Supersymmetric Lattice Models

Kareljan Schoutens
Institute of Physics and QuSoft
University of Amsterdam

ICTS Bangalore – 2 August 2018

The name of the game

QM with *N*=2 supersymmetry

$$Q^2 = 0, \qquad (Q^\dagger)^2 = 0$$

$$[Q,H] = 0, \qquad H = \{Q,Q^\dagger\}, \qquad [Q^\dagger,H] = 0$$

[not to be confused with graded Lie algebra symmetries such as `supersymmetric *tJ*-model']

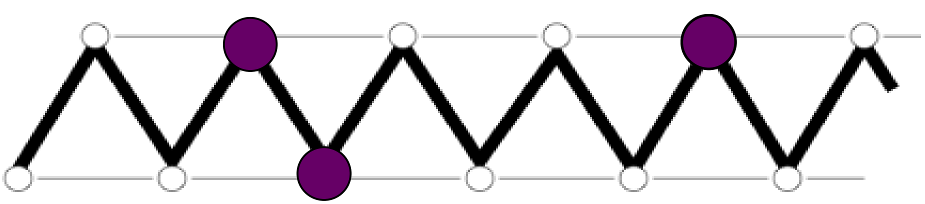
The name of the game

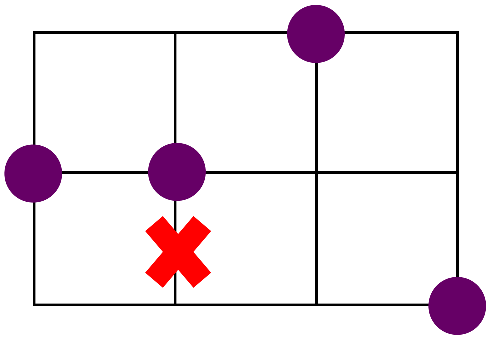
susy QM for lattice fermions

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \qquad i, j \in \Lambda$$

supercharges expressed in fermion operators

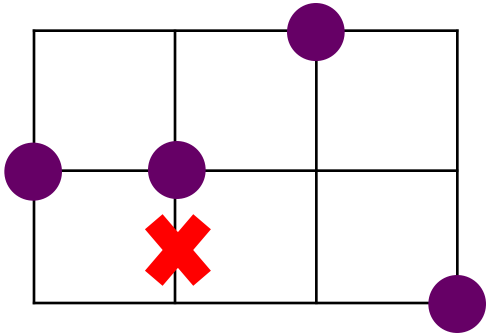
➽ Hamiltonians with kinetic (hopping) terms and strong interactions





The name of the game

features of
susy lattice models



- integrability
- critical behaviour ➟ supersymmetric CFT
- off-critical ➟ kink picture of integrable susy QFT
- superfrustration ➟ proliferation of susy ground states
- dynamics, I ➟ adiabatic driving in susy gs manifold
- dynamics, II ➟ out-of-equilibrium transport, MBL?

Witten index and susy spectra

Basic structure of susy spectra

- ***E* ≥ *0*** for all states
- ***E* > *0*** states are paired into **doublets**

$$\{|\psi\rangle, Q^\dagger|\psi\rangle\}, \quad Q|\psi\rangle = 0$$

- ***E* = *0*** iff a state is a **singlet** under supersymmetry

$$Q|\psi_{\mathrm{gs}}\rangle = 0, \quad Q^\dagger|\psi_{\mathrm{gs}}\rangle = 0$$

Fermion number and Witten index

Supercharges change fermion number F by ± 1

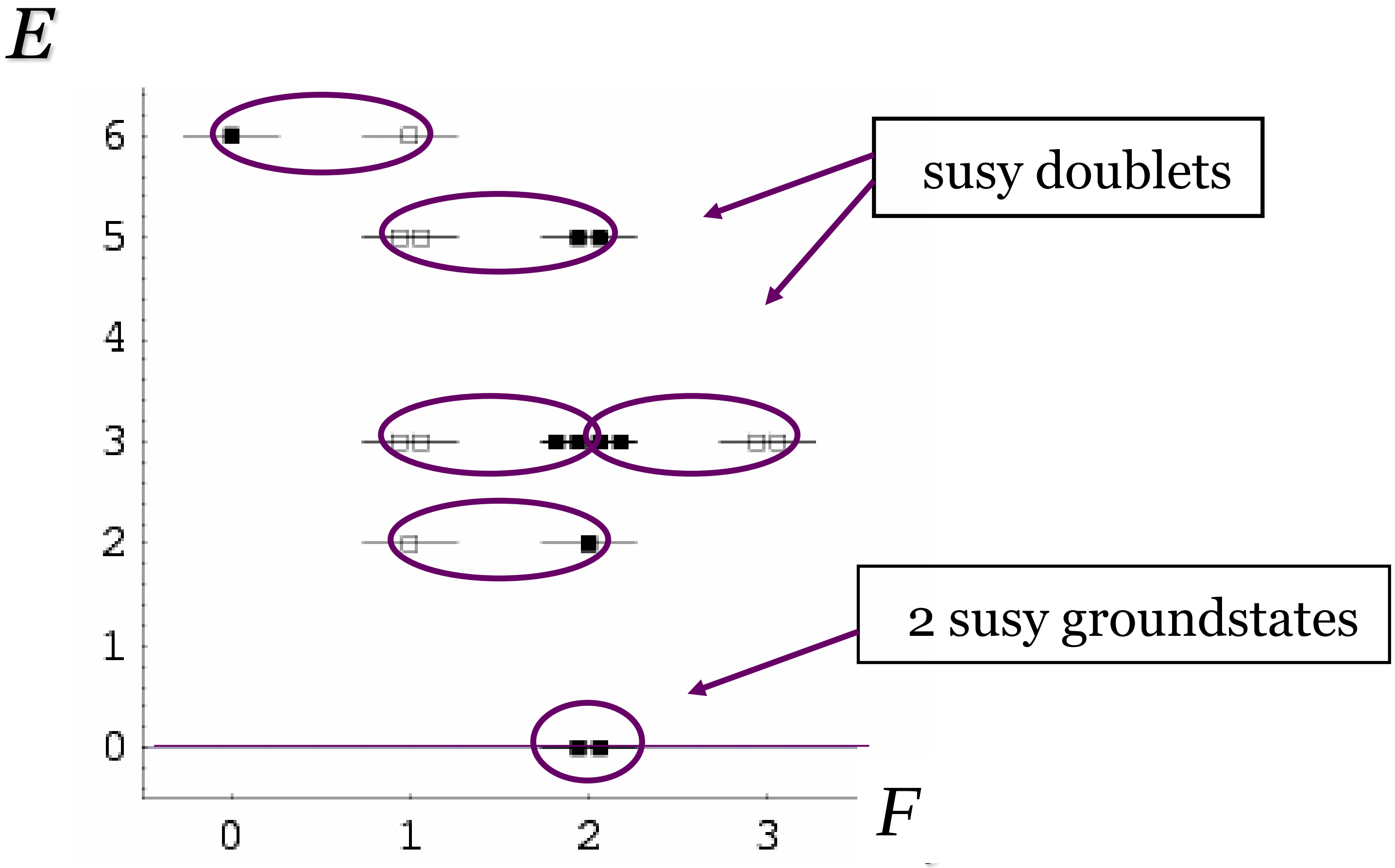
$$[F, Q] = -Q, \quad [F, Q^\dagger] = Q^\dagger$$

