Numerical computation of tidal parameters

I. BACKGROUND

A. Static, spherical stars

Here, we use units where $c = G = 1$. The simplest model of a neutron star is a spherically symmetric, nonrotating perfect fluid, supported against gravitational collapse by its pressure. In Schwarzschild coordinates, the corresponding spacetime metric has the form

$$g_{\alpha\beta}dx^\alpha dx^\beta = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

In the rest frame of the matter, its four-velocity is $u^\alpha = (u^t, 0, 0, 0)$. The value of u^t is fixed by the normalization condition

$$-1 = \vec{u}\cdot\vec{u} = g_{\alpha t}u^t u^\alpha \qquad\qquad \rightarrow u^t = e^{-\Phi} \tag{2}$$

With the stress-energy tensor

$$T^{\alpha\beta} = -pg^{\alpha\beta} + (p+\epsilon)u^\alpha u^\beta \tag{3}$$

and a given equation of state $p = p(\epsilon)$. The structure of the star is determined by Einstein's equations that for a metric of the form (1) reduce to the Tolman-Oppenheimer-Volkoff (TOV) form

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \tag{4a}$$

$$\frac{dp}{dr} = -(\epsilon + p)\frac{(m + 4\pi r^3 p)}{r(r-2m)}, \tag{4b}$$

Here, ϵ is the energy density, p is the pressure, and ρ the baryon rest mass density. Changes in these quantities are related by the second law of thermodynamics with $TdS = 0$ in the form

$$\frac{d\log p}{d\log\rho} = \frac{\epsilon + p}{p}\frac{dp}{d\epsilon} \equiv \Gamma. \tag{5}$$

This defines the adiabatic index Γ characterizing the change in pressure at constant entropy. A neutron star model can be defined by the choice of the central energy density, ϵ_c or mass density ρ_c. The boundary conditions at the center are $m(r = 0) = 0$ and $p = p(\rho_c)$. For numerical integrations one usually starts at a small value away from $r = 0$, say at a very small Δr with the initial conditions

$$m(\Delta r) \approx \frac{4}{3}\pi\epsilon(\rho_c)(\Delta r)^3, \qquad\qquad p(\Delta r) \approx p(\rho_c) \tag{6}$$

The surface of the star is defined to be where the pressure goes to zero,

$$p(r = R) = 0 \tag{7}$$

and this must be solved to find R.

There is a more elegant formulation of the TOV equations in terms of an enthalpy variable that avoids this additional step to find the root of p, see e.g., Lindblom ApJ **398**, 569 (1992), or Lindblom and Indik PRD **86**, 084003 (2012) or PRD **89**, 064003 (2014).

The TOV equations in the form specified above are not in a very convenient form for numerical solutions because, for example, the radius at the edge of the star, R, is not known until the equations are solved, so some iteration procedure is required to seek the edge of the star, where p and ϵ become zero.

1. Formulation in terms of enthalpy

A more efficient formulation is obtained when using the enthalpy as the dependent variable. The enthalpy per unit rest mass is $(\epsilon + p)/\rho$ but it is more convenient to work with the quantity

$$h = \ln\left[\frac{\epsilon + p}{\rho}\right] \tag{8}$$

Integrating in terms of h, instead of the radius, has the advantage that the surface is at a known value $h = 0$ and the equations are nonsingular at this point. Then h satisfies

$$\frac{dp}{dh} = \epsilon + p, \qquad h = \int_0^p \frac{dp'}{\epsilon(p') + p'} \tag{9}$$

For positive energy density and pressure h is monotonically increasing with pressure so we can write the EoS in the form

$$\epsilon = \epsilon(h), \qquad p = p(h) \tag{10}$$

An additional simplification when using h is that there is an algebraic relation between the metric potential and h in the interior. The equations of stellar structure can then be written as

$$\frac{dr}{dh} = -\frac{r(r-2m)}{m + 4\pi r^3 p}, \qquad \frac{dm}{dh} = 4\pi r^2 \epsilon \frac{dr}{dh} \tag{11}$$

Solving the equations in this form begins by specifying initial conditions, at the center of the star where $h = h_c$, and then integrating toward the surface of the star where $h = 0$. Numerically, it is often convenient to start at a small r_0 instead of at $r = 0$, and to use the starting conditions:

$$r(h_c) = r_0, \qquad m(h_c) = \frac{4\pi}{3} r_0^3 \epsilon(h_c) \tag{12}$$

At the surface of the star, the derivative dr/dh is non-zero and bounded, so this formulation completely eliminates the problems associated with solving $p(R) = 0$ to locate the stars surface. The total gravitational mass and radius of the stellar model are obtained in this formulation simply by evaluating the solutions $m(h)$ and $r(h)$ at the surface of the star where $h = 0$: $M = m(0)$ and $R = r(0)$.

