Compactification and Soft Behaviour

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Brief Outline

1. Formulating the problem
2. Review of compactification on S^1
3. Single soft graviton theorem under compactification
4. Some explicit checks

Formulating the problem

Soft Graviton Theorem

- An amplitude M_{n+1} which involves a graviton carrying a soft momentum q, and n arbitrary finite energy particles carrying momenta p_i is related to the amplitude without the soft particle, M_n, by the so called soft graviton theorem

$$\mathcal{M}_{n+1}(q;\{p_i\}) \ = \ \kappa_{d+1}\left[\hat{S}^{(-1)} + \hat{S}^{(0)} + \hat{S}^{(1)}\right]\mathcal{M}_n(\{p_i\}) + O(q^2)$$

- The leading and subleading soft operators are given by

$$\hat{S}^{(-1)} = \epsilon_{MN}\sum_{i=1}^{n}\frac{p_i^M\, p_i^N}{p_i\cdot q} \qquad , \qquad \hat{S}^{(0)} = \epsilon_{MN}\sum_{i=1}^{n}\frac{q_P p_i^M\, J_i^{NP}}{p_i\cdot q}$$

Here κ_{d+1} is the $d+1$ dimensional gravitational coupling constant.

Soft Graviton Theorem

- J_i^{MN} denotes the total angular momentum operator acting on the polarization tensors of finite energy states inside $\mathcal{M}_n$.

- It is given by the sum of the orbital and spin angular momentum

$$J_i^{MN} = L_i^{MN} + S_i^{MN} \;\; ; \;\; L_i^{MN} = p_i^M \frac{\partial}{\partial p_{iN}} - p_i^N \frac{\partial}{\partial p_{iM}}$$

- The spin angular momentum operator S_i^{MN} takes different representations depending on what finite energy state it acts upon. E.g., its action on a spin-2 state is given by

$$(S^{MN}\,\epsilon)_{PQ} = (S^{MN})_{PQ}{}^{AB}\epsilon_{AB} = -\big(\delta_P^M \epsilon^N{}_Q - \delta_P^N \epsilon^M{}_Q + \delta_Q^M \epsilon_P{}^N - \delta_Q^N \epsilon_P{}^M\big)$$

Soft Graviton Theorem

- The leading and the subleading soft operators $\hat{S}^{(-1)}$ and $\hat{S}^{(0)}$ are universal and hence independent of the particular theory we consider.

- Moreover, the above soft theorem statement is valid for any kind of finite energy particles.

- On the other hand, the subsubleading operator $\hat{S}^{(1)}$ is not universal and depends upon the specific interactions of the theory under considerations. (Ladhdha, Sen)

Soft Graviton Theorem

- The universality of the leading and subleading terms allow us to apply the theorem to an arbitrary theory describing gravity in $d + 1$ dimensions and consider the scenario in which one of the direction is compactified on S^1.

- We shall show that the soft graviton theorem in the higher dimension breaks into the soft factorization statement of graviton, vector and scalar fields in the lower dimension.

Review of compactification on S^1

Compactification on S^1

- We consider gravity in $d+1$ dimensions which is described by the Einstein-Hilbert action

$$S = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x \sqrt{-G}\, R$$

  The κ_{d+1} is related to the $(d+1)$ dimensional Newton's coupling constant as $2\kappa_{d+1}^2 = 16\pi G_N$.

- We parametrize the compact direction by z and expand the metric in terms of its fourier modes on the circle as

$$G_{MN} = \sum_{n=-\infty}^{\infty} G_{MN}^{(n)}(x)\; e^{\frac{inz}{R_d}} \qquad ; \qquad z \in [0,\, 2\pi R_d]$$

  where R_d is the radius of the compact direction.

Compactification on S^1

- The compactification ansatz for the metric is taken to be

$$G_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} \; + \; e^{2\beta\phi} A_\mu A_\nu, \quad G_{\mu z} = e^{2\beta\phi} A_\mu, \quad G_{zz} = e^{2\beta\phi}$$

  where α and β are some arbitrary constants

- The inverse metric and the determinant are given by

$$G^{\mu\nu} = e^{-2\alpha\phi} g^{\mu\nu}, \quad G^{\mu z} = -e^{-2\alpha\phi} A^\mu, \quad G^{zz} = e^{-2\beta\phi} + e^{-2\alpha\phi} A_\mu A^\mu$$

$$\det(G_{\mu\nu}) = e^{2(d\alpha+\beta)\phi} \det(g_{\mu\nu})$$

- The fields $g_{\mu\nu}(x, z)$, $\phi(x, z)$ and $A_\mu(x, z)$ depend on the full $d+1$ space-time coordinates.

