Soliton and Some of its Geophysical/Oceanographic Applications

M. Lakshmanan

Centre for Nonlinear Dynamics
Bharathidasan University
Tiruchirappalli – 620024
India

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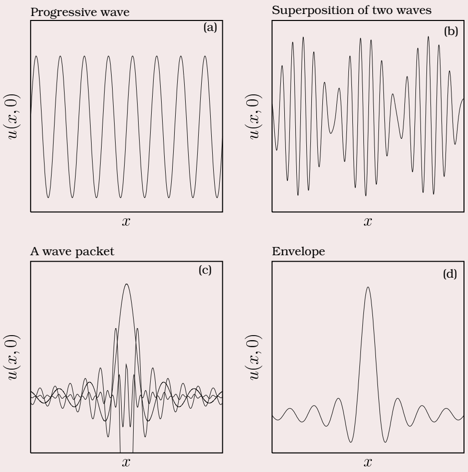
Theme

Soliton is a counter-intuitive wave entity arising because of a delicate balance between dispersion and nonlinearity. Its major attraction is its localized nature, finite energy and preservation of shape under collision $\Longrightarrow$ a remarkably stable structure. Consequently, it has ramifications in as wider areas as hydrodynamics, tsunami dynamics, condensed matter, magnetism, particle physics, nonlinear optics, cosmology and so on. A brief overview will be presented with emphasis on oceanographic applications and potential problems will be indicated.

Plan

Solitons

- Soliton as a counter-intuitive entity
- Linear dispersive waves



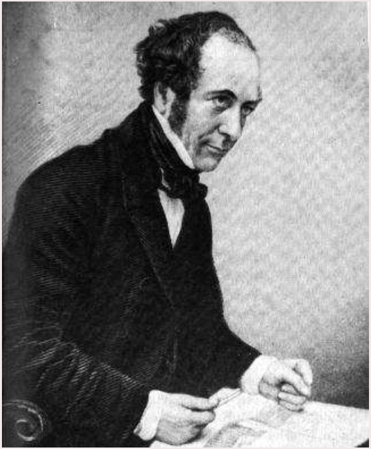
Solitons

- Nonlinear dispersive waves: Solitary waves/solitons
- Many facets of solitons:

a) Tsunami as soliton?	g) Magnetic soliton
b) Internal soliton	h) Topological soliton
c) Rossby soliton	i) Plane soliton
d) Bore soliton	j) Lump soliton
e) Capillary soliton	k) Dromion
f) Optical soliton	l) Light bullet

John Scott-Russel

John Scott-Russel (1808-82)



- Russel's observations - Phenomenological relation:

$$c = \sqrt{g(h+k)}$$

J. S. Russel, British Association Reports (1844)

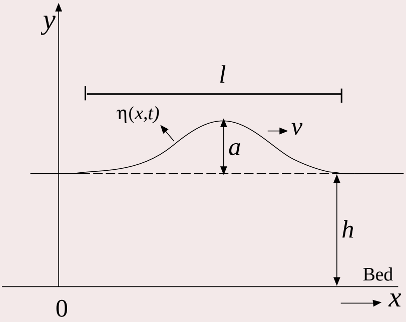
"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon! which I have called the Wave of Translation".

Solitons

Soliton creation

- Scott Russell Aqueduct – Soliton creation – Heriot-Watt University, 1995.
- Korteweg-de Vries Formulation (1895)
- Unidirectional shallow water wave propagation in one dimension
- Incompressible, inviscible fluid

Solitons



$\eta(x,t)$: amplitude

$a << h$ (shallow water)

$h << l$ (long wave) $\Longrightarrow$ Two small parameters, $\epsilon = \frac{a}{h}$, $\delta = \frac{h}{l}$

Solitons

Fluid motion:

$$\vec{V}(x,y,t) = u(x,y,t)\vec{i} + v(x,y,t)\vec{j}$$

$$\Longrightarrow \quad \text{Irrotational:} \quad \vec{\nabla} \times \vec{V} = 0 \quad \Longrightarrow \quad \vec{V} = \vec{\nabla}\phi$$

$\phi(x,y,t)$: velocity potential

(i) Conservation of density:

$$\frac{d\rho}{dt} = \rho_t + \vec{\nabla} \cdot (\rho\vec{V}) = 0, \quad \rho = \rho_0 = \text{constant}$$

$\Longrightarrow \nabla^2\phi(x,y,t) = 0$ – Laplace equation.

