^\mu(z_i)$$

$$p_j^\mu = \omega_j \, q_j^\mu(z_j)$$

$z_i \to z_j$

$\Rightarrow p_i \parallel p_j$

collinear limit

$$\boxed{z_i \to z_j \iff \text{collinear limit of } p_i \,\&\, p_j}$$

$D=4$

Recall collinear limit in gauge amplitudes

$$p_i = x P + \dots$$

$$p_j = (1-x) P + \dots$$

$$p_i + p_j = P \qquad p_i \parallel p_j \parallel P$$

$$\sim \frac{1}{(p_i+p_j)^2} \sim \frac{1}{P^2}$$

$$A_n(1,\dots,i^+,j^+,\dots,n) \longrightarrow \frac{1}{\langle ij\rangle}\frac{1}{\sqrt{x(1-x)}}$$

$$\times A_{n-1}(1,\dots,\underline{P}^+,\dots,n)$$

$\underline{D=2!}$

$$p_i = xP \qquad\qquad \omega_i q_i \overset{!}{=} xP \equiv x\,\omega_p q_p$$

$$p_j = (1-x)P \iff \omega_j q_j = (1-x)P \equiv (1-x)\cdot \omega_p q_p$$

$$P := \omega_p q_p$$

★ $q_i = q_j = q_p$

$$\Rightarrow \quad x = \frac{\omega_i}{\omega_p} \qquad 1 - x = \frac{\omega_j}{\omega_p}$$

$$\omega_p = \omega_i + \omega_j \quad \times$$

$$A_n(1, \ldots, \underset{\substack{\nearrow \\ i}}{(n-1)^+}, \underset{\substack{\nearrow \\ j}}{n^+}) = \underline{\underline{\frac{1}{\langle (n-1) n \rangle}} \frac{1}{\sqrt{x(1-x)}}}$$

$$\times\, A_n(1, \ldots, n-2, P^+)$$

in celestial coords:

$$x = \frac{\omega_{n-1}}{\omega_P}$$

$$1-x = \frac{\omega_n}{\omega_P}$$

$$\omega_P = \omega_n + \omega_{n-1}$$

factor $\frac{1}{z_{n-1,n}} \frac{1}{(\omega_{n-1}\omega_n)^{1/2}}$

$$\times \frac{\omega_P}{(\omega_{n-1}\omega_n)^{1/2}}$$

$$\Rightarrow A_n(1,\ldots,(n-1)^+,n^+) = \frac{1}{z_{n-1,n}} \frac{\omega_P}{\underline{\omega_{n-1}\omega_n}}$$

$$\times A_n(1,\ldots,n-2,P^+)$$ ✗

$$\delta^{(4)}\left(\sum_{i=1}^{n} \omega_i q_i\right) = \delta^{(4)}\left(\sum_{i=1}^{n-2} \omega_i q_i + \right.$$

$$\left. + \underbrace{\omega_{n-1} q_{n-1} + \omega_n q_n}_{\omega_p q_p} \right)$$

consider celestial amplitude $\tilde{A}$

$$\tilde{A}_n\left(1, \dots, (n-1)^+, n^+\right) = \left(\prod_{i=1}^{n-2} \int_0^\infty d\omega_i \; \omega_i^{\overbrace{\Delta_i - 1}^{i\lambda_i}}\right)$$

$$\times \int_0^\infty d\omega_{n-1}\, \omega_{n-1}^{-1+i\lambda_{n-1}} \int_0^\infty d\omega_n\, \omega_n^{-1+i\lambda_n}$$

$$\times \frac{\omega_p}{2_{n-1,n}} \quad A_{n-1}(1,\dots,n-2,P^+) \times d^{(\epsilon)}(\dots)$$

Use: $\int_0^\infty d\omega_{n-1}\, \omega_{n-1}^{-1+i\lambda_{n-1}} \int_0^\infty d\omega_n\, \omega_n^{-1+i\lambda_n} \quad \omega_p$

$$\omega_p = \omega_{n-1} + \omega_n$$

$$\int_0^\infty d\omega_p \int_0^{\omega_p} d\omega_n \,(\omega_p - \omega_n)^{-1+i\lambda_{n-1}} \, \omega_n^{-1+i\lambda_n}$$

$$\times \underline{\omega_p}$$

$$\tilde{\omega}_n = \omega_n / \omega_p$$

$$= \int_0^\infty d\omega_p \; \omega_p^{\underbrace{+1-1+i\lambda_{n-1}-1+i\lambda_n}_{=: i\lambda_p - 1}}$$

$$\int_0^1 d\tilde{\omega}_n \; (1-\tilde{\omega}_n)^{-1+i\lambda_{n-1}} \; \tilde{\omega}^{-1+i\lambda_n}$$