Witten index

$$W = \mathrm{Tr}\left[(-1)^F\right]$$

- W easily evaluated by computing trace over all states
- $E>0$ doublets cancel in W, only $E=0$ singlets contribute
- $W\neq 0$ implies existence of **at least** $|W|$ $E=0$ singlets

Example of a susy spectrum

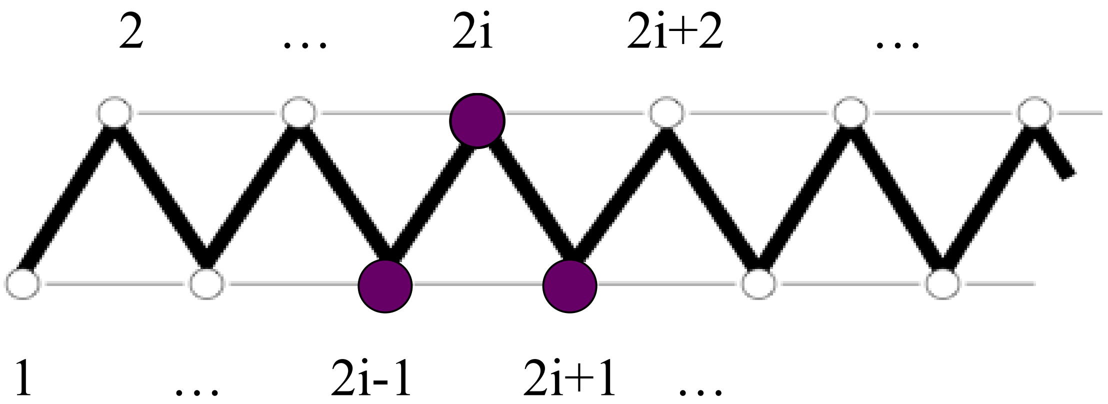


Sampler of susy lattice models

Nicolai model

supercharge

$$Q^{\text{Nic}} = \sum_i c_{2i-1} c^{\dagger}_{2i} c_{2i+1}$$

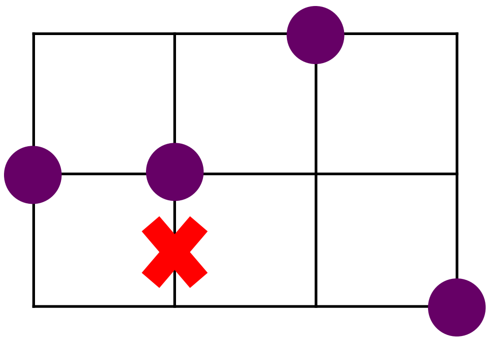


Nicolai 1976

M_1 model

configurations:

lattice fermions with nearest neighbor exclusion



supercharge

takes out particle where possible

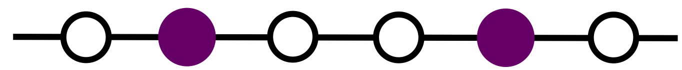
$$Q^{M_1} = \sum_i c_i P_i, \qquad P_i = \prod_{<ij>} (1 - c_j^\dagger c_j)$$

Fendley-KjS-de Boer 2003

M_1 model on 1D lattice

configurations

lattice fermions with nearest neighbor exclusion



supercharge and Hamiltonian

$$Q^{M_1} = \sum_i (1 - n_{i-1}) c_i (1 - n_{i+1}), \qquad n_i = c_i^\dagger c_i$$

n.n. exclusion

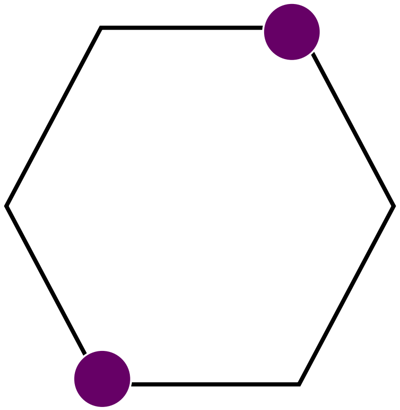
$$H^{M_1} = \sum_i \left[(1 - n_{i-1}) c_i^\dagger c_{i+1} (1 - n_{i+2}) + \text{h.c.} \right] + \sum_i n_{i-1} n_{i+1} - 2F + L$$

