

ICTS Lecture 3

Outline

So far, we have covered extremes of

infinite conductivity (in integrable models), or
zero conductivity (in MBL or Anderson localized phases).

Usually we measure finite conductivities. This could be because
system is coupled to a “bath”, e.g., of phonons.

But what if we have a closed, non-integrable Hamiltonian?

Is the conductivity finite? What determines it?



$$D = \frac{1}{3} \int_0^\infty \langle \mathbf{v}(0) \cdot \mathbf{v}(t) \rangle_T dt \approx v^2 \tau$$

Outline

1. The generic state of 1D metals is a “Luttinger liquid”. We can realize this by adding integrability-breaking terms that retain lattice translation invariance. Conclusion: there are at least *two* different mechanisms for adjustable power-laws in LL transport.
2. We can break translation invariance and integrability by adding a harmonic trap to an integrable system (leaving aside those like Calogero that retain integrability in a trap; see Abanov-Wiegmann, Kulkarni-Polychronakos).
3. What about electrons in real materials and $d > 1$?

1D strategy: start with simple models

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

(Clean XXZ chain + random z-directed Zeeman field)

$$h_i \in (-\Delta, \Delta)$$

Equivalent by Jordan-Wigner transformation to spinless fermions with nearest-neighbor interactions.

$$H = -t \sum_i \left(c_{i+1}^\dagger c_i + h.c. \right) + V \sum_i n_i n_{i+1} + \sum_i \mu_i n_i$$

Advantages:

1. “Solvable” (integrable) without random field. Can add a staggered field to break integrability while keeping translation invariance.
2. Can check predictions with DMRG/matrix product numerics.

One example of the Luttinger liquid idea

Consider the XXZ model when it describes a gapless, linearly dispersing system. ($|\Delta| < 1$).

Some things are *independent of the precise value of interaction*:

for example, the free energy at finite temperature is

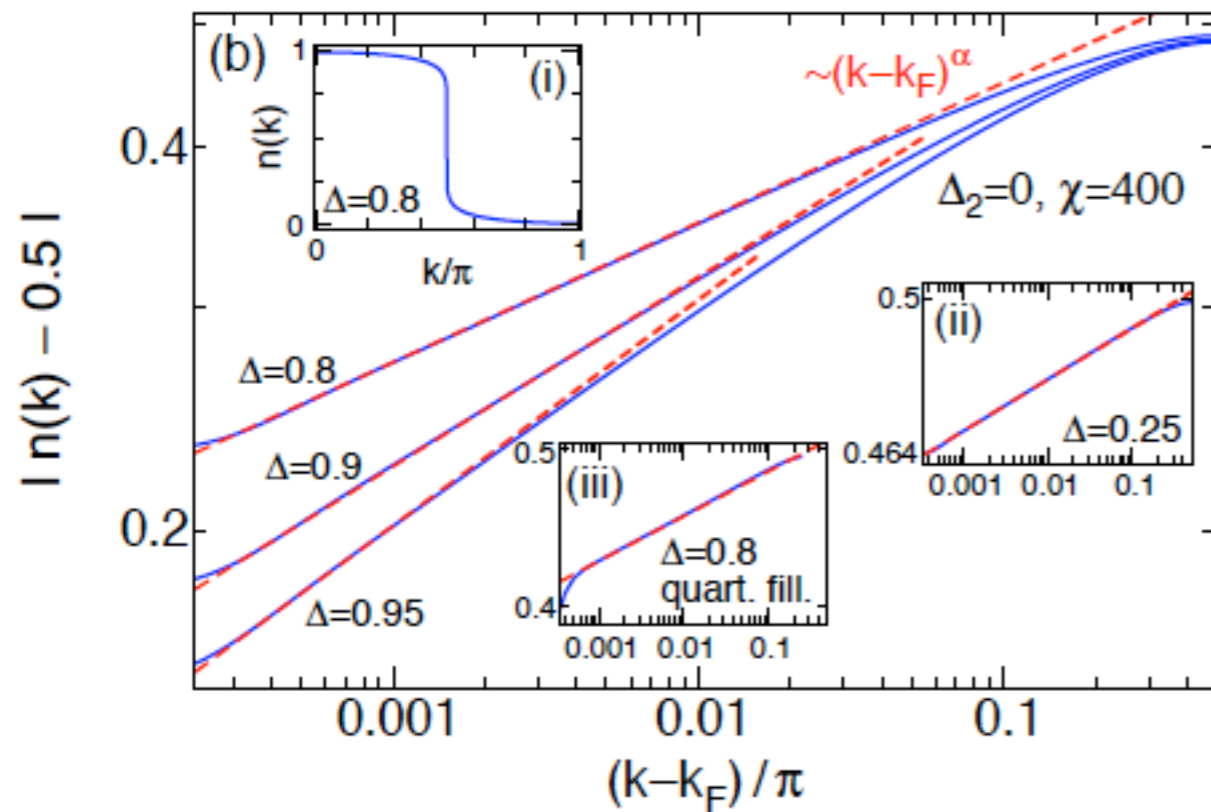
$$f = \frac{F}{L} = f_0 - \frac{\pi}{6} c (kT)^2 \hbar v$$

with central charge $c=1$ everywhere along this line.

Actually interactions are marginal, not irrelevant, and there is a line of critical points that differ in several transport measurements.