Compactification on S^1

For the above metric ansatz, the Einstein-Hilbert action, up to total derivative terms, takes the form (Cho and Zoh)

$$S = \frac{1}{2\kappa_{d+1}^2} \int d^d x \int_0^{2\pi R_d} dz \sqrt{g} \Big[e^{(\beta+(d-2)\alpha)\phi} R_g - \frac{1}{4} e^{((d-4)\alpha+3\beta)\phi} F^{\mu\nu} F_{\mu\nu}$$

$$+ \Big\{ 2\alpha(d-1)(\beta+(d-2)\alpha) - \alpha^2(d-2)(d-1) \Big\} e^{(\beta+(d-2)\alpha)\phi} \partial_\mu \phi \partial^\mu \phi$$

$$+ \frac{1}{4} e^{((d-2)\alpha-\beta)\phi} g^{\mu\nu} g^{\rho\sigma} \Big\{ \partial_z(e^{2\alpha\phi} g_{\mu\rho}) \partial_z(e^{2\alpha\phi} g_{\mu\sigma}) - \partial_z(e^{2\alpha\phi} g_{\mu\nu}) \partial_z(g_{\rho\sigma} e^{2\alpha\phi}) \Big\} \Big]$$

where $F_{\mu\nu}$ denotes the field strength of the vector field A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Compactification on S^1

- We now focus on the zero modes in the KK expansion. These zero modes do not depend on the compact coordinate and represent the massless degrees of freedom in d dimensional theory.

- More precisely, these zero modes describe the metric, a gauge field and a scalar field in d dimensions.

- There are some specific choices for the constants α and β for the zero modes. E.g., if we want to obtain the dimensionally reduced action in the Einstein frame with the canonically normalized scalar kinetic term, we need to choose

$$\beta = (2-d)\alpha \qquad , \qquad \alpha^2 = \frac{1}{2(d-1)(d-2)}$$

Compactification on S^1

- with these choices, the action for the zero modes of the metric reduces to (dropping the zero index from the fields)

$$S_0 = \frac{1}{2\kappa_d^2} \int d^d x \sqrt{-g} \left[R_g - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\sqrt{2(d-1)/(d-2)}\,\phi} \, F_{\mu\nu} F^{\mu\nu} \right]$$

where we defined $\kappa_d^2 = \frac{\kappa_{d+1}^2}{2\pi R_d}$.

- Similarly, for going to the string frame, we need to choose

$$\alpha \;=\; \frac{3}{1-d} \qquad , \qquad \beta \;=\; \frac{d-4}{d-1}$$

- With these choices, for $d+1 = 11$, the action for the zero modes in $d = 10$ becomes

$$S_0 = \frac{1}{2\kappa_{10}^2} \int d^{10} x \sqrt{-g} \left[e^{-2\phi} \left(R_g + 4 \partial_\mu \phi \partial^\mu \phi \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

Compactification on S^1

- The analysis for the non-zero modes is more involved.

(Cho, Zoh; Nappi, Witten; Duff,Dolan)

- Assuming d-dimensional Poincaré invariance of the vacuum, we can impose the conditions

$$\langle g_{\mu\nu} \rangle = \eta_{\mu\nu} \;\; ; \;\; \langle A_\mu \rangle = 0 \;\; ; \;\; \langle e^{\phi} \rangle = 1$$

and expand the metric around this background as

$$G_{\mu\nu} = \eta_{\mu\nu} + 2\kappa_{d+1} S_{\mu\nu}(x,\, z), \;\; G_{zz} = 1 + 2\kappa_{d+1} S_{zz}, \;\; G_{\mu z} = 2\kappa_{d+1} S_{\mu z}$$

- The KK expansion for the non zero modes is given by

$$\tilde{S}_{\mu\nu} = \sum_{n \neq 0} S^{(n)}_{\mu\nu}\, e^{\frac{inz}{R_d}} \;\; ; \;\; \tilde{S}_{\mu z} = \sum_{n \neq 0} S^{(n)}_{\mu z}\, e^{\frac{inz}{R_d}} \;\; ; \;\; \tilde{S}_{zz} = \sum_{n \neq 0} S^{(n)}_{zz}\, e^{\frac{inz}{R_d}}$$

where $\tilde{S}$ denotes the non zero modes of the KK expansion of the metric.