✗ $\lambda_p := \lambda_n + \lambda_{n-1}$

$$= \frac{\Gamma(i\lambda_{n-1})\,\Gamma(i\lambda_n)}{\Gamma(i\lambda_{n-1}+i\lambda_n)} \int_0^\infty d\omega_p \; \omega_p^{\,i\lambda_p - 1}$$

$$\Rightarrow \boxed{\tilde{A}_n(1,\dots,(n-1)^+,n^+) = \frac{B(i\lambda_{n-1},i\lambda_n)}{z_{n-1}-z_n} \times \tilde{A}_{n-1}(1,\dots,n-2,P)}$$

$$\lambda_p = \lambda_n + \lambda_{n-1}$$

↪ Extract from correlator local behaviour on CS^2 for $z_n \to z_{n-1}$

$$\boxed{O_{\lambda_{n-1},+}(z_{n-1}, \bar{z}_{n-1})\, O_{\lambda_n,+}(z_n, \bar{z}_n) = \frac{B(i\lambda_{n-1}, i\lambda_n)}{z_{n-1}-z_n}\, O_{\underset{\lambda_{n-1}+\lambda_n}{\underset{\|}{\lambda_p}},+}(z_p, \bar{z}_p)}$$

in full color sum:

$$\boxed{\mathcal{O}^{a}_{\Delta_1,t}(z,\bar z)\,\mathcal{O}^{b}_{\Delta_2,t}(w,\bar w) = \frac{B(i\lambda_1, i\lambda_2)}{z-w}\, f^{abc}\, \mathcal{O}^{c}_{\Delta_1+\Delta_2-1,t}(z_p,\bar z_p)}$$

Define :
$$\lim_{\substack{\Delta \to 1 \\ \lambda \to 0}} O^{a}_{\Delta,+}(z,\bar{z}) := j^{a}(z)$$

current

↳ OPE

$$\boxed{j^{a}(z)\, j^{b}(z) \sim \underline{\frac{1}{z-w}}\, f^{abc} j^{c}(w)}$$

gives rise to holomorphic Kac-Moody Algebra

↳ non-Abelian (global) symmetry

of CFT on CS^2

Operator product expansion

Celestial conformal field theory (CCFT)

$$
\begin{aligned}
\mathcal{O}^a_{\Delta_1,-1}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+1}(w,\bar{w}) = {} & \frac{C_{(-,+)-}(\Delta_1,\Delta_2)}{z-w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),-1}(w,\bar{w}) \\
& + \frac{C_{(-+)+}(\Delta_1,\Delta_2)}{\bar{z}-\bar{w}}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),+1}(w,\bar{w}) \\
& + C_{(-+)--}(\Delta_1,\Delta_2)\,\frac{\bar{z}-\bar{w}}{z-w}\ \delta^{ab}\,\mathcal{O}_{(\Delta_1+\Delta_2),-2}(w,\bar{w}) \\
& + C_{(-+)++}(\Delta_1,\Delta_2)\,\frac{z-w}{\bar{z}-\bar{w}}\ \delta^{ab}\,\mathcal{O}_{(\Delta_1+\Delta_2),+2}(w,\bar{w})\ + reg\,.
\end{aligned}
$$

Derive from collinear limits of D=4 EYM amplitudes

Fan, Fotopoulos, St.St., Taylor, Zhu (2019)

↕ D=4 S-matrix constrains OPE
or vice versa

Derive from first principles and consistency conditions

Pate, Raclariu, Strominger, Yuan (2019)

} extended
BMS
symmetry

Symmetries

At null infinity $\mathscr{I}^{\pm}$ more (hidden) symmetries present to constrain S-matrix

→ non-trivial consistency on amplitudes

$$z_i \to \frac{az_i + b}{cz_i + d}$$

$$\boxed{SL(2,\mathbf{C})_{z_i}: \ \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\}) \longrightarrow (cz_i + d)^{\Delta_i + J_i} \, (\bar{c}\bar{z}_i + \bar{d})^{\Delta_i - J_i} \, \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\})}$$

$$P_{-1/2,-1/2} = e^{(\partial_h + \partial_{\bar{h}})/2} = P^0 + P^3$$

St.St., Taylor (2018)

$$\boxed{P^{(j)}_{-1/2,-1/2}: \ \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\}) \longrightarrow \tilde{\mathscr{A}}_n(\{\Delta_j + 1, J_i\})}$$

comprises into translation operator P^{μ} *shifts conformal dimension* Δ_j

celestial gravitational amplitudes appear
as gauge amplitudes translated in space-time