hopping

n.n.n. repulsion

M_1 model on 6 site chain

$$W = \mathrm{Tr}\left[(-1)^F\right]$$



$F = 0$: 1 state

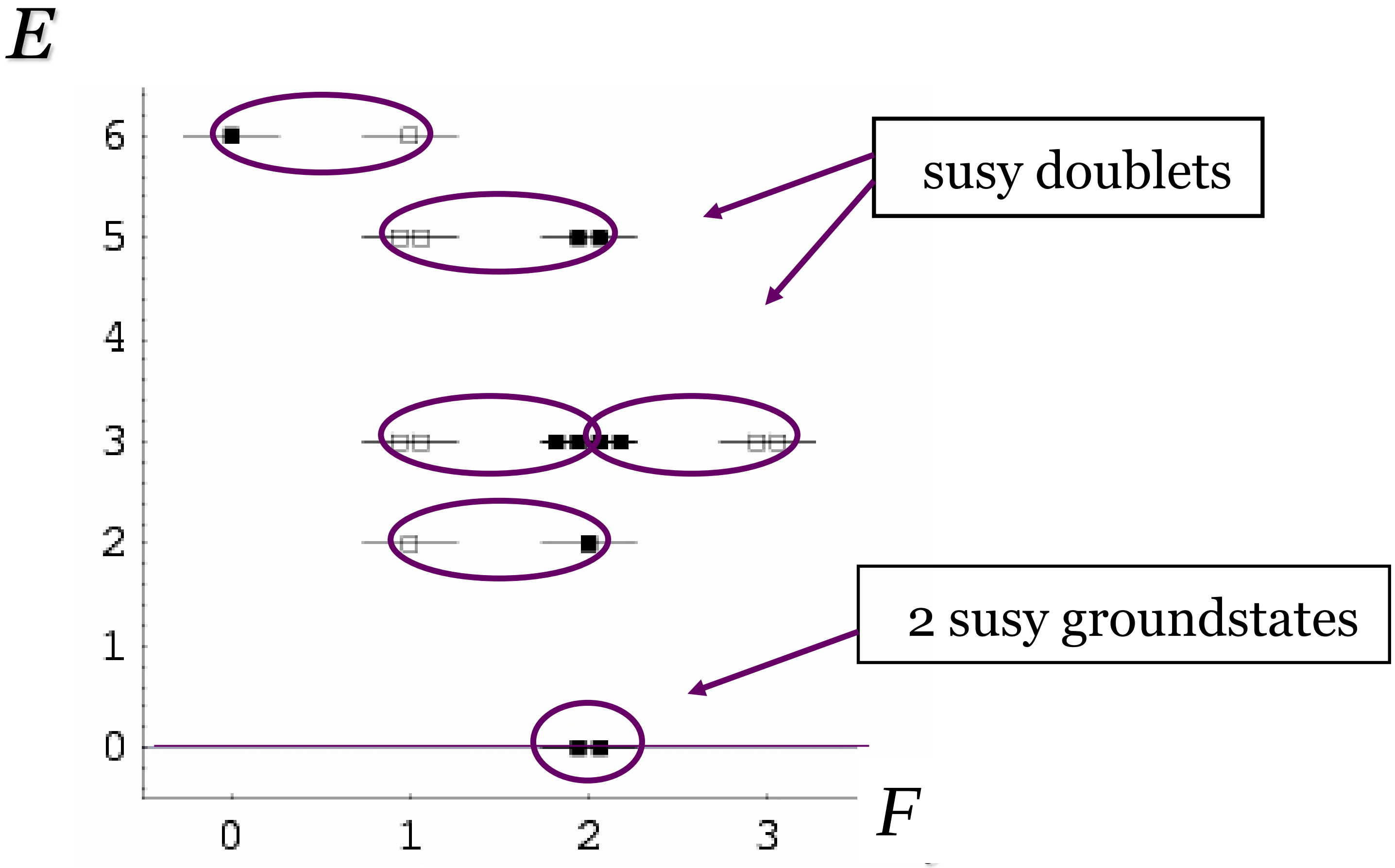
$F = 1$: 6 states

$F = 2$: 9 states

$F = 3$: 2 states

$$\Rightarrow W = 1 - 6 + 9 - 2 = 2$$

M_1 model, 6 site chain, W=2



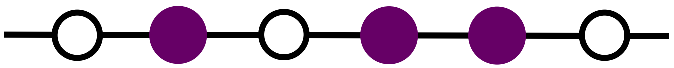
M_k model in 1D

Fendley-Nienhuis-KjS 2003

configurations

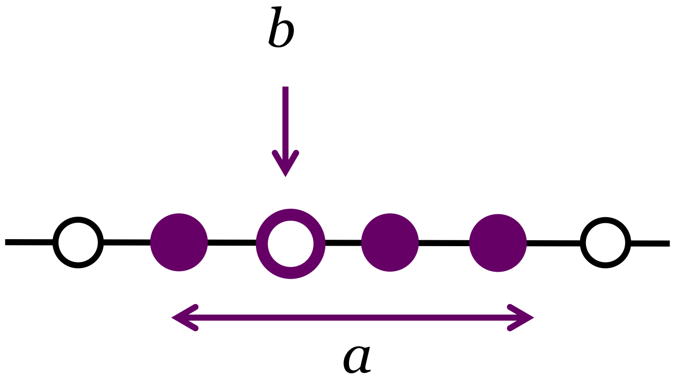
lattice fermions **up to *k***

nearest neighbors occupied



supercharge

$$Q^{M_k} = \sum_i \sum_{a,b} \lambda_{[a,b],i} d_{[a,b],i}$$



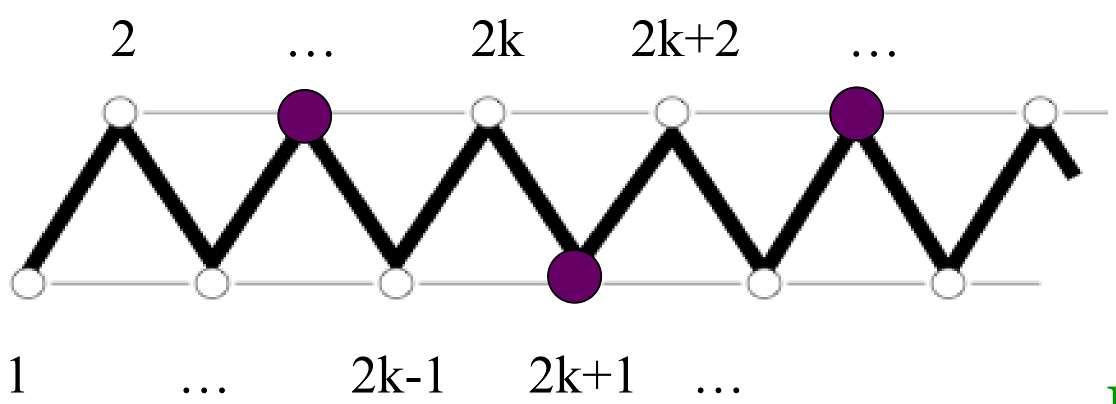
annihilates particle at position b from string of length a

Coupled free fermion chains

supercharge

moves particle from bottom to top chain

$$Q = c_2^\dagger c_1 + \sum_{k=1}^{L-1} \left(e^{i\alpha_{2k-1}\pi/2} c_{2k}^\dagger + e^{i\alpha_{2k}\pi/2} c_{2k+2}^\dagger \right) c_{2k+1}$$

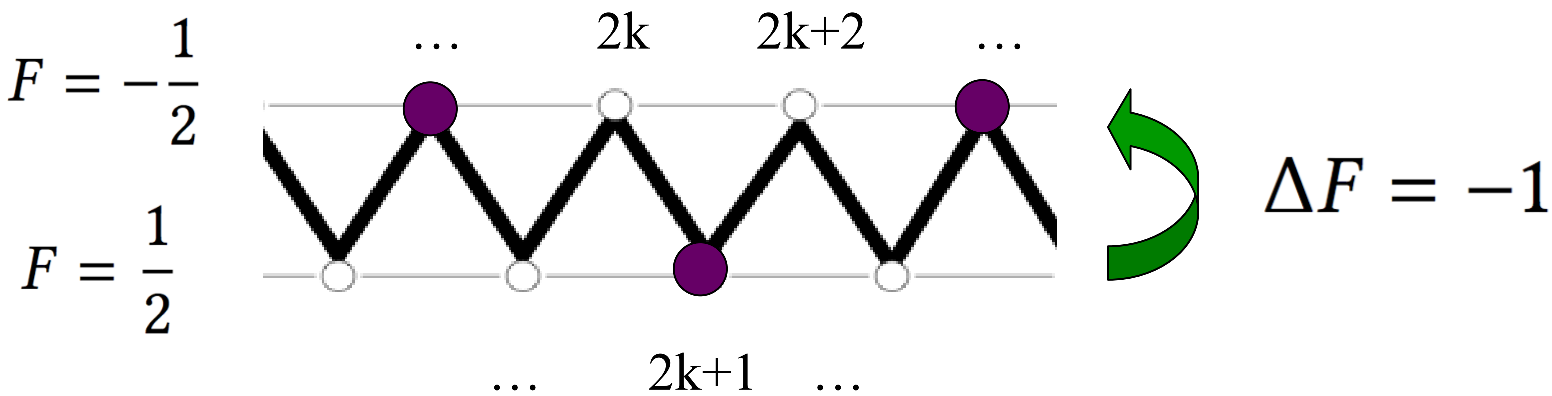


$$\alpha_k \equiv \sum_{j=1}^{k} (-1)^j n_j$$

Fendley-KjS 2007

Coupled free fermion chains

particles on lower (upper) chain have $F = \pm 1/2$ (semions)