Compactification on S^1

- The $d+1$ dimensional parametrization invariance allows us to gauge fix the fields $\tilde{S}_{\mu z}$ and $\tilde{S}_{zz}$ to zero. (Nappi, Witten)

- This corresponds to fixing the non zero modes of the scalar and gauge fields to zero.

- Essentially, we can gauge away these fields because $S_{\mu z}^{(n)}$ and $S_{zz}^{(n)}$ act as Goldstone fields and $S_{\mu\nu}^{(n)}$ eats them to become a massive spin 2 field.

Compactification on S^1

- For the rescaled non zero modes $\phi^{(n)}_{\mu\nu} = \sqrt{2\pi R_d}\, S^{(n)}_{\mu\nu}$ one gets, for each level n of the Kaluza-Klein mode expansion and at lowest order in the field expansion, the Fierz-Pauli lagrangian in d dimension

$$\begin{aligned}\mathcal{L}^{(n)} &= \frac{1}{2}\partial_\mu \phi^{(-n)}_{\nu\rho}\partial^\mu \phi^{(n)\,\nu\rho} - \partial_\mu \phi^{(-n)\,\mu\nu}\partial^\rho \phi^{(n)}_{\rho\nu} - \frac{1}{2}\partial_\mu \phi^{(-n)}\partial^\mu \phi^{(n)} \\ &+\frac{1}{2}\partial_\mu \phi^{(-n)\,\mu\nu}\partial_\nu \phi^{(n)} + \frac{1}{2}\partial_\mu \phi^{(n)\,\mu\nu}\partial_\nu \phi^{(-n)} \\ &+\frac{m_n^2}{2}(\phi^{(-n)}_{\mu\nu}\phi^{(n)\,\mu\nu} - \phi^{(-n)}\phi^{(n)})\end{aligned}$$

  with $\phi = \phi^\mu{}_\mu$ and $m_n^2 = \frac{n^2}{R_d^2}$.

- The $\phi^{(n)}_{\mu\nu}$ represent an infinite tower of massive modes with masses given by $m_n^2 = \frac{n^2}{R_d^2}$.

Compactification on S^1

- These massive KK modes are also charged with respect to the massless $U(1)$ gauge field $A_\mu^{(0)}$.

- This happens because the zero mode of the diffeomorphism along the compact direction, namely, $\delta z = -\xi^z(x^\mu)$, becomes a local gauge transformation for the d-dimensional vector field, $A_\mu^{(0)} \to A_\mu^{(0)} + \partial_\mu \xi^z(x^\mu)$.

- Under this transformation, the massive modes $S_{\mu\nu}^{(n)}$ transform as $S_{\mu\nu}^{(n)} \to S_{\mu\nu}^{(n)} e^{-in\xi^z/R_d}$ and therefore carry the charge $e \equiv p_z = \frac{n}{R_d}$ with respect to this $U(1)$ group.

Compactification on S^1

- The interaction between the massive KK modes and the massless gauge field is usually introduced by minimal coupling procedure $\partial_\mu \to D_\mu = \partial_\mu + i\hat{e}_n \hat{A}_\mu$.
- Due to the ordering ambiguity in covariant derivatives, the minimal coupling procedure turns out to be ambiguous and this ambiguity is parametrized by a constant g called the gyromagnetic ratio.
- The minimally coupled Fierz-Pauli Lagrangian is thus given by (Deser,Waldron;...)

$$\mathcal{L} = \frac{1}{2} D_\mu \phi^*_{\nu\rho} D^\mu \phi^{\nu\rho} - D_\mu \phi^{*\mu\nu} D^\rho \phi_{\rho\nu} - \frac{1}{2} D_\mu \phi^* D^\mu \phi + \frac{1}{2} D_\mu \phi^{*\mu\nu} D_\nu \phi$$
$$+ \frac{1}{2} D_\mu \phi^{*\mu\nu} D_\nu \phi + \frac{m_n^2}{2} (\phi^*_{\mu\nu} \phi^{\mu\nu} - \phi^* \phi) - i\hat{e} g \phi^{*\rho\mu} F_{\mu\nu} \phi^\nu{}_\rho$$

with $\phi^* \equiv \phi^{(-n)}$.

Compactification on S^1

- For the compactification of 11 dimensional theory on S^1, the massive KK modes form the short 256-dimensional susy multiplets and are all BPS states.

- As mentioned above, the mass (or equivalently conserved U(1) charge) of these states is given by $m = |n|/R_d$.