Soft theorems

In usual QFT soft theorems $E_n \to 0$ play an important role in consistency and structure of amplitudes (in fact, soft theorems completely constrain almost all amplitudes)

in D=4: $p_{n+1} \to 0$

$$M_{n+1} \longrightarrow \left(\underbrace{\frac{1}{\epsilon^3} S_{\mathrm{G}}^{(0)}}_{\text{Weinberg (1965)}} + \underbrace{\frac{1}{\epsilon^2} S_{\mathrm{G}}^{(1)} + \frac{1}{\epsilon} S_{\mathrm{G}}^{(2)}}_{\text{Cachazo, Strominger (2014)}} + \ldots \right) M_n$$

$$A_{n+1} \longrightarrow \left(\frac{1}{\epsilon^2} S_{\mathrm{YM}}^{(0)} + \frac{1}{\epsilon} S_{\mathrm{YM}}^{(1)} + \ldots \right) A_n$$

soft theorems imply
Ward identities for asymptotic symmetries

Strominger (2013)

on CS^2: $\omega_{n+1} \to 0$

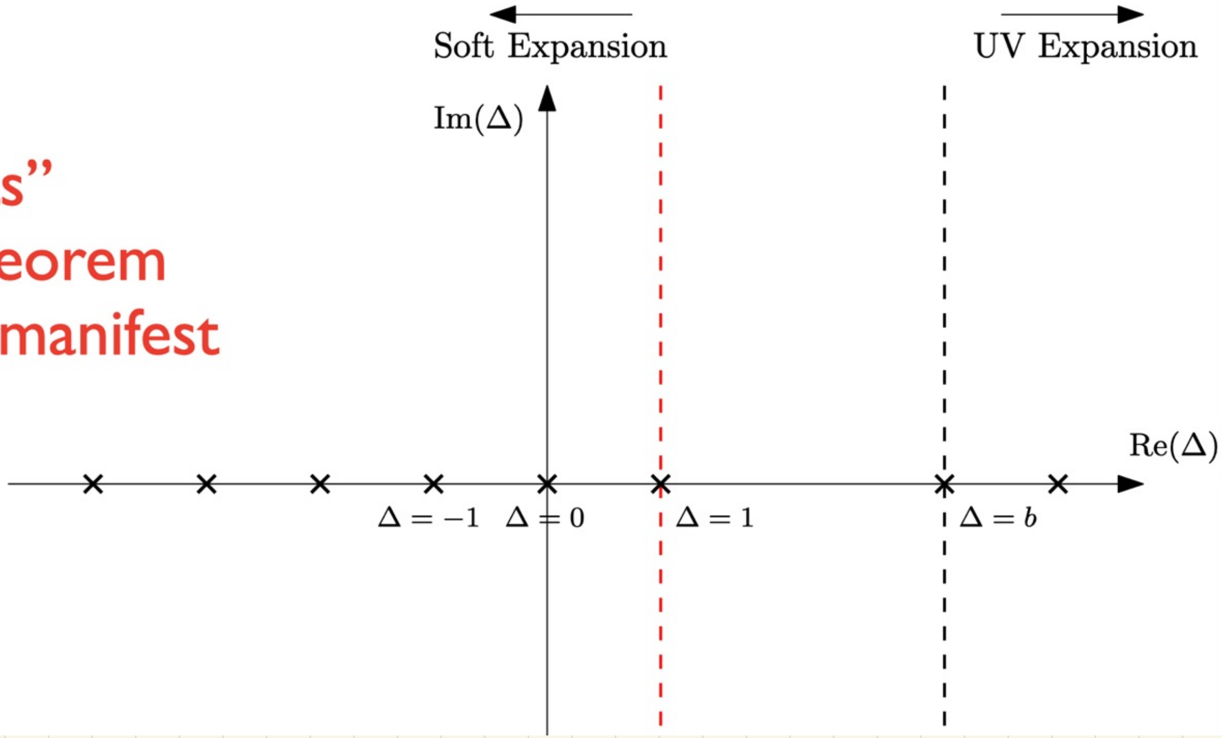
$$\mathcal{M}_{n+1} \longrightarrow \Big(\underbrace{\omega_{n+1}^{-1}\; S_{\mathrm{G}}^{(0)}}_{\Delta_{n+1} \to 1} + \underbrace{\omega_{n+1}^{0}\; S_{\mathrm{G}}^{(1)}}_{\Delta_{n+1} \to 0} + \underbrace{\omega_{n+1}\; S_{\mathrm{G}}^{(2)}}_{\Delta_{n+1} \to -1} + \dots \Big)\; \mathcal{M}_n$$