Witten index $\quad W = \mathrm{Tr}\left[(-1)^F\right]$

2^L $E=0$ groundstates realized via band of L tightly bound pairs between top and bottom chains

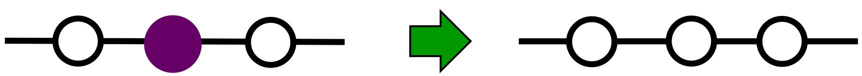
Particle hole symmetric M_1 model

supercharge

$$Q = \sum_i (d_i + e_i^\dagger)$$

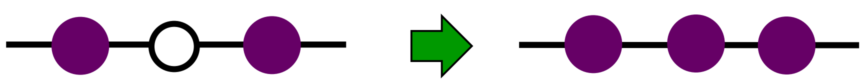
$$d_i = p_{i-1} c_i p_{i+1}$$

$$p_i = 1 - n_i$$



$$e_i^\dagger = n_{i-1} c_i^\dagger n_{i+1}$$

$$n_i = c_i^\dagger c_i$$



de Gier-Feher-Nienhuis-Rusaczonek 2015

A curious mapping

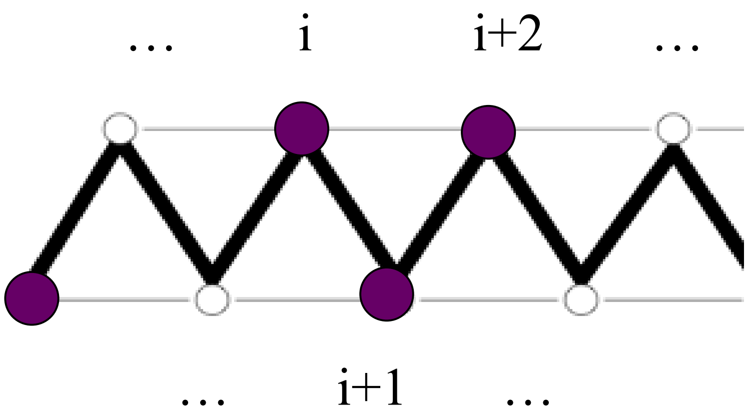
coupled fermion chain model (FS) and particle-hole symmetric M_1 model (FGNR) turn out to be equivalent

empty ladder :	$\lvert 0000\,0000\,0000\rangle_{FS}$	$\leftrightarrow$	${}_0\lvert 1100\,1100\,1100\rangle_{FGNR}$
single FS semion :	$\lvert 0000\,1000\,0000\rangle_{FS}$	$\leftrightarrow$	${}_0\lvert 1100\,0011\,0011\rangle_{FGNR}$
single FS pair :	$\lvert 0000\,1100\,0000\rangle_{FS}$	$\leftrightarrow$	${}_0\lvert 1100\,0100\,1100\rangle_{FGNR}$
lower leg filled :	$\lvert 1010\,1010\,1010\rangle_{FS}$	$\leftrightarrow$	${}_0\lvert 0000\,0000\,0000\rangle_{FGNR}$
single FGNR particle :	$\lvert 1010\,0110\,1010\rangle_{FS}$	$\leftrightarrow$	${}_0\lvert 0000\,1000\,0000\rangle_{FGNR}$
upper leg filled :	$\lvert 0101\,0101\,0101\rangle_{FS}$	$\leftrightarrow$	${}_0\lvert 1010\,1010\,1010\rangle_{FGNR}$
upper leg plus semion :	$\lvert 1101\,0101\,0101\rangle_{FS}$	$\leftrightarrow$	${}_0\lvert 0101\,0101\,0101\rangle_{FGNR}$
filled ladder :	$\lvert 1111\,1111\,1111\rangle_{FS}$	$\leftrightarrow$	${}_0\lvert 0110\,0110\,0110\rangle_{FGNR}$

Feher-Garbali-de Gier-KjS 2017

Models with Q cubic in c_i

Z_2 Nicolai model



$$Q^{\mathbb{Z}_2} = \sum_i c_i c_{i+1} c_{i+2}$$

Sannomiya-Katsura-Nakayama 2017

N=2 susy SYK model

$$Q^{\mathrm{SYK}} = \sum_{ijk} J_{ijk} c_i c_j c_k$$

random couplings on full graph

Fu-Gaiotto-Maldacena-Sachdev 2017

Features

- integrability and field theory connections
- superfrustration
- adiabatic driving and topological response

- out-of equilibrium dynamics, MBL?

Moriya 2016
Padmanabhan-Rey-Teixeira-Trancanelli 2017

Integrability and field theory connections

Integrability

M_1 model at λ=1 integrable by Bethe Ansatz – can be understood via mapping to XXZ chain at Δ=-1/2

Fendley-Nienhuis-KjS 2003

Hagendorf et al found 1-parameter family of models $M_k[\lambda]$ with couplings $\lambda_{[a,b],j}$ such that

- all couplings repeat for j modulo $(k+2)$
- $M_k[\lambda]$ model integrable by nested Bethe Ansatz for all λ
- $M_k[\lambda]$ is critical for $\lambda=1$

Hagendorf-Fokkema-Huijse 2014

Hagendorf-Huijse 2015

Integrability

couplings $\lambda_{[a,b],j}$ for $M_3[\lambda]$

$$y = \sqrt{\frac{1+\sqrt{5}}{2}}$$

λ=1

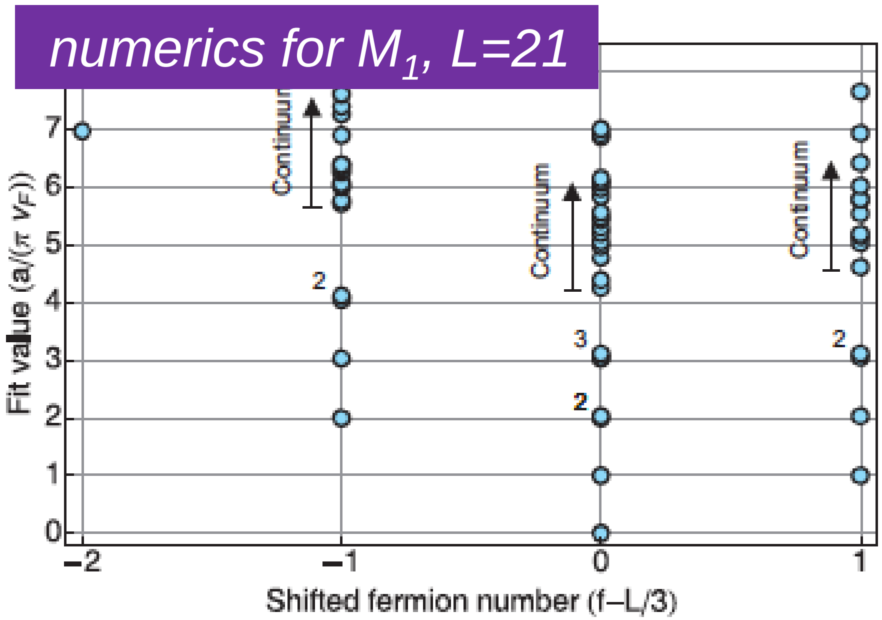
$$\lambda_{[1,1],j} = y, \quad \lambda_{[2,1],j} = \lambda_{[2,2],j} = 1, \quad \lambda_{[3,1],j} = \lambda_{[3,3],j} = y, \quad \lambda_{[3,2],j} = 1/y^2$$