- It turns out that in type IIA string theory in 10 dimensions, there are objects with precisely the same properties, namely D0 branes. (Witten; Sen)

Compactification on S^1

- They also form the short 256-dimensional representation of susy algebra. The tension (or mass) of the D0 branes is given by $1/(g_s\sqrt{\alpha'})$. Hence, being the BPS states, they also carry the U(1) charge in multiples of $1/(g_s\sqrt{\alpha'})$.

- This means that a single D0 brane can be identified with the $n = 1$ KK modes for the radius of compactification $R_{10} = g_s\sqrt{\alpha'}$. The higher KK modes are then identified with the bound states of D0 branes.

Soft graviton theorem and compactification

Identifying the physical polarizations

- The polarization tensor of the $d+1$ dimensional on-shell graviton field $S_{MN} = \epsilon_{MN} e^{ip_M x^M}$, satisfies the following conditions

$$p_M \epsilon^{MN} = 0 = \epsilon^N_{\ N} \;\; ; \;\; \epsilon_{MN} = \epsilon_{NM} \;\; ; \;\; p^2 = 0$$

  with $M, N = 0 \ldots d$.

- We denote by $\mu, \nu = 0, \cdots, d-1$ the indices along the d dimensional non compact space-time and by z the index along the compact direction.

- The mass shell condition is

$$p^2 = p_\mu p^\mu + p_z^2 = 0$$

Identifying the physical polarizations

- The conditions on the $d+1$ dimensional polarization tensor can be written as

$$p_\mu \epsilon^{\mu\nu} + p_z \epsilon^{z\nu} = 0 \quad , \quad p_\mu \epsilon^{\mu z} + p_z \epsilon^{zz} = 0 \quad , \quad \epsilon^\mu{}_\mu + \epsilon^z{}_z = 0$$

- We denote the d dimensional graviton polarization by $\varepsilon_{\mu\nu}$. It is demanded to satisfy

$$p^\nu \varepsilon_{\nu\mu} = \varepsilon^\nu{}_\nu = 0 \ ; \ \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu} \ ,$$

- The d dimensional vector and scalar fields will be denoted by ε_μ and $\hat{\phi}$ respectively.
- For the soft particles, we need to necessarily set the component of the momentum along the compact direction to be zero, i.e., $p_z = 0$, since we want it to remain massless under the compactification.

Identifying the physical polarizations

- The identification between the d dimensional polarization tensors and the $d+1$ dimensional polarization tensors is taken to be

$$\epsilon_{\mu\nu}(p^\mu) \;=\; \frac{\kappa_d}{\kappa_{d+1}}\left(\varepsilon_{\mu\nu}(p^\mu) + \frac{2\alpha}{\sqrt{2}}\hat{\phi}(p^\mu)\,\eta^{\perp}_{\mu\nu}\right) + O(\kappa_d^2)$$

$$\epsilon_{\mu z}(p^\mu) \;=\; \frac{\kappa_d}{\sqrt{2}\kappa_{d+1}}\varepsilon_\mu(p^\mu) + O(\kappa_d^2) \;\;;\;\; \epsilon_{zz}(p^\mu) = 2\beta\frac{\kappa_d}{\sqrt{2}\kappa_{d+1}}\hat{\phi}(p^\mu)$$

where,

$$\eta^{\perp}_{\mu\nu} \equiv \eta_{\mu\nu} - p_\mu\bar{p}_\nu - p_\nu\bar{p}_\mu \quad ; \quad p\cdot\bar{p} = 1 \quad ; \quad p^\mu\eta^{\perp}_{\mu\nu}(p) = 0 \qquad ;$$

where $\bar{p}^\mu$ is a reference null vector.

Identifying the physical polarizations

- The relation between the graviton polarizations in $d+1$ and d dimension involves the transverse metric $\eta^{\perp}_{\mu\nu}$. The reason is as follows:

- At the massless level, the $d+1$ dimensional fields do not depend upon momentum p_z along the compact direction.

- Hence, the transversality condition of the $d+1$ dimensional polarization tensor, namely, $p_M \epsilon^{MN} = 0$ immediately gives $p_\mu \epsilon^{\mu\nu} = 0$.

Identifying the physical polarizations

- Now, the d dimensional polarization tensor $\varepsilon_{\mu\nu}$ is also demanded to satisfy the same relation, namely, $p_\mu \varepsilon^{\mu\nu} = 0$.
- However, both these conditions are not compatible with each other if we use $\eta_{\mu\nu}$ in the identification between the two polarizations.
- By using $\eta^{\perp}_{\mu\nu}$, both the transversality conditions become compatible with each other.