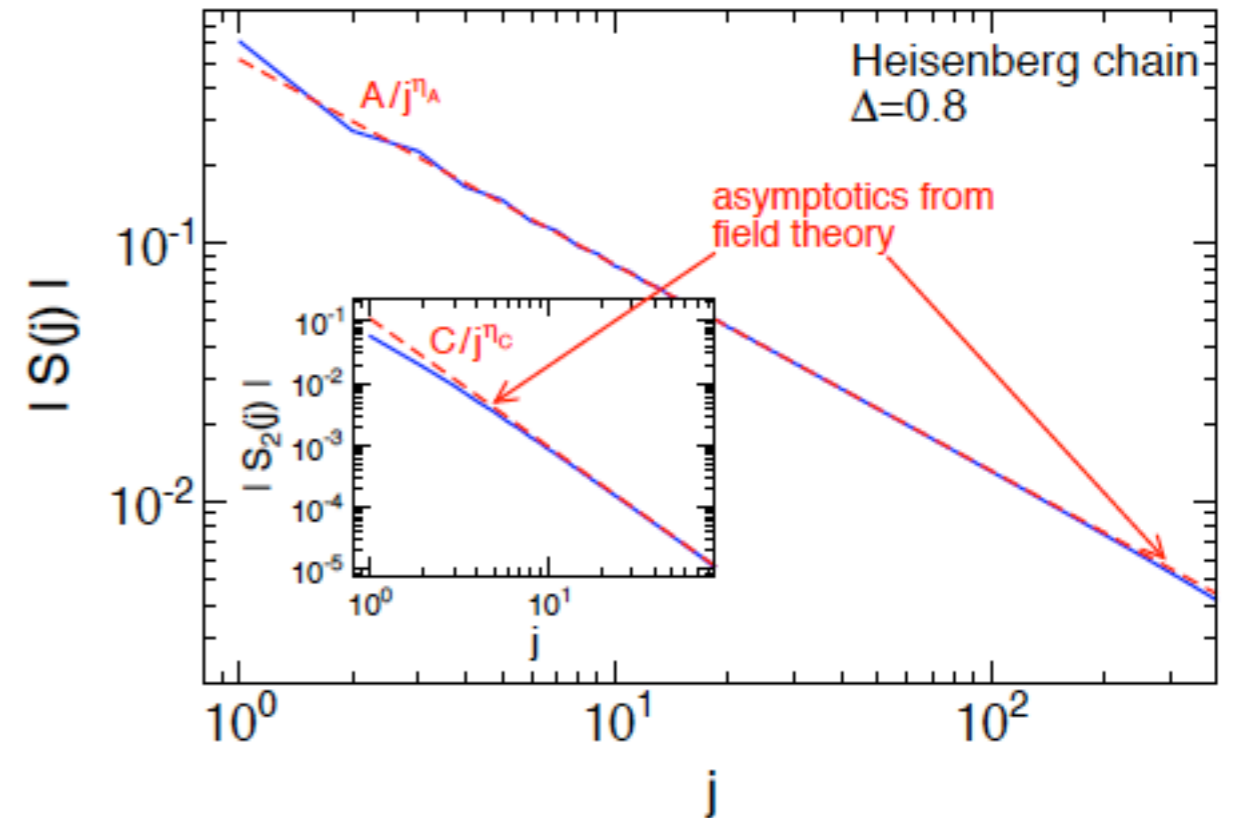
These are Luttinger liquids with varying Luttinger parameter.

Tests of Luttinger liquid behavior in the XXZ model



Momentum distribution $n(k)$

(C. Karrasch and JEM, PRB)



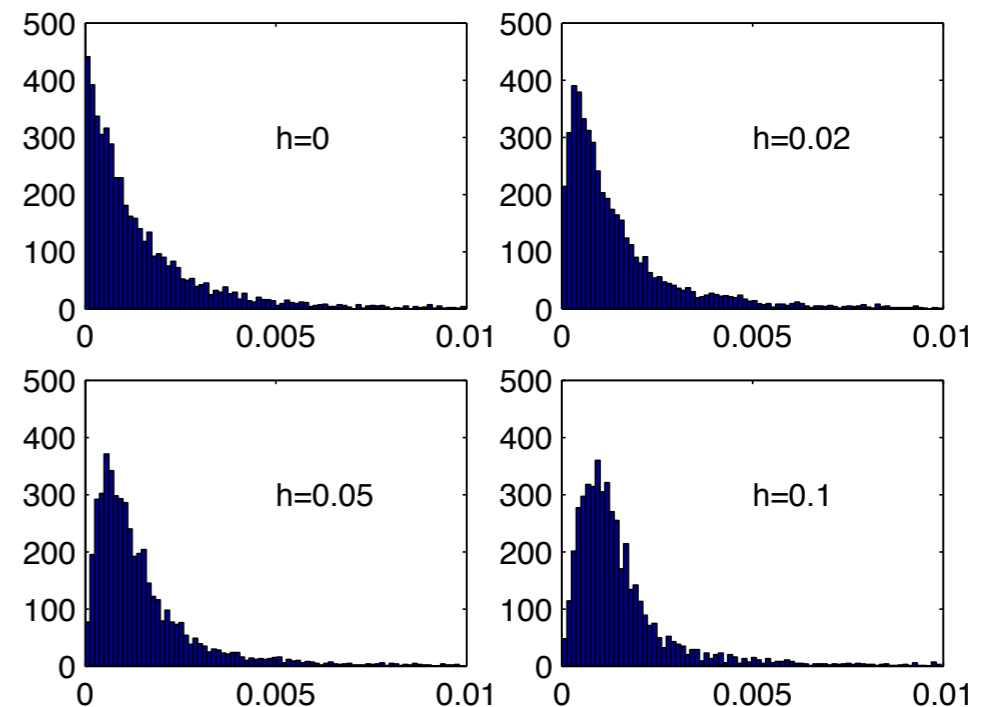
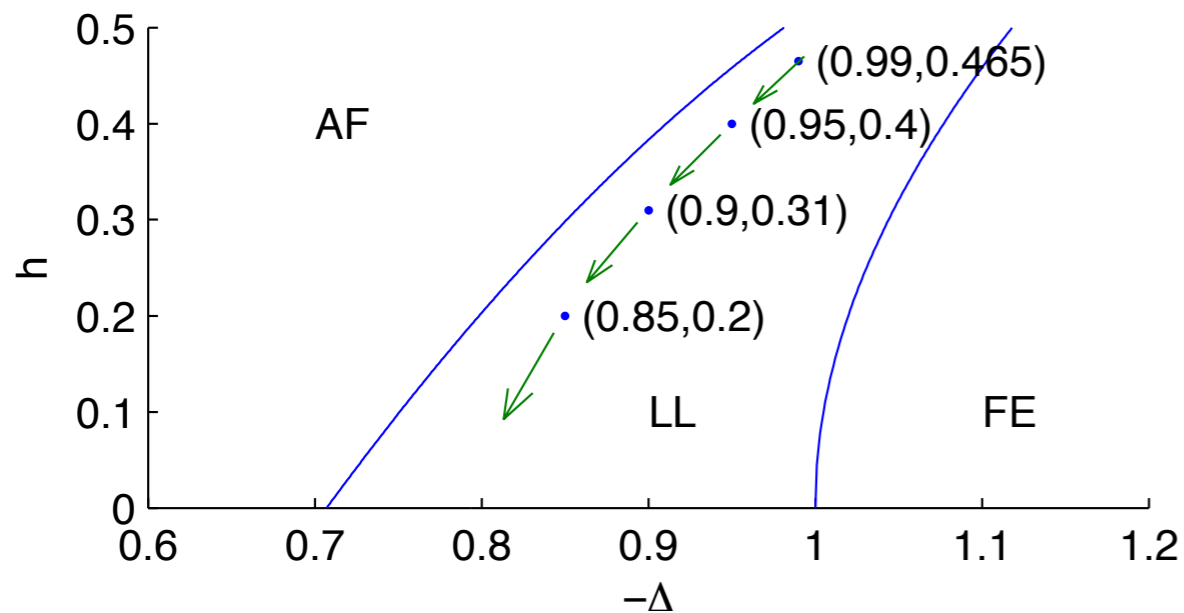
Check of leading staggered and uniform correlators against Lukyanov and Terras