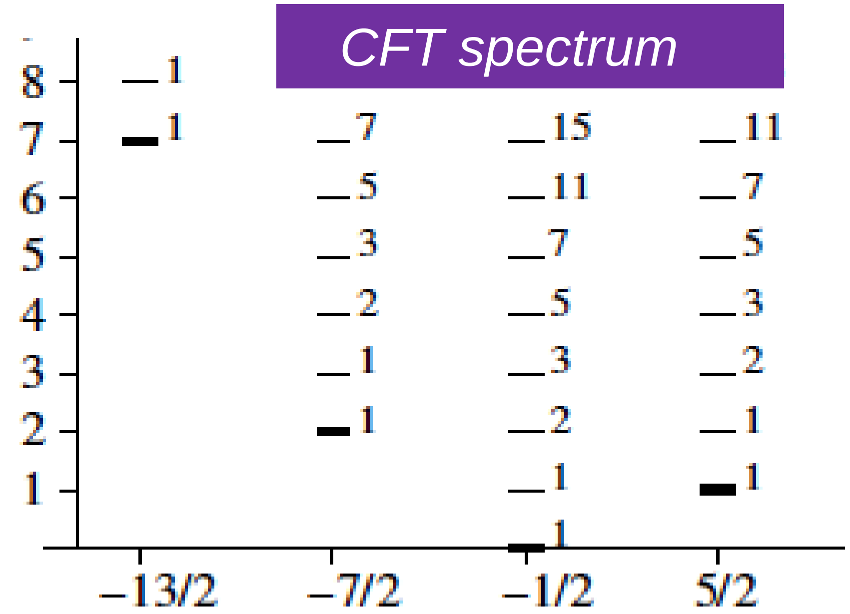
λ≪1

$\lambda_{[1,1],j}$:	…	1	$\sqrt{2}$	$\sqrt{2}\lambda$	$\sqrt{2}$	1	…
$\lambda_{[2,1],j}$:	…	1	1	λ	$\sqrt{2}$	λ	…
$\lambda_{[2,2],j}$:	…	λ	$\sqrt{2}$	λ	1	1	…
$\lambda_{[3,1],j}$:	…	1	λ	λ	1	$\lambda/\sqrt{2}$	…
$\lambda_{[3,2],j}$:	…	$\lambda/\sqrt{2}$	1	$\lambda^2/\sqrt{2}$	1	$\lambda/\sqrt{2}$	…
$\lambda_{[3,3],j}$:	…	$\lambda/\sqrt{2}$	1	λ	λ	1	…

Criticality

$M_k[\lambda=1]$ is critical – low energy physics given by k-th minimal model of $N=2$ superconformal field theory





Huijse 2010; Fokkema-KjS 2017

Integrable massive QFT

$M_k[\lambda<1]$ connects to $N=2$ supersymmetric integrable massive QFT, with superpotentials of Chebyshev form

$k=1$: sine-Gordon at $N=2$ susy point
$k=2$: $N=1$ supersymmetric sine-Gordon at $N=2$ susy point

lattice model excitations at $\lambda \ll 1$ are kinks between $W=k+1$ possible $E=0$ states – they are in 1-1 correspondence with kinks in the $N=2$ integrable QFT

Fokkema-KjS 2017

Integrable massive QFT

susy lattice model $M_3[\lambda]$

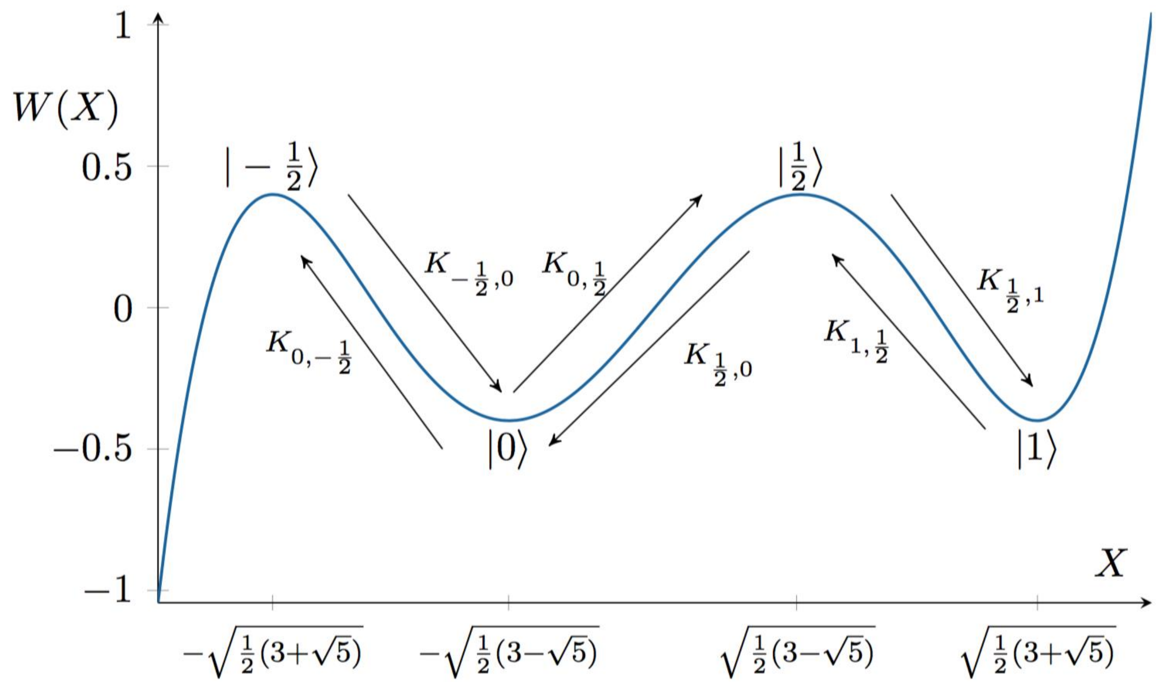
susy ground-states at $\lambda \ll 1$

$$|1\rangle = \dots 1\underset{\wedge}{1}10011\underset{\wedge}{1}100\dots, \quad |\tfrac{1}{2}\rangle = \dots(\cdot\underset{\wedge}{1}\cdots)(\cdot\underset{\wedge}{1}\cdots)\dots,$$

$$|0\rangle = \dots 0\underset{\wedge}{1}01101\underset{\wedge}{0}11\dots, \quad |-\tfrac{1}{2}\rangle = \dots\underset{\wedge}{0}(\cdot 11\cdot)\underset{\wedge}{0}(\cdot 11\cdot)\dots,$$