Soft factorization under compactification

- Using the identification between the polarization tensors, the soft graviton amplitude $\mathcal{M}_{n+1}$ can be expressed as the sum of three terms in d dimensions

$$
\begin{aligned}
\mathcal{M}_{n+1}(q,\epsilon;\epsilon_i,p_i) &= \kappa_{d+1}\epsilon_{MN}\mathcal{M}^{MN}_{n+1}(q;\epsilon_i,p_i) \\
&= \kappa_d\,\varepsilon_{\mu\nu}\mathcal{M}^{\mu\nu}_{n+1}(q,\varepsilon;\varepsilon_i,p_i) \\
&\quad +\kappa_d\varepsilon_\mu\mathcal{M}^{\mu}_{n+1}(q,\varepsilon;\varepsilon_i,p_i) \\
&\quad +\sqrt{2}\,\alpha\hat{\phi}\mathcal{M}^{\phi}_{n+1}(q,\varepsilon;\varepsilon_i,p_i)+O(\kappa_d^2)
\end{aligned}
$$

where we have denoted $\mathcal{M}^{\mu}_{n+1}\equiv\mathcal{M}^{\mu z}_{n+1}$ and defined

$$
\mathcal{M}^{\phi}_{n+1} = \kappa_d\,\eta^{\perp}_{\mu\nu}\left(\mathcal{M}^{\mu\nu}_{n+1}(q,\varepsilon;\varepsilon_i,p_i)-\eta^{\mu\nu}\mathcal{M}^{zz}_{n+1}(q,\varepsilon;\varepsilon_i,p_i)\right)
$$

Soft factorization for leading term

- The finite energy states inside $\mathcal{M}_{n+1}$ in the right hand side depend upon the momentum along the d non compact directions. The massive states also depend upon the compact direction through their mass/charge.

- By replacing the $d+1$ dimensional graviton polarization ϵ_{MN} in terms of the d dimensional polarizations in the leading soft theorem in $d+1$ dimensions, we find the structure mentioned above.

- In other words, the $d+1$ dimensional soft theorem statement breaks into soft factorizations for three particles, namely graviton, vector and scalar, in d dimension.

Soft factorization for leading term

- For the graviton, we get

$$\mathcal{M}^{g}_{n+1}(q,\{p_i\}) \equiv \kappa_d\, \varepsilon_{\mu\nu}\mathcal{M}^{\mu\nu}_{n+1} = \kappa_d \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} p_i^{\mu} p_i^{\nu}}{(p_i \cdot q)} \mathcal{M}_n(\{p_i\}),$$

- For the scalar, we get

$$\mathcal{M}^{\phi}_{n+1}(q,\{p_i\}) = \kappa_d \left\{ 2 + \sum_{i=1}^{n} \left[\frac{(2-d)(p_i^z)^2 + p_i^2}{(p_i \cdot q)} \right] \right\} \mathcal{M}_n(\{p_i\}),$$

- Finally, for the vector field, we get

$$\mathcal{M}^{A}_{n+1}(q,\{p_i\}) = \sqrt{2}\,\kappa_d \sum_{i=1}^{n} \frac{e_i \varepsilon_{\mu} p_i^{\mu}}{(p_i \cdot q)} \mathcal{M}_n(\{p_i\}) \qquad ; \quad e_i = p_i^z$$

Soft factorization for subleading term

- For the subleading term also, the analysis is similar though more involved.
- We replace the $d+1$ dimensional graviton polarization in terms of the d dimensional polarizations in the subleading term of the soft graviton theorem. This gives,

$$\begin{aligned}
&\mathcal{M}_{n+1}(\epsilon, q; \epsilon_i, p_i) \\
&= \kappa_d \sum_{i=1}^{n} \left[\frac{\varepsilon_{\mu\nu} p_i^{\mu} q_{\rho} J_i^{\nu\rho}}{(p_i \cdot q)} + \frac{\varepsilon_{\mu} q_{\rho} (p_i^{\mu} J_i^{z\rho} + p_i^{z} J_i^{\mu\rho})}{\sqrt{2}(p_i \cdot q)} \right. \\
&\qquad \left. + \frac{2(\beta p_i^{z} q_{\rho} J_i^{z\rho} + \alpha \eta^{\perp}_{\mu\nu} p_i^{\mu} q_{\rho} J_i^{\nu\rho}) \hat{\phi}}{\sqrt{2}(p_i \cdot q)} \right] \mathcal{M}_n(\varepsilon_i, p_i)
\end{aligned}$$