B. Exercises: Equilibrium configuration

1. The first step in obtaining a neutron star model is deciding on the equation of state used to describe its matter. There are many different calculations of the equation of state, using various different approximations, but the calculations for most of these are so complicated that they are simply output as tables of values. Here we will use the SLy equation of state, described in Douchin and Haensel A&A **380**, 151 (2001) and freely available in tabular form here. (This is a standard *soft* EOS, i.e., one that gives stars with small radii, and thus small tidal deformabilities. The name comes from Skyrme-Lyon, the name of the nucleon-nucleon interaction model used.)

   Note that this table gives the baryon number density, which you do not need here, and denotes the energy density, which we call ϵ, by rho. Your code thus needs to be able to read in an equation of state table, and interpolate it [in the "inverse form" of energy density versus pressure, $\epsilon(p)$], so you can compute the pressure as a function of the enthalpy, $p(h)$, and thus also (by composition) the energy density as a function of the enthalpy, $\epsilon(h)$.

   There are fancy methods of interpolation that respect the first law of thermodynamics [e.g., Haensel and Prószyński ApJ **258**, 306 (1982)], but these are only necessary for very high accuracy work. For our purposes, it is sufficient to use a simple linear interpolation of the logarithms of p and ϵ, where one interpolates the logarithms of these quantities since they range over many orders of magnitude. It also works well to compute the enthalpy on a mesh of p values and again perform linear interpolation of the logarithms to compute $p(h)$.

2. Now you can write a numerical code to integrate the TOV equations (4). Think about the units you want to use. A convenient choice are units with $G = c = M_\odot = 1$. Alternatively, you can use only geometric units or restore the factors of G and c to use cgs or SI units. The conversion factors are given in the Appendix. For example, the $c = G = M_\odot = 1$ units are related to cgs units by $\mathrm{cm} = c^2/(GM_\odot)$ and $g = 1/M_\odot$ where the rhs contains numerical values in cgs units. (Note that the SLy EOS table is given in cgs units.)

3. For the SLy equation of state, compute the mass and radius of the NS for different choices of the central density. Find the maximum mass of a stable neutron star. (This is the maximum mass for which the star's gravitational mass increases with its central density, or central enthalpy.) Plot your results for the mass-radius curve, with the mass in units of solar masses, and the radius in km. Note that you can evaluate $m(h)$ and $r(h)$ at the enthalpy corresponding to a small density in the EOS table to obtain their surface values, since the EOS table does not go down to zero density.

II. COMPUTATION OF LOVE NUMBERS

As discussed in the lectures, the Love numbers are computed by solving for the perturbed structure and spacetime geometry of the NS. This is accomplished by decomposing the perturbations to the metric components and the fluid variables into spherical harmonics in the Regge-Wheeler gauge, and substituting into Einstein's equations, the four-velocity normalization condition, and the Bianchi identities/stress-energy conservation. Similar to the Newtonian case that you considered in Ex. 3, in the limit of linear, static, even-parity perturbations, the system of perturbation equations simplifies to a single ODE for a variable y that is the logarithmic derivative of the perturbation to the analog of the Newtonian potential in the metric; see, e.g., Sec. VIII in Landry and Poisson PRD **89**, 124011 (2014). The resulting ODE for the ℓth multipole is given by

$$ry' + y(y-1) + \frac{2}{f}\left[1 - \frac{3m}{r} - 2\pi r^2(\epsilon + 3p)\right] y - \frac{1}{f}\left[\ell(\ell+1) - 4\pi r^2(\epsilon+p)\left(3 + \frac{d\epsilon}{dp}\right)\right] = 0 \tag{13}$$

Here, the prime denotes the derivative with respect to r (we have suppressed the functional dependence on r in y, m, ϵ, and p) and we omitted the subscripts ℓ on y. Additionally, $f := 1 - 2M/R$, where M is the star's gravitational mass and R is its areal (i.e., Schwarzschild coordinate) radius.

The boundary condition at the center is

$$y(r=0) = \ell \tag{14}$$

Given a numerical solution in the interior the next step is to evaluate it at the surface to obtain

$$Y = y(r=R). \tag{15}$$

The exterior solution has long been known in terms of special functions (associated Legendre functions, which are a special class of hypergeometric functions). From the asymptotic behavior of these functions at $r \gg R$ one can identify the piece that falls off as $r^{-(\ell+1)}$ associated with the ℓth mass multipole moment, and the piece that grows as r^ℓ corresponding to the tidal field.