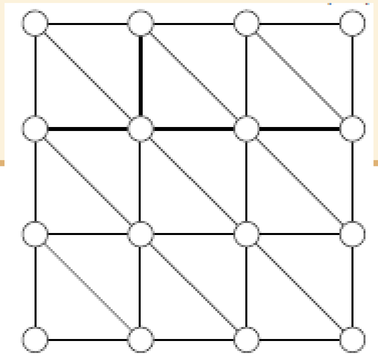
massive *N=2* integrable QFT

groundstates as extrema of $k=3$ Chebyshev superpotential



Superfrustration

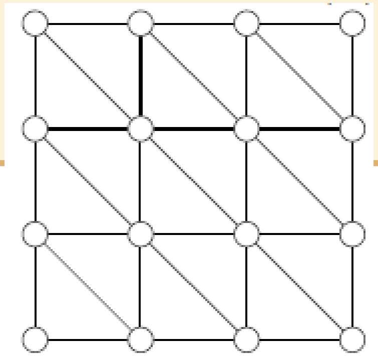
M_1 model, 2D triangular lattice



Witten index for $N \times M$ sites with periodic BC

	1	2	3	4	5
1	1	1	1	1	1
2	1	-3	-5	1	11
3	1	-5	-2	7	1
4	1	1	7	-23	11
5	1	11	1	11	36

M_1 model, 2D triangular lattice



Witten index for $N{\times}M$ sites with periodic BC

	1	2	3	4	5	9	10
1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	-5	57
3	1	-5	-2	7	1	-2	-65
4	1	1	7	-23	11	-29	-279
5	1	11	1	11	36	811	-1064
6	1	9	-14	25	-49	1462	-4911
7	1	-13	1	-69	211	-7055	5237
8	1	-31	31	193	-349	-28517	50849
9	1	-5	-2	-29	881	31399	313315
10	1	57	-65	-279	-1064	313315	950592
11	1	67	1	859	1651	499060	2011307
12	1	-47	130	-1295	-589	-2573258	-3973827
13	1	-181	1	-77	-1949	-10989458	-49705161
14	1	-87	-257	3641	12611	4765189	-232675057
15	1	275	-2	-8053	-32664	134858383	-702709340

\`superfrustration’

van Eerten 2005

Proliferation of susy ground states

ground state counting problem

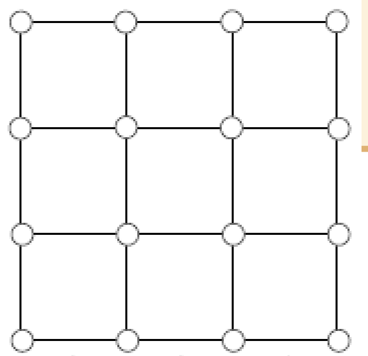
- unsolved for most $D>1$ lattices (including triangular)

important clue

- gs counting problem equivalent to determining (dimension of) the homology of the operator Q
- can use methods from math literature (spectral sequences & tic-tac-toe lemma, homological perturbation lemma) to make progress

Fendley-KjS 2005, Huijse-KjS 2010

M_1 model, 2D square lattice



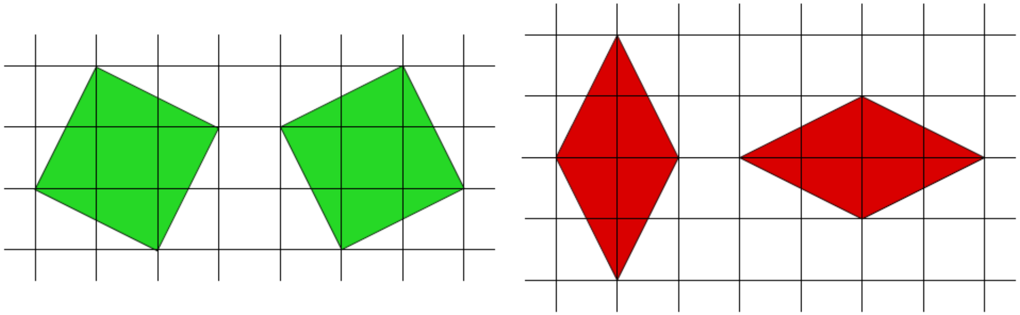
Witten index for $N \times M$ sites with periodic BC

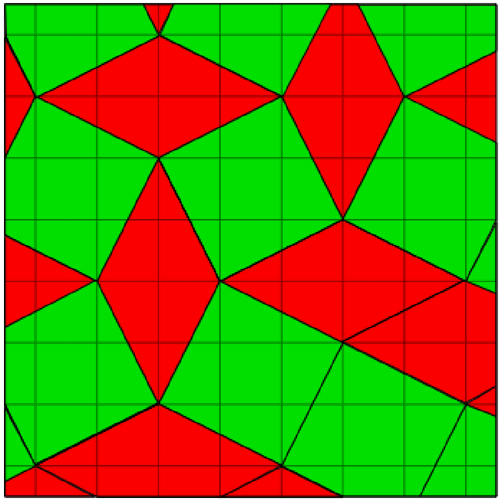
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3
3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1
4	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7
5	1	1	1	1	−9	1	1	1	1	11	1	1	1	1	−9	1	1	1	1	11
6	1	−1	4	3	1	14	1	3	4	−1	1	18	1	−1	4	3	1	14	1	3
7	1	1	1	1	1	1	1	1	1	1	1	1	1	−27	1	1	1	1	1	1
8	1	3	1	7	1	3	1	7	1	43	1	7	1	3	1	7	1	3	1	47
9	1	1	4	1	1	4	1	1	40	1	1	4	1	1	4	1	1	76	1	1
10	1	−1	1	3	11	−1	1	43	1	9	1	3	1	69	11	43	1	−1	1	13
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	3	4	7	1	18	1	7	4	3	1	166	1	3	4	7	1	126	1	7
13	1	1	1	1	1	1	1	1	1	1	1	1	−51	1	1	1	1	1	1	1
14	1	−1	1	3	1	−1	−27	3	1	69	1	3	1	55	1	451	1	−1	1	73
15	1	1	4	1	−9	4	1	1	4	11	1	4	1	1	174	1	1	4	1	11

Fendley-KjS-van Eerten 2005

M_1 model, 2D square lattice

Number of gs related to rhombus tilings of the lattice, with $F = N_t$





Theorem [Jonsson, Fendley, Huijse-KjS 2009]

$$\# \text{ GS } = t_{even} + t_{odd} - (-1)^{(\theta_m+1)p} \theta_{d_-} \theta_{d_+}$$

with $d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$, $\theta_{3p} = 2$, $\theta_{3p\pm1} = -1$