Soft factorization for subleading term

- The terms corresponding to soft graviton, vector and scalar fields can be simplified to give a more familiar looking expressions.
- The subleading term for the soft graviton is universal. Hence, we just get the standard result in this case.
- For the soft vector field, the expression simplifies to give

$$\mathcal{M}_{n+1} = \sqrt{2}\,\kappa_d \sum_{i=1}^{n} \left[\frac{e_i \varepsilon_\mu\, q_\nu (2L_i^{\mu\nu} + S_i^{\mu\nu})}{2\, p_i \cdot q} + \frac{\varepsilon_\mu q_\nu p_i^\sigma\, (\Sigma_{\sigma\rho})^{\mu\nu} S_i^{z\rho}}{2\, p_i \cdot q} \right] \mathcal{M}_n$$

where,

$$(\Sigma_{\sigma\rho})_{\mu\nu} \quad \equiv \quad \eta_{\sigma\mu}\, \eta_{\rho\nu} - \eta_{\sigma\nu}\, \eta_{\rho\mu}$$

Soft factorization for subleading term

- The action of the broken generator $S_i^{z\rho}$ on the finite energy states can be worked explicitly. Its action on the massive spin 2 states annihilates them.
- On the massless fields, its' action is given by

$$
\begin{aligned}
S_i^{\rho z}\left[\varepsilon^i_{\mu\nu}\mathcal{M}_n^{\mu\nu}\right] &= -\frac{1}{\sqrt{2}}\varepsilon_i^{\sigma}\left[\eta_{\nu\sigma}\,\eta^{\rho}_{\mu}+\eta^{\rho}_{\nu}\,\eta_{\sigma\mu}-\frac{2\,\eta_{\sigma\rho}\eta_{\mu\nu}}{(d-2)}\right]\mathcal{M}_n^{\mu\nu} \\
S_i^{\rho z}\frac{\varepsilon^i_{\mu}}{\sqrt{2}}\mathcal{M}_n^{\mu} &= \left(\varepsilon^{\rho}_{i\;\mu}-\sqrt{\frac{d-1}{d-2}}\eta^{\rho}_{\mu}\,\hat{\phi}_i\right)\mathcal{M}_n^{\mu} \\
S_i^{\rho z}\hat{\phi}_i &= \sqrt{\frac{2(d-1)}{d-2}}\;\varepsilon_i^{\rho}
\end{aligned}
$$

- Clearly, it converts one type of particle into another type and implies the existence of specific interaction vertices.

Soft factorization for subleading term

- For the scalar field, the subleading contribution is simplified to give

$$\mathcal{M}_{n+1}^{\phi} = \kappa_d \left\{ 2 - \sum_{i=1}^{n} \left[p_i \cdot \frac{\partial}{\partial p_i} + \frac{(2-d)e_i^2 - m_i^2}{p_i \cdot q} q \cdot \frac{\partial}{\partial p_i} - (2-d) e_i \frac{\partial}{\partial e_i} \right] \right\} \mathcal{M}_n$$

where $e_i = p_i^z = \frac{n}{R_d}$.

Some explicit checks

- We now consider the case of soft vector field in d dimension in some more details.

- In particular, we shall consider the soft vector field interacting with finite energy graviton, vector, scalar and massive spin 2 fields.

- We start by considering the interaction of soft vector field with the massless fields.

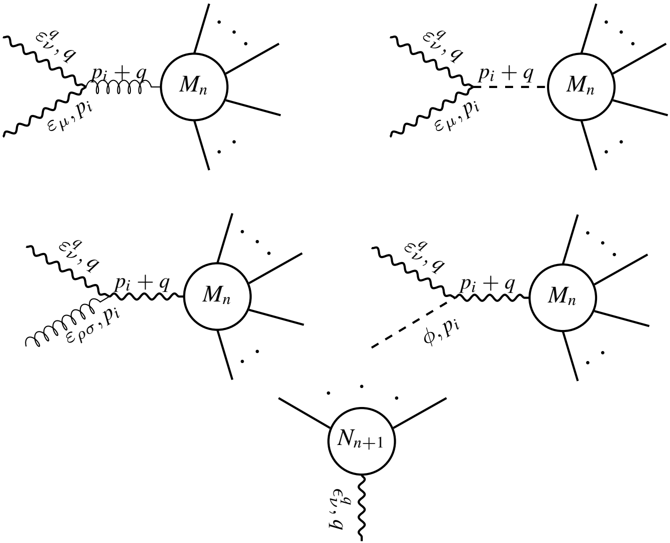
Soft factorization with massless fields

- The interaction between graviton, scalars and the 1-form field is described by the following effective action

$$S_0 = \frac{1}{2\kappa_d^2} \int d^d x \sqrt{-g} \left[R_g - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\sqrt{2(d-1)/(d-2)}\,\phi} \, F_{\mu\nu} F^{\mu\nu} \right]$$

- This implies that there are total 5 diagrams which can contribute to amplitudes with one soft vector.