Similar to the Newtonian case you considered in Ex. 3, the tidal field can be eliminated by considering the logarithmic derivative of the metric function. Finally, matching the interior and exterior solutions at the surface of the NS determined the Love numbers [1]. For the simple cases with $\Gamma \neq 0$ considered here, the result from the numerical solution for Y from the interior can be substituted directly into the algebraic expression for the Love numbers k_ℓ that are generalizations of the Newtonian results you derived in Ex. 3. The general result is given in terms of the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ and is

$$2k_\ell = \frac{RA_1' - [Y - \ell - 4M/(R-2M)]A_1}{[Y + \ell + 1 - 4M/(R-2M)]B_1 - RB_1'} \tag{16}$$

where

$$A_1 = {}_2F_1(-\ell, -\ell+2; -2\ell; 2M/R) \tag{17a}$$
$$B_1 = {}_2F_1(\ell+1, \ell+3; 2\ell+2; 2M/R) \tag{17b}$$

and primes on A_1 and B_1 denote derivatives with respect to R. These derivatives can easily be evaluated in terms of the hypergeometric functions themselves using the identity

$$\frac{d}{dz}{}_2F_1(a,b;c;z) = \frac{ab}{c}{}_2F_1(a+1,b+1;c+1;z). \tag{18}$$

Alternatively, if you prefer typing to hypergeometric functions, explicit expressions for the lowest order multipoles are

$$\begin{aligned} k_2 = {} & \frac{8}{5}(1-2C)^2 C^5 [2C(Y-1)-Y+2] \\ & \times \left\{ 2C\left(4(Y+1)C^4+(6Y-4)C^3+(26-22Y)C^2+3(5Y-8)C-3Y+6\right) \right. \\ & \left. -3(1-2C)^2(2C(Y-1)-Y+2)\log\left(\frac{1}{1-2C}\right)\right\}^{-1}, \end{aligned} \tag{19a}$$

$$\begin{aligned} k_3 = {} & \frac{8}{7}(1-2C)^2 C^7 \left[2(Y-1)C^2-3(Y-2)C+Y-3\right] \\ & \times \left\{ 2C\left[4(Y+1)C^5+2(9Y-2)C^4-20(7Y-9)C^3+5(37Y-72)C^2-45(2Y-5)C+15(Y-3)\right] \right. \\ & \left. -15(1-2C)^2\left(2(Y-1)C^2-3(Y-2)C+Y-3\right)\log\left(\frac{1}{1-2C}\right)\right\}^{-1}, \end{aligned} \tag{19b}$$

where $C = M/R$ is the compactness of the neutron star.

A. Exercises: Computation of Love numbers

1. Extend your TOV code to compute y for $\ell = 2$ from (13). You can either change variables in the ODE to enthalpy, or convert everything to r (since you now know the domain over which you are solving the equation, for a given stellar model). You can invert $r(h)$ to find $h(r)$ using logarithmic interpolation, as before.
2. Compute the corresponding Love number k_2 from (16) or (19a).
3. Use the same setup for which you computed the mass-radius curve, and compute $\lambda = 2k_2R^5/3$. Generate a plot of λ as a function of mass. Here you should plot λ in cgs units and the mass in solar masses, so you can compare with the SLy curve in Fig. 2 of Hinderer *et al.* PRD **81**, 123016 (2010).

1. Appendix: units

In general relativity, the equations often simplify when using units adapted to the situation, typically the convenient choice is geometrized units with $G = c = 1$. For neutron stars, it is convenient to further specialize the geometric units to units where $M_\odot = 1$, so overall the units are $G = c = M_\odot = 1$. The conversion factors are given in the table below

Dimension	**cgs**	**Geometrized**	$c = G = M_\odot = 1$ **Units**
Length	$d = d(\text{cm})$	$d^*(\text{cm}) = d$	$d_*(-) = d/M_\odot^*$
Time	$t = t(\text{s})$	$t^*(\text{cm}) = ct$	$t_*(-) = ct/M_\odot^*$
Mass	$M = M(\text{g})$	$M^*(\text{cm}) = (G/c^2)M$	$M_*(-) = (G/M_\odot^* c^2)M$
Mass Density	$\rho_0 = \rho_0(\text{g cm}^{-3})$	$\rho_0^*(\text{cm}^{-2}) = (G/c^2)\rho_0$	$\rho_{0,*}(-) = (GM_\odot^{*,2}/c^2)\rho_0$
Energy	$E = E(\text{erg})$	$E^*(\text{cm}) = (G/c^4)E$	$E_*(-) = (G/M_\odot^* c^4)E$
Energy Density	$e = e(\text{erg cm}^{-3})$	$e^*(\text{cm}^{-2}) = (G/c^4)e$	$e_*(-) = (GM_\odot^{*,2}/c^4)e$
Pressure	$P = P(\text{dyne cm}^{-2})$	$P^*(\text{cm}^{-2}) = (G/c^4)P$	$P_*(-) = (GM_\odot^{*,2}/c^4)P$
Specific Internal Energy	$\epsilon = \epsilon(\text{erg g}^{-1})$	$\epsilon^*(-) = (1/c^2)\epsilon$	$\epsilon_*(-) = (1/c^2)\epsilon$

Taken from online notes, which you can also refer to for further details. The quantity $M_\odot^* = Gc^{-2}M_\odot = 1.476 \times 10^5$cm is the mass of the sun in geometrized units.

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[1] For objects with a discontinuous transition of the density in the interior and exterior such as quark stars this matching must be performed more carefully, see the last part of Ex. 3