Ground state counting – recent results

Z_2 Nicolai model

The number of *E*=0 states (OBC)

N	3	4	5	6	7	8	9	10	11	12	13
deg.	6	12	20	36	64	112	200	352	624	1104	1952

THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

They are generated by the recursion:

$$a_n = 2a_{n-2} + 2a_{n-3}, \qquad a_0 = 1,\ a_1 = 2,\ a_2 = 4$$

$$Z \sim (1.77)^N$$

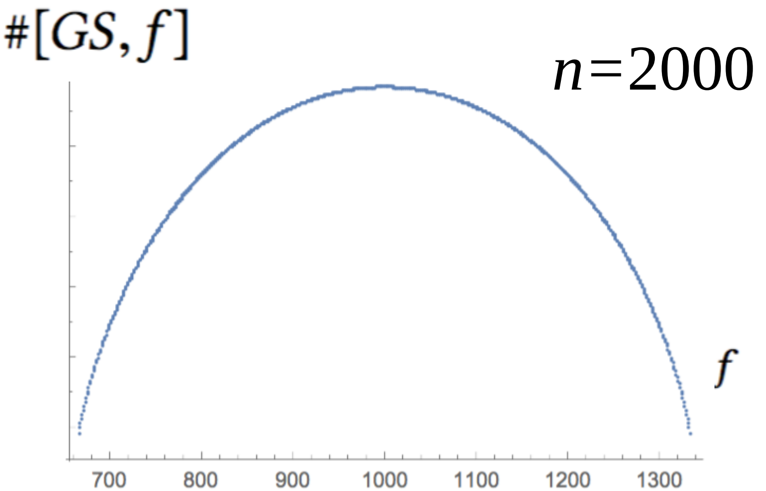
slide H. Katsura in talk at UvA, Nov 2017

Ground state counting – recent results

Z_2 Nicolai model

generating function

$$P_n^{\mathbb{Z}_2}(z) := \sum_f \#[GS, f]\, z^f$$



Theorem 2.2.1. *The polynomials $P_n^{\mathbb{Z}_2}(z)$, $n \geq 3$, can be determined by the recursion*

$$P_n^{\mathbb{Z}_2}(z) = 2zP_{n-2}(z) + (z + z^2)P_{n-3}^{\mathbb{Z}_2}(z)$$

with the initial values given by

$$P_0^{\mathbb{Z}_2}(z) := 1, \qquad P_1^{\mathbb{Z}_2}(z) := 1 + z \quad \text{and} \quad P_2^{\mathbb{Z}_2}(z) = 1 + 2z + z^2.$$

conjecture in Sannomiya-Katsura-Nakayama 2017

proof in La-Shadrin-KjS to appear

Ground state counting – recent results

Nicolai model

■ Number of E=0 ground states (N: odd, OBC)

N	3	5	7	9	11	13
deg.	6	20	64	208	672	2176

Grows exponentially with system size!

THE ON–LINE ENCYCLOPEDIA
OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Sorry, but the terms do not match anything in the table.

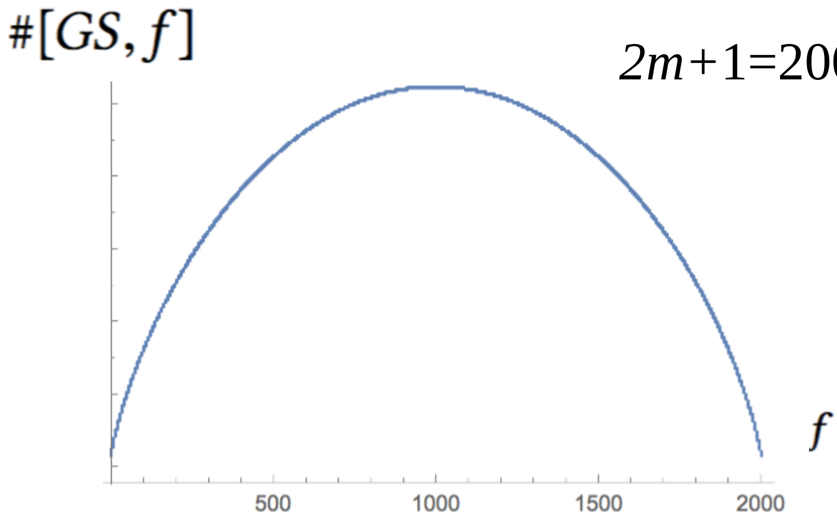
slide H. Katsura in talk at UvA, Nov 2017

Ground state counting – recent results

Nicolai model

generating function

$$P_{2m+1}(z) := \sum_f \#[GS, f]\, z^f$$



Theorem 2.1.1. *The polynomials* $P_{2m+1}(z)$, $m \geq 3$, *can be determined by the recursion*

$$P_{2m+1}(z) = (1+z^2)P_{2m-1}(z) + (z + 2z^2 + z^3)P_{2m-3}(z)$$

with the initial values given by

$$P_3(z) = 1 + 2z + 2z^2 + z^3 \quad \text{and} \quad P_5(z) = 1 + 3z + 6z^2 + 6z^3 + 3z^4 + z^5.$$

La-Shadrin-KjS to appear

Adiabatic driving and topological response

Adiabatic dynamics for susy groundstates

- susy charges can be `staggered' by parameters λ_i

$$Q = \sum_i \lambda_i Q_i$$

- *E=0* groundstates protected by Witten index, but their expressions change with λ_i

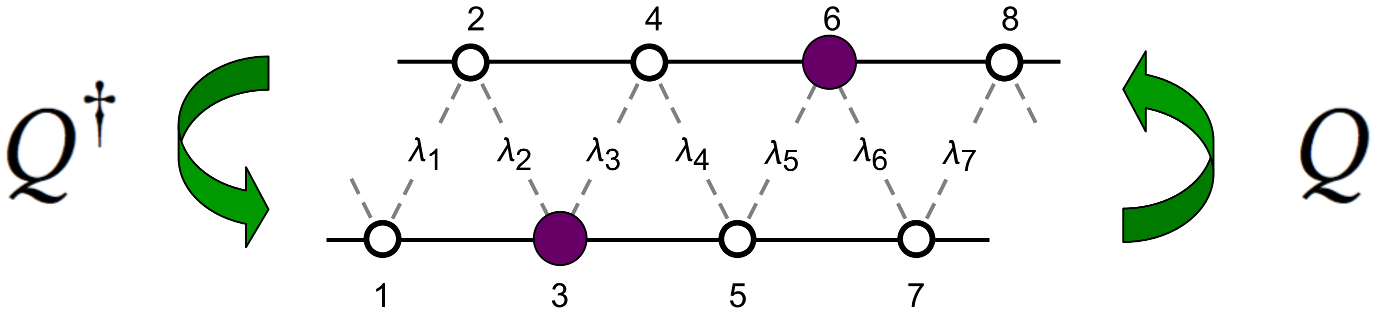
- close loops $\lambda_i[t]$ represented by non-Abelian holonomies in *U(G)*