Current ground-state applications moving to 2D: FQHE, spin liquids, ...

Next: try to solve an open problem of dynamical properties at finite temperature.

Staggered field non-integrability

$$H = \sum_{i=1}^L \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z + (-1)^i h S_i^z \right]$$



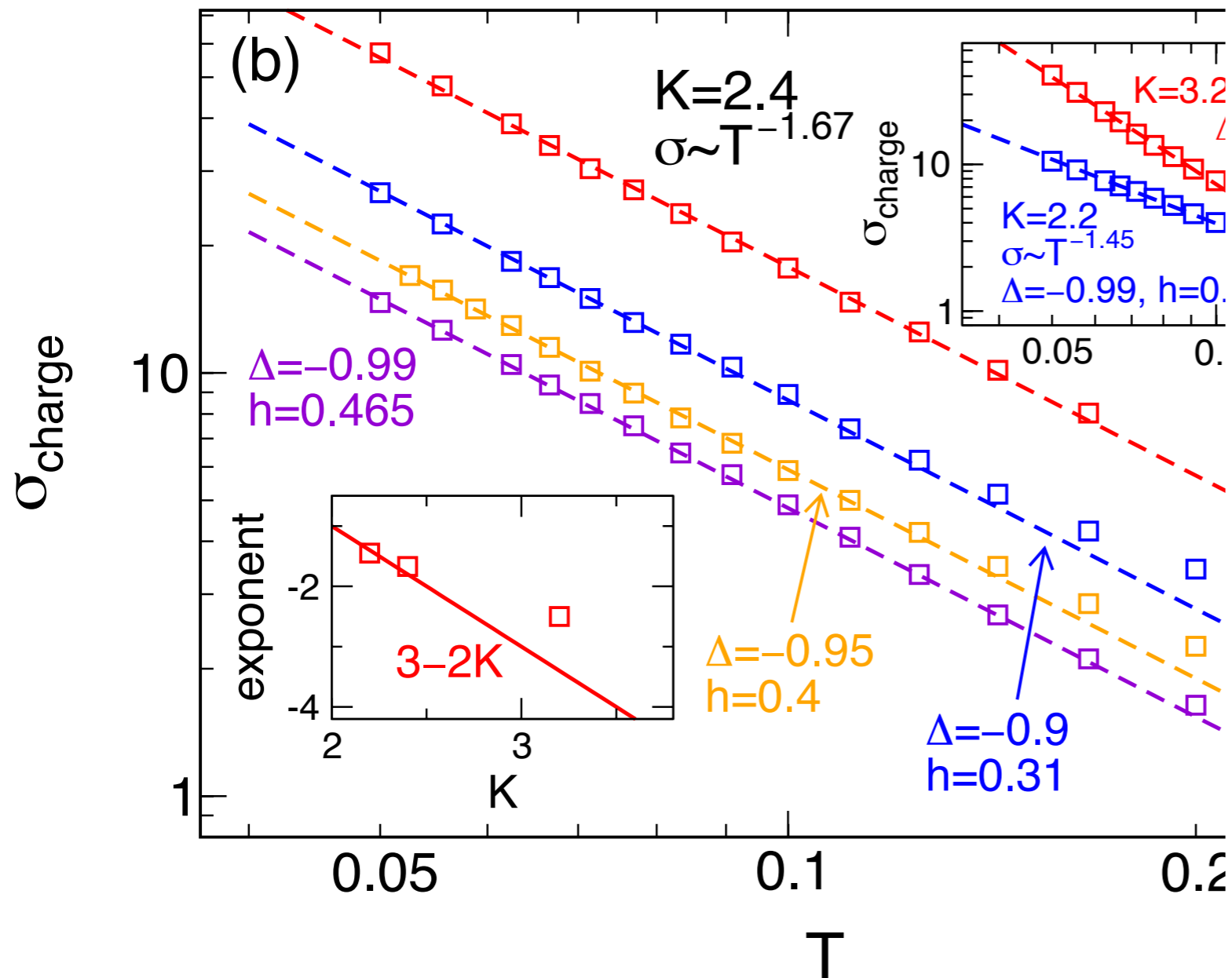
In one region, of the phase diagram, h is irrelevant (system remains Luttinger liquid), and we can track RG flow

Level statistics become Wigner-Dyson (level repulsion) rather than Poisson

Argument for Poisson statistics: two nearby states are likely to be in different symmetry sectors, and hence do not repel each other as they are not mixed by a perturbation.