$$\mathcal{A}_{n+1} \longrightarrow \Big(\underbrace{\omega_{n+1}^{-1}\; S_{\mathrm{YM}}^{(0)}}_{\Delta_{n+1} \to 1} + \underbrace{\omega_{n+1}^{0}\; S_{\mathrm{YM}}^{(1)}}_{\Delta_{n+1} \to 0} + \dots \Big)\; \mathcal{A}_n$$

$\Delta \to 0,1,\dots$

in Mellin space "soft-limits"
reproduce Weinberg's soft theorem
and more symmetries become manifest

…



Kapec, Mitra, Raclariu, Strominger (2016)

Donnay, Puhm, Strominger (2018)

explicit field realization

(i) energy-momentum tensor T(z):

soft-graviton $\Delta \to 0$

$$T(z) := \tilde{\mathcal{O}}_{\Delta=2,J=+2}(z,\bar{z}) = \frac{3}{\pi}\int d^2w\ (z-w)^{-4}\ \mathcal{O}_{\Delta=0,J=-2}(w,\bar{w})$$

$$(h,\bar{h}) = (2{,}0)$$

Fotopoulos, Taylor (2019)

shadow transformation:

$$\tilde{\mathcal{O}}^a_{\tilde{\Delta},\tilde{J}}(z,\bar{z}) = \tilde{\mathcal{O}}^a_{2-\Delta,-J}(z,\bar{z}) = \frac{(\Delta+J-1)}{\pi}\int_{\mathbf{C}} \frac{d^2w}{(z-w)^{2-\Delta-J}(\bar{z}-\bar{w})^{2-\Delta+J}}\ \mathcal{O}^a_{\Delta,J}(w,\bar{w})$$

Ferrara, Grillo, Parisi, Gatto (1972)
Dolan, Osborn (2012)

then:

$$\boxed{\langle T(z)\ \prod_{i=1}^{n} O_{\Delta_i}(z_i,\bar{z}_i)\rangle = \sum_{i=1}^{n}\left(\frac{h_{O_i}}{(z-z_i)^2}+\frac{\partial_{z_i}}{z-z_i}\right)\langle\prod_{i=1}^{n} O_{\Delta_i}(z_i,\bar{z}_i)\rangle}$$

OPE:

$$\boxed{\begin{aligned}&T(w)T(z) = \frac{2T(z)}{(w-z)^2}+\frac{\partial_z T(z)}{w-z}+\dots\\&T(w)\bar{T}(z) = reg\,.\end{aligned}}$$

$$\longrightarrow \quad L_n, \bar{L}_m \qquad\qquad c=0$$

Fotopoulos, St.St., Taylor, Zhu (2019)

(ii) supertranslation operator P(z):

soft-graviton $\Delta \to 1$

$$P(z,\bar{z}) := \partial_{\bar{z}} \mathcal{O}_{\Delta=1,J=+2}(z,\bar{z}) \qquad (h,\bar{h}) = (\tfrac{3}{2},\tfrac{1}{2})$$

then:

$$\boxed{\left\langle P(z_0) \prod_{j=1}^{n} \mathcal{O}_{\Delta_j,l_j}(z_j,\bar{z}_j) \right\rangle = \frac{1}{4} \sum_{i=1}^{n} \frac{c_i(\Delta_i)}{c_i(\Delta_i+1)} \frac{1}{z_0 - z_i} \left\langle \prod_{n=1}^{n} \mathcal{O}_{\Delta_j,l_j}(z_j,\bar{z}_j) \right\rangle \Bigg|_{\Delta_i \to \Delta_i + 1}}$$

OPE:

$$T(w)P(z) = \frac{3}{2(w-z)^2} P(z) + \frac{1}{w-z} \partial_z P(z) + reg\,.$$

In addition to Virasoro symmetry, we construct <u>all supertranslation generators</u> acting on primary fields