Solitons

(ii) Euler's Equation:

$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = -\frac{1}{\rho_0}\vec{\nabla}p - g\vec{j} \implies \phi_t + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \frac{p}{\rho_0} + gy = 0$$

(iii) Boundary conditions:

$$\phi_y(x,0,t) = 0$$

$$y = h + \eta(x,t); \quad \phi_{1y} = \eta_t + \eta_x\phi_{1x}$$

$$p_1 = 0 \implies u_{1t} + u_1 u_{1x} + v_1 v_{1x} + g\eta_x = 0$$

Solitons

(iv) Taylor Expansion:

$$\phi(x,y,t) = \sum_{n=0}^{\infty} y^n \phi_n(x,t)$$

$$u_1 = \phi_{1x} = f - \frac{1}{2} y_1^2 f_{xx} + \ldots,$$

$$v_1 = \phi_{1y} = -y_1 f_x + \frac{1}{6} y_1^3 f_{xxx} + \ldots, \quad f = \frac{\partial \phi_0}{\partial x}$$

(v) Introducing ϵ and δ:

$$u_1 = \epsilon c_0 u_1', \quad v_1 = \epsilon \delta c_0 v_1', \quad f = \epsilon c_0 f'$$

$$y_1 = h + \eta(x,t) = h\left(1 + \epsilon \eta'\left(x', t'\right)\right)$$

Solitons

$$\eta_t + f_x + \epsilon\eta f_x + \epsilon f\eta_x - \frac{1}{6}\delta^2 f_{xxx} = 0$$

$$f_t + \epsilon f f_x + \frac{ga}{\epsilon c_0^2}\eta_x - \frac{1}{2}\delta^2 f_{xxt} = 0$$

(vi) Perturbation analysis:

$$f = f^{(0)} + \epsilon f^{(1)} + \delta^2 f^{(2)} + \text{higher order}$$

$$f^{(0)} = \eta, \quad f^{(1)} = -\frac{1}{4}\eta^2, \quad f^{(2)} = \frac{1}{3}\eta_{xx}$$

$$\eta_t + \eta_x + \frac{3}{2}\epsilon\eta\eta_x + \frac{\delta^2}{6}\eta_{xxx} = 0$$

Solitons

(vii) The standard contemporary form:

$$u_t \pm 6uu_x + u_{xxx} = 0$$

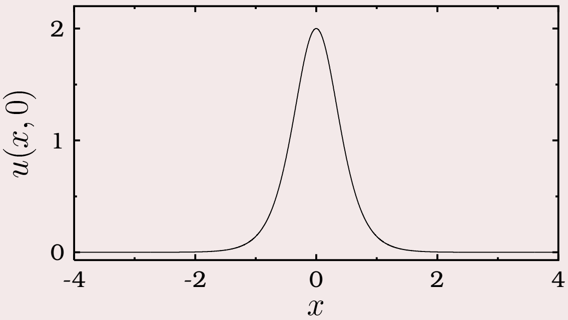
(ix) Cnoidal and solitary wave solution:

$$u(x,t) = 2f(\zeta), \quad \zeta = (x - ct) \implies -c\frac{\partial f}{\partial \zeta} + 12f\frac{\partial \rho}{\partial \zeta} + \frac{\partial^3 f}{\partial \zeta^3} = 0$$

$$\implies \quad f(\zeta) = \alpha_3 - (\alpha_3 - \alpha_2)\mathrm{sn}^2(\sqrt{\alpha_3 - \alpha_1}(x - ct), m)$$

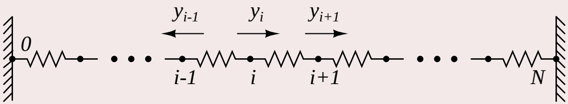
$$\implies \quad \frac{c}{2}\mathrm{sech}^2\frac{\sqrt{c}}{2}(x - ct + \delta), \quad \delta = \text{constant } (m = 1)$$

Solitons



III. FPU Problem (1955) and Zabusky-Kruskal Analysis (1965)

Wave propagation in anharmonic lattices



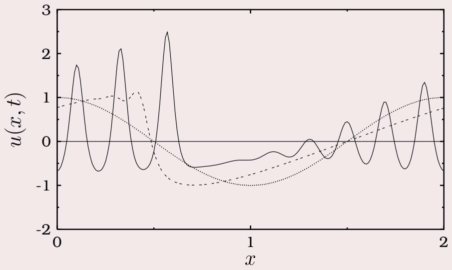
Equation of motion:

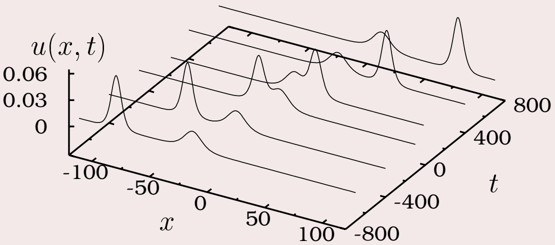
$$m\frac{\partial^2 y_i}{\partial t^2} = f\left(y_{i+1} - y_i\right) - f\left(y_i - y_{i-1}\right), \quad i = 1, 2, ..., N-1,$$

$$y_i \longrightarrow y(x,t), \quad y_{i+1} = y(x,t) \pm \frac{\partial y}{\partial x} \pm \frac{a^2}{2}\frac{\partial^2 y}{\partial x^2} \pm + \ldots$$

$\Longrightarrow$ KdV equation: $\qquad u_t + 6uu_x + u_{xxx} = 0$

Zabusky-Kruskal Numerical Analysis: Birth of soliton





Soliton

Solitary wave which retains its shape and speed under collision except for a phase shift – Elastic collision.

Other ubiquitous equations

Boussinesq equation

$$u_t + uu_x + g\eta_x - \frac{1}{3}h^2 u_{txx} = 0$$
$$\eta_t + [u(h+\eta)]_x = 0$$

Benjamin-Bona-Mahoney (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0$$

Camassa-Holm equation

$$u_t + 2\kappa u_x + 3uu_x - u_{xxt} = 3u_x u_{xx} + uu_{xxx}$$

Other ubiquitous equations

Kadomtsev-Petviashville (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0$$

($\sigma^2 = -1$: KP-I, $\sigma^2 = +1$: KP-II)

Modified KdV equation

$$u_t \pm 6u^2 u_x + u_{xxx} = 0$$

Sine-Gordon (sG) equation (in light-cone coordinates)

$$u_{xt} + m^2 \sin u = 0 \ , \quad m^2 = \text{constant}$$

Other ubiquitous equations

Nonlinear Schrödinger equation

$$iq_t + q_{xx} \pm 2|q|^2 q = 0\,, \quad q \in C$$

Continuous Heisenberg ferromagnetic spin equation

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}, \quad \vec{S} = (S_1, S_2, S_3)\,, \quad \vec{S}^2 = 1$$

Toda lattice equation

$$\ddot{u}_n = \exp\left[-\left(u_n - u_{n-1}\right)\right] - \exp\left[-\left(u_{n+1} - u_n\right)\right], \quad i = 1, 2, .., N$$

Mathematical theory of solitons: Cauchy IVP

KdV equation $\Longrightarrow$ Linearization

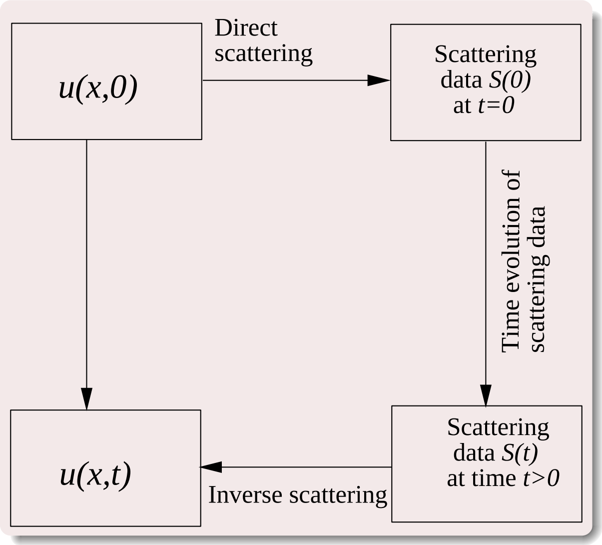
Given: $u(x,0)$: Initial value

(i) $\psi_{xx} + [\lambda - u(x,t)]\psi = 0, \quad u(x,t) \longrightarrow 0$, at $|x| \to \infty$