“Conventional” conductivity scaling

$$\sigma = \lim_{t_M \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{1}{LT} \operatorname{Re} \int_0^{t_M} \langle J(t) J(0) \rangle dt.$$



For K not too large, linear prediction is self-consistent and power-laws are observed that are consistent with bosonization predictions.

Conductivity diverges at low temperature as the integrability-breaking perturbation is irrelevant.

(Huang, Karrasch, Moore PRB 2013)

What if we break integrability by imposing an external global potential?

The hard-rod gas in a harmonic trap is equivalent to N one-dimensional harmonic oscillators with hard-core repulsive interaction. The Hamiltonian reads

$$H = \sum_{j=1}^N \left[\frac{1}{2} p_j^2 + V(x_j) \right] + \sum_{j < k} U(x_j - x_k) \quad (1)$$

with

$$V(x) = \frac{1}{2} \omega^2 x^2, \quad U(\delta x) = \begin{cases} 0 & |\delta x| > a \\ \infty & |\delta x| \leq a, \end{cases} \quad (2)$$

where $a > 0$ denotes the rod length, and x_j and p_j denote positions and momenta (we set $m = 1$). Upon re-scaling time as $t \rightarrow t\omega$, we may set $\omega = 1$ without loss of generality. Starting from a configuration such that $x_{j+1} - x_j \geq a$, $j = 1, \dots, N - 1$, the gas evolves as N decoupled oscillators, until the next collision (*i.e.*, $x_{j+1} - x_j = a$ for some j) in which the rods j and $j + 1$ exchange their velocities spontaneously. Such a dynamics can be efficiently and exactly simulated. There are two integrable limits. Upon removing the trap, one recovers the usual hard-rod gas. Its momentum distribution is conserved and its dynamics map to those of N independent particles. Meanwhile, in the limit of vanishing rod length $a = 0$, we obtain N decoupled harmonic oscillators. Yet, in the presence of both trap and interaction, we find no other conserved quantities besides the total energy and the center-of-mass energy which we set to 0.

A quantum Newton's cradle

Toshiya Kinoshita¹, Trevor Wenger¹ & David S. Weiss¹

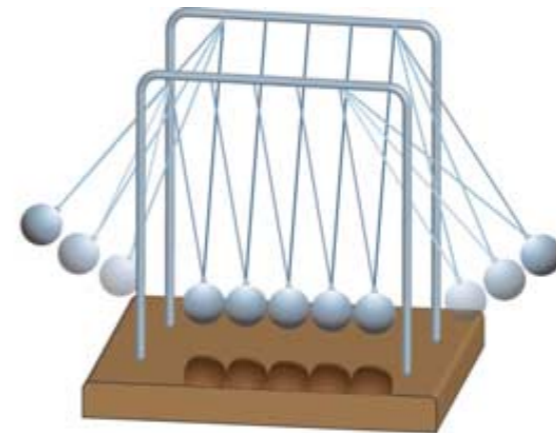
It is a fundamental assumption of statistical mechanics that a closed system with many degrees of freedom ergodically samples all equal energy points in phase space. To understand the limits of this assumption, it is important to find and study systems that are not ergodic, and thus do not reach thermal equilibrium. A few complex systems have been proposed that are expected not to thermalize because their dynamics are integrable^{1,2}. Some nearly integrable systems of many particles have been studied numerically, and shown not to ergodically sample phase space³. However, there has been no experimental demonstration of such a system with many degrees of freedom that does not approach thermal equilibrium. Here we report the preparation of out-of-equilibrium arrays of trapped one-dimensional (1D) Bose gases, each containing from 40 to 250 ⁸⁷Rb atoms, which do not noticeably equilibrate even after thousands of collisions. Our results are probably explainable by the well-known fact that a homogeneous 1D Bose gas with point-like collisional interactions is integrable. Until now, however, the time evolution of out-of-equilibrium 1D Bose gases has been a theoretically unsettled issue⁴⁻⁶, as practical factors such as harmonic trapping and imperfectly point-like interactions may compromise integrability. The absence of damping in 1D Bose gases may lead to potential applications in force sensing and atom interferometry.

To see qualitatively why 1D gases might not thermalize, consider the elastic collision of two isolated, identical mass classical particles in one dimension. Energy and momentum are conserved only if they

the prevailing density¹⁴. The collisions that we study satisfy this criterion well. Our observations extend from the Tonks–Girardeau regime, where only pairwise collisions can occur¹⁵, to the intermediate coupling regime, where there can be three- (or more) body collisions¹⁵⁻¹⁷. In both regimes, atoms that are set oscillating and colliding in a trap do not appreciably thermalize during our experiment.

We start our experiments with a Bose–Einstein condensate (BEC) loaded into the combination of a blue-detuned two-dimensional (2D) optical lattice and a red-detuned crossed dipole trap (see Methods). The combination of light traps makes a 2D array of distinct, parallel Bose gases, with the 2D lattice providing tight transverse confinement and the crossed dipole trap providing weak axial trapping¹¹. The dynamics within each tube of the 2D array are strictly 1D because the lowest transverse excitation, $\hbar\omega_r$ (where $\omega_r/2\pi = 67$ kHz is the transverse oscillation frequency), far exceeds all other energies in

a



Integrability vs. thermalization in 1D Bose gas

Hard rods in 1D

This is a famous example of an integrable classical model where hydrodynamical approaches have a long history.

We study it because it has the same type of integrable kinetic theory (Boltzmann equation) as Lieb-Liniger or XXZ, but long-time numerics are much easier than in the quantum problems.

Simple guess:

at short times, the system will look like the integrable system;

at long times, the system will thermalize.