Soft factorization with massless fields



Soft factorization with massless fields

- The exchange diagrams can be evaluated explicitly. The diagram withput any pole in soft momenta can be evaluated by imposing the gauge invariance of the full amplitude $q_\mu M^\mu_{n+1} = 0$. The final result is

$$
\begin{aligned}
M_{n+1} \;=\;& 2\kappa_d \sum_{i=1}^{n_A} \frac{\varepsilon^q_\nu p_{i\rho} q_\sigma (\Sigma^{\rho\mu})^{\nu\sigma} \mathcal{S}^i_\mu}{2p_i \cdot q} M_n^{(h_i)}(p_1, \ldots, p_i, \ldots p_n) \\
& -2\kappa_d \sum_{i=1}^{n_d} \frac{\varepsilon^q_\nu p_{i\rho} q_\sigma (\Sigma^{\rho\mu})^{\sigma\nu} \mathcal{S}^i_\mu}{2p_i \cdot q} M_n^{(\hat{\phi}_i)}(p_1, \ldots, p_i, \ldots p_n) \\
& 2\kappa_d \sum_{i=1}^{n_A} \frac{\varepsilon^q_\nu(q) p_{i\mu} q_\rho (\Sigma^\mu{}_\sigma)^{\nu\rho} \mathcal{S}^\sigma_i}{2p_i \cdot q} M_n(p_1, \ldots, p_i, \ldots p_n) + O(q)
\end{aligned}
$$

Soft factorization with massless fields

- The action of the operators $\mathcal{S}^{\mu}$ on various polarization tensors is given by

$$\mathcal{S}^{\nu}\,\varepsilon^{\mu} = -\left[\varepsilon^{\nu\mu}(p_i) - \eta^{\nu\mu}\hat{\phi}\sqrt{\frac{d-1}{d-2}}\right]; \quad \mathcal{S}^{\mu}\hat{\phi} = -\sqrt{\frac{d-1}{d-2}}\varepsilon^{\mu}$$

$$\mathcal{S}^{\nu}\varepsilon^{\rho\sigma} = \frac{1}{2}\left[\eta^{\sigma\nu}\,\eta^{\rho\mu} + \eta^{\rho\nu}\,\eta^{\sigma\mu} - \frac{2\eta^{\mu\nu}\eta^{\rho\sigma}}{d-2}\right]\varepsilon_{\mu}$$

- If we compare these equations with the action of the broken generators $S^{z\nu}$ on the massless states, we immediately see that they coincide if we identify $\mathcal{S}^{\rho}$ with the angular momentum operator $S^{z\rho}/\sqrt{2}$ associated to broken generators of the $d+1$ dimensional Lorentz group.

Soft factorization with massive spin-2 field

- We now consider the interaction between the soft vector and the massive KK fields.
- The non interacting Lagrangian for the massive KK modes following from the compactification turns out to be Fierz-Pauli Lagrangian. (Cho, Zoh;...)
- We shall consider the minimally coupled Fierz Pauli Lagrangian to describe the interaction between the KK modes and massless vector field. (Porrati, Rahman;...)

Soft factorization with massive spin-2 field

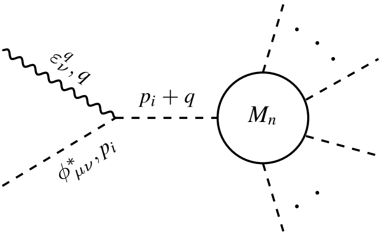


Figure: Diagrams contributing to the amplitude with a soft vector field interacting with Kaluza-Klein states.