(ii) $\psi_t = -4\psi_{xxx} + 3u\psi_x + 6u_x\psi$

$\Longrightarrow (\psi_{xx})_t \equiv (\psi_t)_{xx} \Longrightarrow$ KdV

Inverse Scattering Transform method



Inverse Scattering Transform method

1-soliton solution:

$$u = -2\kappa^2 \operatorname{sech}^2 \left[\kappa \left(x - 4\kappa^2 t\right) + \delta\right]$$

2-soliton solution:

$$u(x,t) = -2\left(\kappa_2^2 - \kappa_1^2\right) \frac{\kappa_2^2 \operatorname{cosech}^2\gamma_2 + \kappa_1^2 \operatorname{sech}^2\gamma_1}{\left(\kappa_2 \coth\gamma_2 - \kappa_1 \tanh\gamma_1\right)^2}$$

N-soliton solution

- Bilinearization: $u = 2\frac{\partial^2}{\partial x^2}\log F$
  - $$F_{xt}F - F_xF_t + F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 = 0$$
- Bäcklund transformation
- Infinite number of conservation laws/symmetries
- Geometric structures/Group theoretic
- Hamiltonian/Bi-Hamiltonian structure
- Completely integrable infinite dimensional nonlinear dynamical systems

VI. Geophysical applications

A. Tsunami as solitons?

Geophysical applications

Tsunami as solitons?

Geophysical applications

Tsunami as solitons?

- 2004 Indian ocean / 1960 Chilean earth quake
- Typical $\lambda \simeq 200$ kms, $h \simeq 1$ to 4 kms, $a \simeq 60$ cms
- Shallowness condition: $\epsilon = \frac{a}{h} \simeq 10^{-4} \ll 1$, $\delta^2 = \frac{h^2}{l^2} \simeq 10^{-4}$
- Propagation of tsunami waves as KdV like soliton

Recent generalization

- Phase modulated solitary waves controlled by boundary conditions at the bottom.

  (A. Mukherjee and M.S. Janaki, Phys. Rev E(2014)).
- Lump solitons & Rogue waves controlled by ocean currents.

  (A. Kundu, A. Mukherjee, T. Naskar, Proc. R. Soc. Lond. (2014))

Geophysical applications

B. Internal solitons?

- Seafarers through strait of Malacca often note that in the Andaman sea bands of strongly increasing surface roughness occur.
  $\Longrightarrow$ Internal solitons (SAR images of ERS-1/2 satellites) (Solitary wave like distortion of the boundary layer between warm upper sea and cold lower depths)
- Peculiar striations of 100 km long, separated by 6-15 kms and grouped in packets of 4-8 of the surface of Andaman and Sulu seas.
  $\Longrightarrow$ Secondary phenomena of internal solitons.
- Internal solitons are traveling edges of warm water with enormous energy affecting oil rigs

Geophysical applications

B. Internal solitons

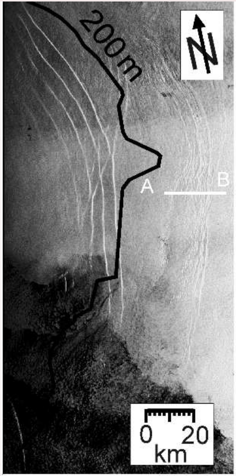
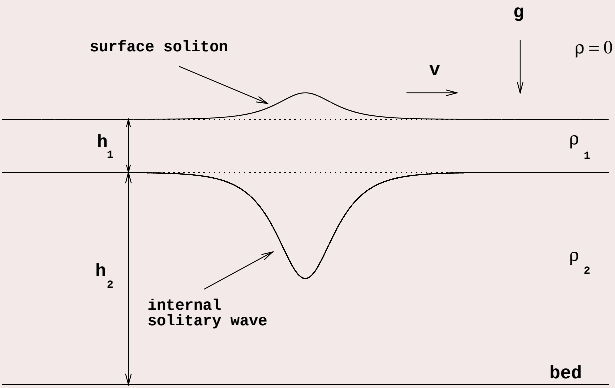


Figure: SAR image of the Andaman Sea acquired by the ERS-2 satellite showing two internal solitary wave packets