Three regimes, none thermal

1. integrable hydrodynamics
2. development of chaos
3. hydrodynamical (not thermal) steady state

Figures from Cao et al., PRL 2018

**Incomplete Thermalization from Trap-Induced Integrability Breaking:
Lessons from Classical Hard Rods**

Xiangyu Cao,¹ Vir B. Bulchandani,¹ and Joel E. Moore^{1,2}

¹*Department of Physics, University of California, Berkeley, Berkeley, California 94720, USA*

²*Materials Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

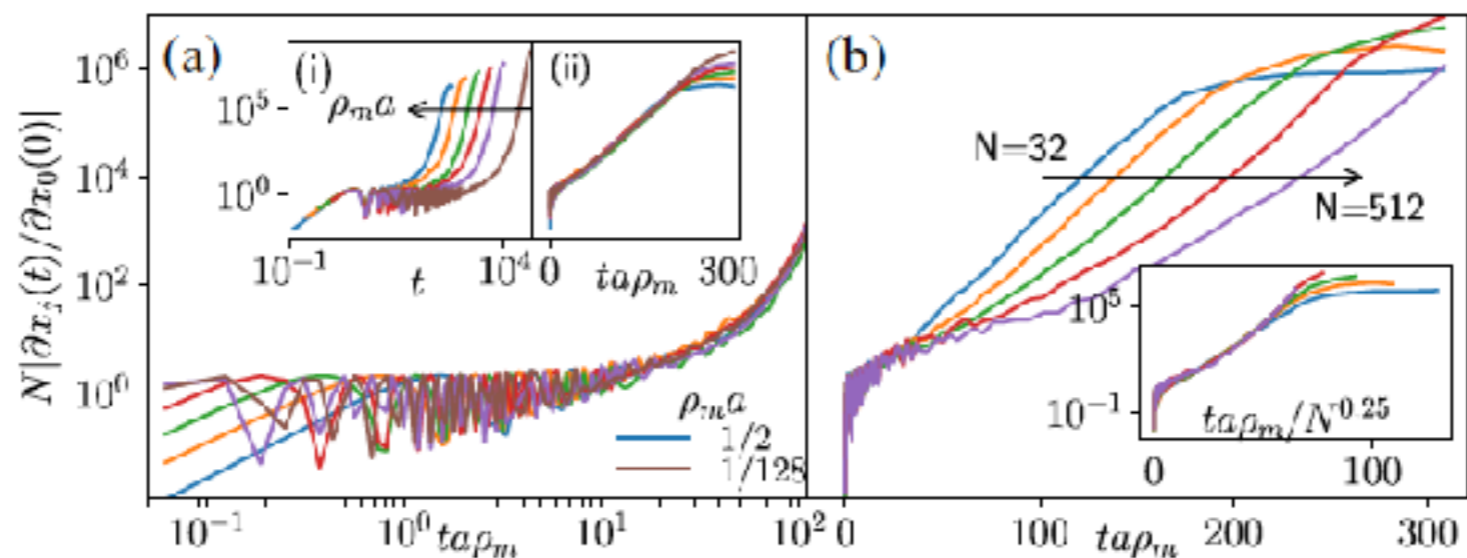
Large-N limit: continuum hydrodynamics (Percus)

$$\partial_t \rho + \partial_x (v \rho) = 0, \quad v[\rho](p) = p + \frac{a \int_{p'} (p - p') \rho(x, p')}{1 - a \int_{p'} \rho(x, p')}$$

Can add forcing from external potential to this Boltzmann-like equation.

Hydrodynamics works until a time scale determined by the initial density in units of rod length. Then non-integrability takes over and we see exponential separation of trajectories.

Strength of chaos: Lyapunov exponent observed to scale as $\gamma \sim N^{-0.25}$



However, final state does not seem to be thermal (Maxwellian)