Soft factorization with massive spin-2 field

- The 3-point vertex giving the interaction among the vector with momentum q and polarization ε^μ and two Kaluza-Klein states at the same level n, having momenta and polarizations $(k_2, \phi_{\mu\nu})$ and $(k_3, \phi^*{}_{\rho\sigma})$ is

$$
\begin{aligned}
&V_{\tau;\rho\sigma;\mu\nu}(q, k_3, k_2) \\
&= \frac{i}{2}\hat{e}\Bigg[\frac{1}{2}(\eta_{\rho\mu}\eta_{\sigma\nu} + \eta_{\rho\nu}\eta_{\sigma\mu} - 2\eta_{\rho\sigma}\eta_{\nu\mu})(k_2 - k_3)_\tau + \frac{1}{2}\eta_{\tau\rho}\eta_{\mu\nu}(k_2 - k_3)_\sigma \\
&+\frac{1}{2}\eta_{\tau\sigma}\eta_{\mu\nu}(k_2 - k_3)_\rho + \frac{1}{2}\eta_{\tau\mu}\eta_{\rho\sigma}(k_2 - k_3)_\nu + \frac{1}{2}\eta_{\tau\nu}\eta_{\rho\sigma}(k_2 - k_3)_\mu \\
&-\frac{1}{2}\eta_{\tau\rho}(\eta_{\sigma\nu}k_{2\mu} + \eta_{\mu\sigma}k_{2d\nu}) - \frac{1}{2}\eta_{\tau\sigma}(\eta_{\rho\nu}k_{2\mu} + \eta_{\rho\mu}k_{2\nu}) + \frac{1}{2}\eta_{\tau\mu}(k_{3\rho}\eta_{\sigma\nu} + \eta_{\rho d}k_{3\sigma}) \\
&+\frac{1}{2}\eta_{\tau\nu}(\eta_{\sigma\mu}k_{3\rho} + \eta_{\rho\mu}k_{3\sigma}) + \frac{g}{2}\eta_{\tau\mu}(q_\sigma\eta_{\rho\nu} + q_\rho\eta_{\sigma\nu}) + \frac{g}{2}\eta_{\tau\nu}(q_\sigma\eta_{\rho\mu} + q_\rho\eta_{\sigma\mu}) \\
&-\frac{g}{2}\eta_{\tau\sigma}(q_\mu\eta_{\rho\nu} + q_\nu\eta_{\rho\mu}) - \frac{g}{2}\eta_{\tau\rho}(q_\mu\eta_{\sigma\nu} + q_\nu\eta_{\mu\sigma})\Bigg]
\end{aligned}
$$

Soft factorization with massive spin-2 field

- The propagator of the massive states is given by

$$D^{\mu\nu\rho\sigma} = \frac{i}{p^2 + m^2}\left[\Pi^{\mu\rho}\Pi^{\nu\sigma} + \Pi^{\mu\sigma}\Pi^{\nu\rho} - \frac{2\,\Pi^{\mu\nu}\Pi^{\rho\sigma}}{d-1}\right]$$

where,

$$\Pi^{\mu\nu} = \eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$$

- The exchange diagram is evaluated by the above Feynman rules and the contact diagram is evaluated by imposing the gauge invariance of the full amplitude $q_\mu M^\mu(q, p_i) = 0$.

Soft factorization with massive spin-2 field

- The final result is given by

$$M_{n+1} = \varepsilon_\mu \sum_{i=1}^{n} \hat{e}_i \left[\frac{p_i^\mu}{p_i \cdot q} + \frac{q_\rho}{2p_i \cdot q} (2L_i^{\mu\rho} + gS_i^{\mu\rho}) \right] M_n(p_i) + O(q)$$

- This equation is consistent with the result obtained from the compactification of the soft graviton amplitude when we specialize that to the case of the massive spin 2 finite energy states.

- However, this matching requires us to take the gyromagnetic ratio to be $g = 1$.

Soft factorization with massive spin-2 field

- In general, the elementary particles have gyromagnetic ratio $g = 2$. (Weinberg; Jackiw; Ferrara, Porrati, Telegdi)

- However, this holds when one demands the tree level unitarity and good high energy behaviour. More precisely, this holds when the energy regime satisfies (Weinberg)

$$M_{pl} \;>\; E \;>\; \frac{M}{Q}$$

- The KK modes satisfy the BPS condition and hence they are out of this regime.

Soft factorization with massive spin-2 field

- Thus, there is no contradiction with the value of gyromagnetic ratio $g = 1$ for the massive KK modes interacting with the photon fields. (Hosoya, Ishikawa, Ohkuwa, Yamagishi)

- This is also consistent with the value of gyromagnetic ratio for the D0 brane in type IIA theory considered as black hole carrying spin. (Duff, Liu, Rahmfeld)

Thank You