Fotopoulos, St.St., Taylor, Zhu (2019)

construct:

$$P_{n-\frac{1}{2},-\frac{1}{2}} = \frac{1}{i\pi(n+1)}\oint dw\, w^{n+1}\left[T(w), P_{-\frac{1}{2},-\frac{1}{2}}\right]$$

$$P_{n-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{i\pi(m+1)}\oint d\bar{w}\, \bar{w}^{m+1}\left[\bar{T}(\bar{w}), P_{n-\frac{1}{2},-\frac{1}{2}}\right]$$

$$P_{-1/2,-1/2} = e^{(\partial_h+\partial_{\bar{h}})/2} = P^0 + P^3$$

we find:

$$\boxed{\left[P_{n-\frac{1}{2},m-\frac{1}{2}}, \phi^{h,\bar{h}}(z,\bar{z})\right] = z^n \bar{z}^m \phi^{h+\frac{1}{2},\bar{h}+\frac{1}{2}}(z,\bar{z})}$$

→ $P_{k,l}, \bar{P}_{k,l}$

→ <u>local (or extended) BMS algebra:</u>

$$\boxed{\begin{aligned} &[P_{ij}, P_{k,l}] = 0\ , \\ &[L_n, P_{k,l}] = \left(\frac{1}{2}n - k\right) P_{n+k,l} + n(n^2-1)\ C_{n,k}\ , \\ &[\bar{L}_n, P_{k,l}] = \left(\frac{1}{2}n - l\right) P_{k,n+l} + n(n^2-1)\ \bar{C}_{n,l}\ . \end{aligned}}$$

$m, n \in \mathbf{Z},\ i, j, k, l \in \mathbf{Z}+\frac{1}{2}$

Barnich (2017)

Conformal soft-theorems ⟷ Ward identities ⟷ BMS algebra

relation to P(z):

combine: $$\mathscr{P}(z,\bar{z}) := \sum_{n,m\in\mathbb{Z}} P_{n-\frac{1}{2},m-\frac{1}{2}}\, z^{-n-1}\bar{z}^{-m-1}$$

operator with $(h,\bar{h}) = (\frac{3}{2},\frac{3}{2})$

$$P(z) = \frac{1}{8\pi i}\oint d\bar{z}\, \mathscr{P}(z,\bar{z})$$

$$\overline{P}(\bar{z}) = \frac{1}{8\pi i}\oint dz\, \mathscr{P}(z,\bar{z})$$

but $\mathscr{P}$ includes all BMS supertranslations

BMS group

BMS$^{\pm}$ group
= symmetry of asymptotically flat D=4 space-time at null infinity $\mathscr{I}^{\pm}$

- transforms <u>one asymptotically flat solution</u>
  <u>to a new physically inequivalent</u> one

- <u>large diffeomorphisms</u>,
  which take one asymptotically flat solution into another
  (not isometries of flat space)

BMS group = Lorentz group + supertranslations

originally proposed in order to investigate
the flow of energy at infinity
due to propagating gravitational waves

Bondi, Burg, Metzner (1962)
Sachs (1962)

Extended BMS group on celestial sphere

global BMS symmetry on celestial sphere	local BMS symmetry on celestial sphere
<u>Lorentz group:</u> global conformal transformations on celestial sphere SL(2,C) $z \to \frac{az+b}{cz+d}$ $L_{-1} = \partial$ $L_0 = z\partial + h$ $L_1 = z^2\partial + 2hz$	local conformal transformations = superrotations T(z) $[L_m, L_n] = (m-n)\, L_{m+n}$ $[\bar{L}_m, \bar{L}_n] = (m-n)\, \bar{L}_{m+n}$
<u>global space-time translation:</u> Abelian subgroup of supertranslations $P_{-1/2,-1/2} = e^{(\partial_h+\partial_{\bar{h}})/2}$ $P_{1/2,1/2} = z\ e^{(\partial_h+\partial_{\bar{h}})/2}$ $P_{-1/2,1/2} = \bar{z}\ e^{(\partial_h+\partial_{\bar{h}})/2}$ $P_{-1/2,1/2} = \|z\|^2\ e^{(\partial_h+\partial_{\bar{h}})/2}$	local space-time translations =supertranslations P(z) $P_{n-\frac{1}{2},m-\frac{1}{2}} \quad n, m \in \mathbf{Z}$

→ Symmetries of the celestial OPEs and correlators
S-matrix (non-trivial consistency)

- *Extended Super BMS Algebra*

Further Directions

- understand Virasoro central charge (-one-loop ?)
- establish double-copy structure
  (elaborate on gauge/gravity connections)
- high-energy (large λ) limit: string world-sheet = celestial sphere
  celestial $CFT_2 \simeq$ string world-sheet CFT_2
- understanding the nature of 2D CFT on celestial sphere would enable a holographic description of flat spacetime
- uplift AdS_3/CFT_2 holography to $\mathscr{M}_4$
  towards flat space-time holography