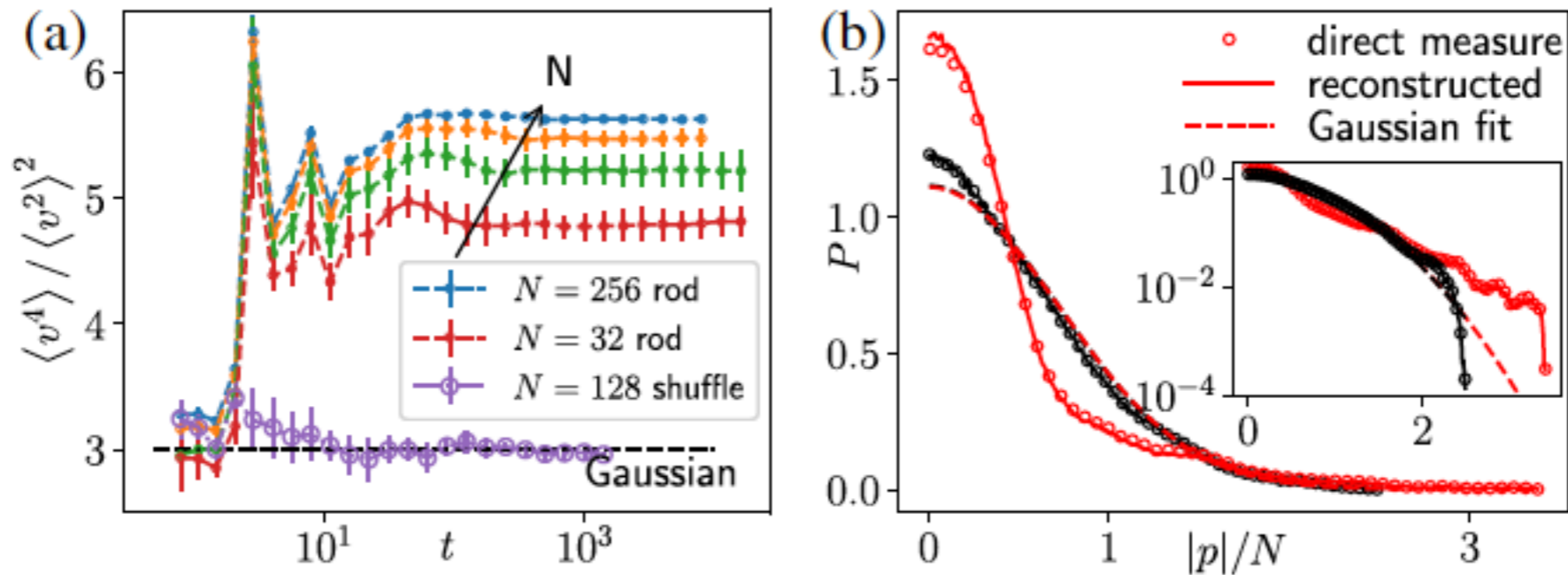
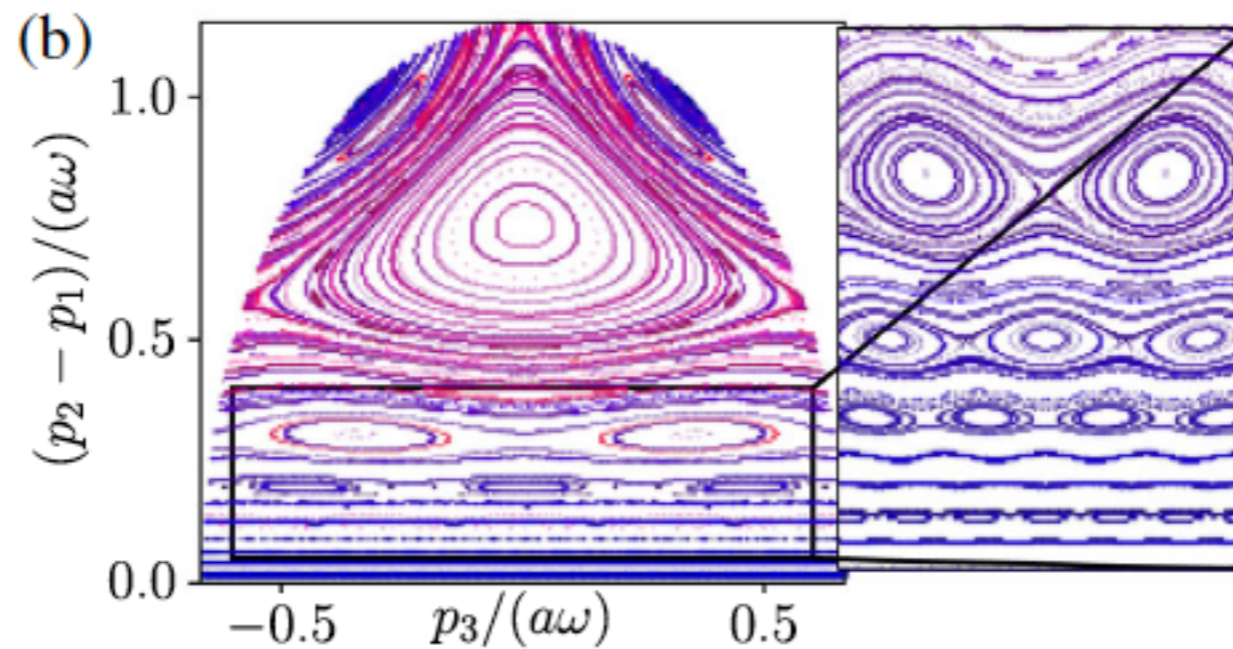


FIG. 5. (a) (Non)-Gaussianity of the velocity distribution of the time-averaged ensemble, as revealed by the moment-ratio test. (b) Comparing the velocity distribution with the reconstructed one from the density, assuming that the late-time ensemble solves the stationary THRE [Eq. (8)]. The two long-time ensembles are obtained from two squeezed (red) and circular (black) ICs, both with $\rho_m a = 1/2$.

Why?

Go back to three rods



3D phase space: make Poincaré map by looking just after a collision to reduce to 2 parameters. Orbits do not look integrable (since fractal structure) but also do not look micro canonical (ergodic over all possibilities of constant energy).

So the final state need not be thermal. What is it?

A constraint on final ensemble

We find that the final ensemble, for more than a few rods, is a solution of the steady-state hydrodynamical equation at every (x,p) :

$$\partial_x (v[\rho]\rho) - \partial_x V \partial_p \rho = 0$$

We do not see additional thermalization on the accessible time scale (hundreds of thousands of periods for small number of rods).

So at least in this classical problem the hydrodynamical approach is not just useful for time evolution, but gives (partial) information about the final ensemble.

Electron hydrodynamics?

Why don't we normally treat electrons in a solid as fluid-like?

Real solids are not perfect: momentum is not a conserved quantity.

Solids are not generally isotropic, either—they can break spatial and time-reversal symmetries.

Finally, electrons are charged, which makes them a somewhat unusual fluid.

Hydrodynamics of 2D electrons

In materials that are very clean, momentum relaxation may take a relatively long time. It might be better to view electrons as a *fluid* rather than as independently scattering quasiparticles.



Thomas Scaffidi
(UCB/Toronto)

Solid-state electrons where fluid properties measured

2DEGs (Molenkamp & others, 1990s)

Graphene (P. Kim; A. Geim)

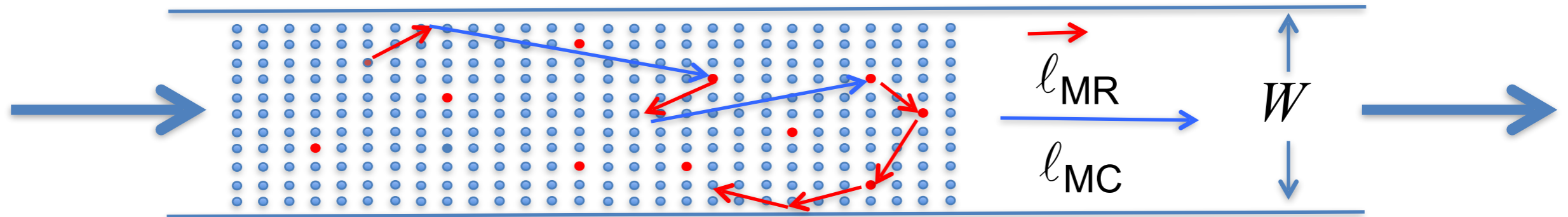
Layered crystals (A. Mackenzie)

...