Geophysical applications

B. Internal solitons



Geophysical applications

B. Internal solitons

- Study of Osborne and Birch (1980) of underwater currents by an oil rig drilling at 36000 ft depth in the Andaman sea $\sim$ KdV solitons
- Other observations
- Theory: Consider two incompressible, immiscible fluid, densities ρ_1, ρ_2 and depths h_1, h_2
- KdV
- Intermediate long wavelength (ILW)
- Benjamin-Ono

Geophysical applications

C. Rossby solitons

- In the atmosphere of a rotating planet, fluid particle $\Longrightarrow$ rotation rate (Coriolis freq)/latitude
- Motion in N/S direction inhibited by constant angular momentum

Rossby Waves:

- Large scale atmospheric waves caused by variation of Coriolis frequency with latitude $\Longrightarrow$ KdV equation can model Rossby waves (E-W zonal flow) (long wave/incompressible fluid/β plane approximation etc)
  Experimental confirmation: Eight years of Topex/Poseidon altimeter observations. (Pacific dynamics) – 1998

Geophysical applications

D. Bore solitons

- When a tidal wave/tsunami/storm surge hits an estuary $\Longrightarrow$ hydraulic jump (step-wise perturbations like a shock wave) in the water height and speed which propagates upstream.
- Far less dangerous but very similar is the bore (mascaret in French), a tidal wave which can propagate in river for considerable distances.

Examples

- Seine river / Hooghly river
- 1983 Japan sea tsunami $\Longrightarrow$ bores ascended many rivers
- IHOP of Kansas

Geophysical applications

Figure: Tidal bore at the mouth of the Aragnau River (Brazil)

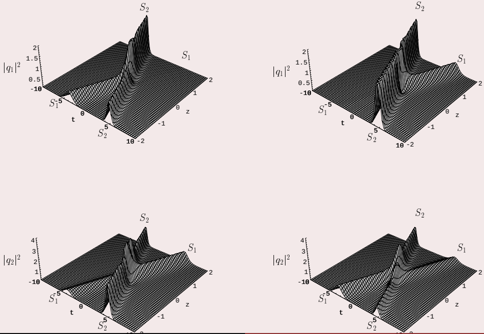
E. Capillary solitons
F. Resonant three and four wave solitons
G. Optical solitons

VII. Solitons in other situations

A. Multicomponent solitons:

- Manakov model

$$
\begin{aligned}
iq_{1,z} + q_{1,tt} + 2(|q_1|^2 + |q_2|^2)q_1 &= 0,\\
iq_{2,z} + q_{2,tt} + 2(|q_1|^2 + |q_2|^2)q_2 &= 0.
\end{aligned}
$$

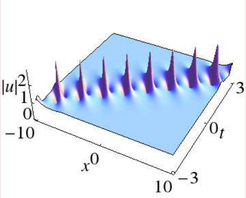


Breathers and Rogue Waves

- Nonlinear Schrödinger Equation

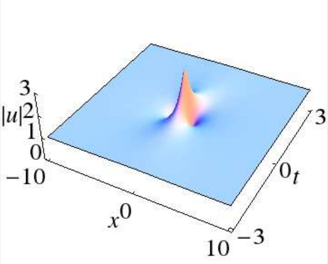
$$iu_t + u_{xx} + q|u|^2u = 0, \quad u \in \mathbb{C}$$

- Coupled NLS Manakov Equation



Breather solution of NLS equation

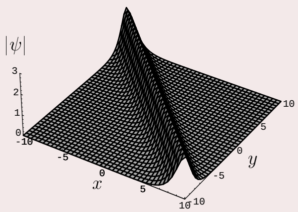
Breathers and Rogue Waves

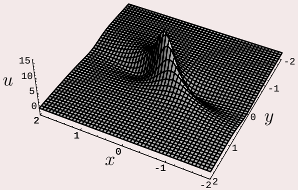


Rogue wave solution of NLS equation

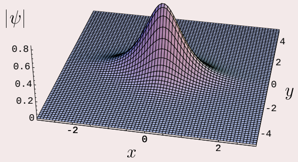
- Rogue waves in BECs/Optics

B. (2+1)- dimensions:





- Plane solitons
- Lump solitons



- Dromions

C. (3+1)- dimensions

- Localized solutions?

VIII. Conclusion

- Solitons are very stable nonlinear entities
- Occur in (1+1) and (2+1) dimensions
- Have considerable physical relevance
- What happens in (2+1) and (3+1) dimensions?

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