Hydro in clean electron systems (slide from A. Mackenzie)

R.N. Gurzhi, JETP 44, 771 (1963); Usp. Fiz. Nauk 94, 689 (1968)

Key point introduced by Gurzhi: In solids, hydrodynamic effects can be parameterised in terms of the relationship between three length scales: momentum relaxing mfp ℓ_{MR} , momentum conserving mfp ℓ_{MC} and sample dimension (here W).



$$\ell_{MR} \ll \ell_{MC}, W$$

Standard ohmic theory applies;
 R is determined entirely by solid resistivity ρ and usual geometrical factors

$$\ell_{MC} \ll W \ll \ell_{MR}$$

Hydrodynamic theory applies; R is determined entirely by fluid viscosity η , boundary scattering and 'Navier-Stokes' geometrical factors

Pioneering measurements on microfabricated narrow 2DEG channels:

L.W. Molenkamp & M.J.M de Jong, Phys. Rev. B 49, 5038 (1994)

How are electron fluids different from normal fluids?

One active area, originally motivated by AdS but derivable without gravity dual: (Kitaev, Maldacena-Shenker-Stanford, 2015; Sachdev-Ye-Kitaev models):

The Lyapunov exponent for short-time onset of chaos is bounded

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar}$$

Other ways electron fluids have different hydrodynamics:

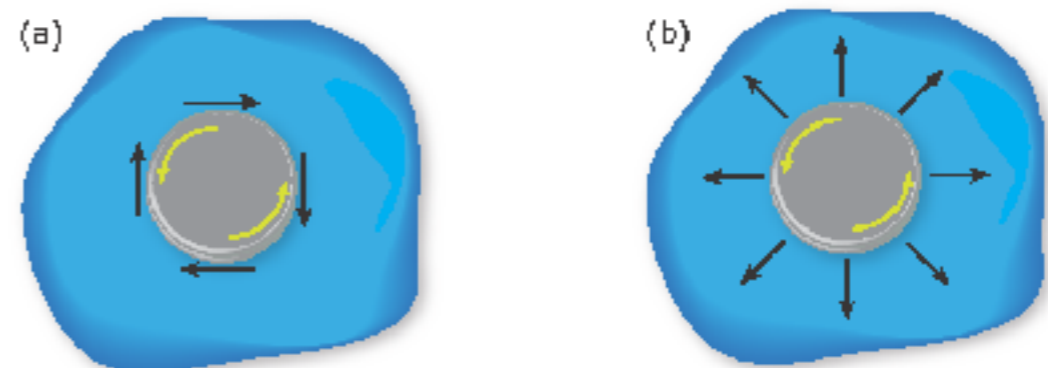
“**Hall viscosity**” in topological states: (Avron; Read; Gurarie...) stress tensor is

$$T_{ij} = p\delta_{ij} - \lambda_{ijkl}\xi_{kl} - \eta_{ijkl}\dot{\xi}_{kl}$$

and T-breaking allows an odd contribution $\eta_{ijkl}^{(A)} = -\eta_{klij}^{(A)}$

What is Hall viscosity in T-breaking gapless systems?

Allowed by symmetry...



From Hughes-Leigh-Parrikar

Hydrodynamics of electrons

What makes electron fluids different from classical fluids?

In 2D and 3D, can induce broken T by a magnetic field and have a new kind of viscosity, “Hall viscosity”

Is significant, and could be observable, in simple *metals*:
T. Scaffidi, N. Nandi, B. Schmidt, AP Mackenzie, JEM, PRL 17

In the quantum Hall regime there are two contributions in the q^2 correction to Hall conductance

$$\sigma_{xy}(q) = \sigma_{xy}(0) + O(q^2)$$

that are comparable (Hoyos-Son): one from Hall viscosity and one from (inverse) internal compressibility. In a metal, the internal compressibility part is small and the Hall viscosity follows from a Boltzmann calculation.

Hydrodynamical calculation

$$\partial_t \vec{v} = \eta_{xx} \nabla^2 \vec{v} + \eta_{xy} \nabla^2 \vec{v} \times \vec{z} + \frac{e}{m} (\vec{E} + \vec{v} \times \vec{B})$$

Scaling in bulk, Alekseev:

$$\eta_{xx} = \eta \frac{1}{1 + (2 \frac{l_{MC}}{r_c})^2}$$
$$\eta_{xy} = \eta \frac{2 \frac{l_{MC}}{r_c}}{1 + (2 \frac{l_{MC}}{r_c})^2}$$

Can incorporate boundary effects of diffuse scattering in a thin channel by going back to Boltzmann eq:

Prediction:

$$\rho_{xy} = \rho_{xy}^{\text{bulk}} \left(1 - 6 \left(\frac{l_{MC}}{W} \right)^2 \right) \text{ for } B \rightarrow 0$$

We want $l_{MC} \ll l_{MR}$ for hydrodynamics. Crucially, the hydrodynamic regime can happen only if W is somewhat larger than l_{MC} . In order to measure this effect, one needs to choose W to be as small as possible, but still larger than l_{MC} . If we have, say, $W = 5l_{MC}$, one then expects a relative change in the slope of the Hall resistivity at zero field of the order $\simeq 25\%$, which should be measurable.

Possibly just seen in graphene, Geim et al., arXiv:1806.01606.

Crystal background modification to quantum transport theory

For the velocity in Boltzmann equation

$$\partial_t f_1 + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_1 = \int w'(f'_1 f'_2 - f_1 f_2) d\mathbf{p}_2 d\mathbf{p}'_1 d\mathbf{p}'_2.$$

we should really use the semiclassical velocity of a wave packet:

$$\frac{dx^a}{dt} = \frac{1}{\hbar} \frac{\partial \epsilon_n(\mathbf{k})}{\partial k_a} + \mathcal{F}_n^{ab}(\mathbf{k}) \frac{dk_b}{dt}.$$

Second term is anomalous velocity or “Berry phase” piece.

This is just one example of how, even in a perfect crystal, the non-isotropy can modify the long-distance physics.

Active topic in quantum hydrodynamics above 1D: linear-response relaxation

An example of recent progress on a long-standing question:

Are there intrinsic limits on how fast a system can relax to equilibrium?

Related to conductivity via the Kubo formula: how rapidly does the current-current correlation decay in time?

Also related to existence of “hydrodynamical” regimes of electron transport where quasiparticle scattering is not the right picture.

Some past formulations:

Mott-Ioffe-Regel: mean free path must be at least the lattice spacing

Sachdev: $\tau \geq \frac{\hbar}{k_B T}$ quantum $e^{-iHt/\hbar} \leftrightarrow e^{-H/(k_B T)}$ thermal

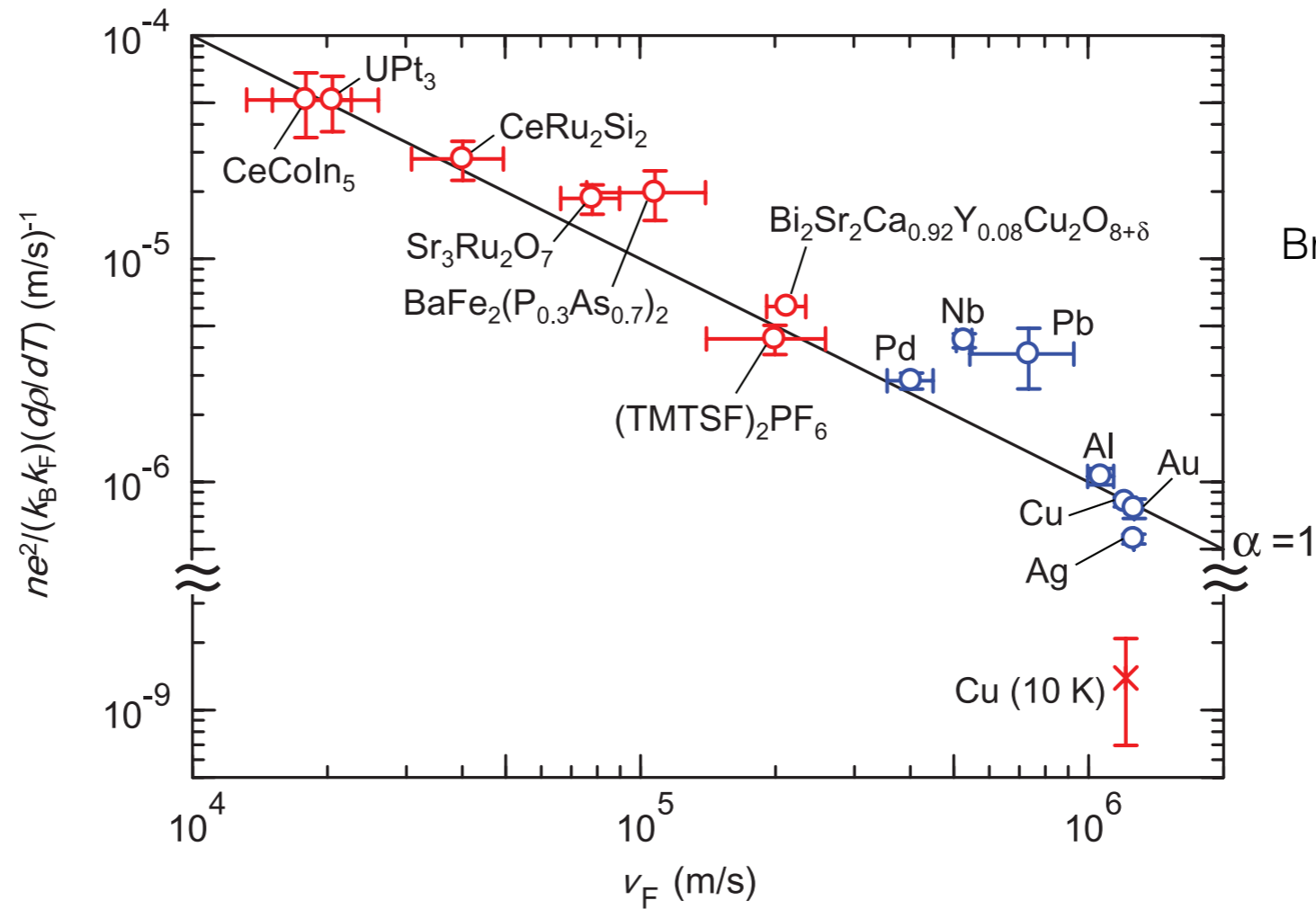
Kovtun-Son-Starinets: the viscosity is bounded below* $\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$

Hartnoll: reinterpret viscosity bound as a lower limit on diffusion constant

Near-equilibrium

Why these bounds matter: nature seems to contain such behavior

$$\tau \geq \frac{\hbar}{k_B T}$$



Bruin et al. (Mackenzie),
Science 2013

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

is satisfied (I believe) in all experimental liquids,
and within ~ 10 of saturation in helium and QGP,
but there exist violations in exotic theories

Some take-away messages

Lecture I: in integrable models, transport is ballistic and relaxation is to a “generalized Gibbs ensemble”.

Flow between local GGEs seems to be well described by Bethe-Boltzmann equation, which can be checked against a few exact far-from-equilibrium results for XXZ, and against numerics.

Lecture II: the many-body-localized phase can be understood as a new type of integrable model. There are *local* conserved quantities (not sums of local densities).

The weak interactions between these lead to slow logarithmic dynamics that can be observed either in entanglement or in revival rate.

Lecture III: dynamics without integrability is hard. Transport in Luttinger liquids is controlled by leading irrelevant perturbations (a difference from higher dimensions).

Integrability breaking by a trap, in a classical problem, seems to lead to only partial thermalization on observable time scales. Are there quantum problems that retain some integrable features without being fully integrable?

Thank you for your participation!