Wilczek-Zee 1984

Topological pumping

van Voorden-Kjs to appear

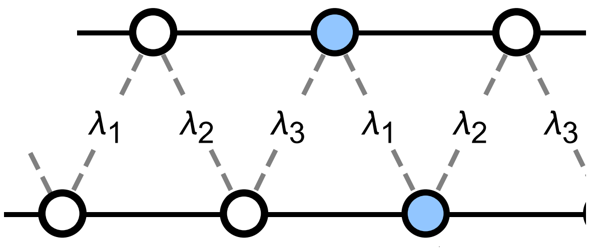
- susy model for coupled chains staggered as



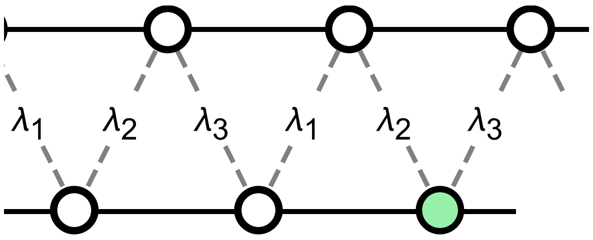
- for $\lambda_i = 111111..$ total of 2^L susy groundstates generated by flat band of L local pairs

Topological pumping

- for λ_i = ..100100.. susy gs generated by flat bands of
  - *2L/3* local pairs on sites [3i+1, 3i+2]



  - *2L/3* single particles on sites [3i]



Topological pumping

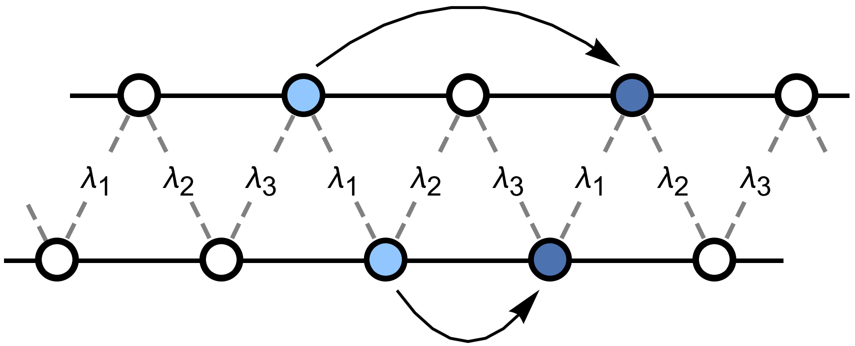
- under adiabatic cycle for λ_i

..100100.. → ..010010.. → ..001001.. → ...

local pairs move

one unit cell to the right

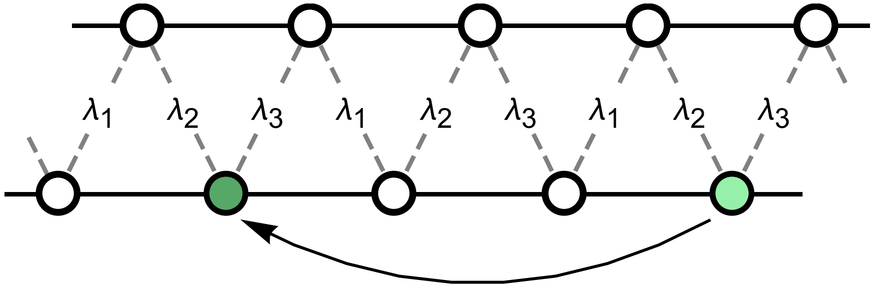
$$C_p = +1$$



single particles move

2 unit cells to the left

$$C_s = -2$$

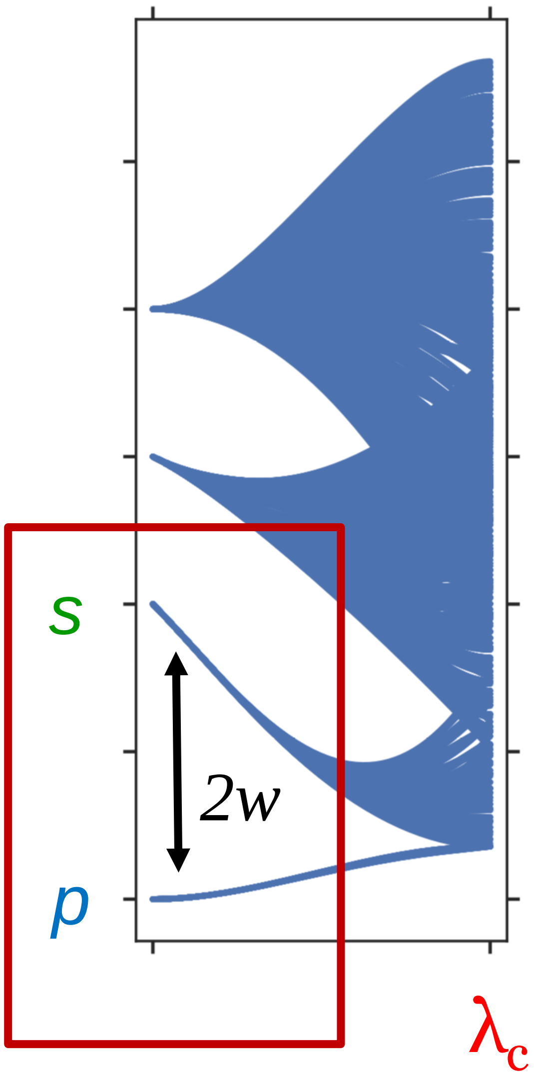


Topological pumping

- to stabilize topological pumping, need to split the E=0 bands of pairs and singles
- add to Hamiltonian

$$H_w = w \sum_i (1 - \lambda_{i-1})(1 - \lambda_i) n_i$$

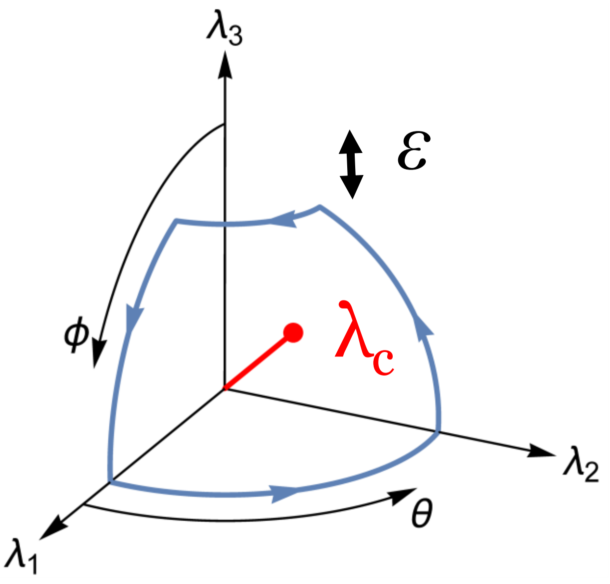
- this opens gap between bands of pairs (p) and singles (s), away from isotropic point λ_c

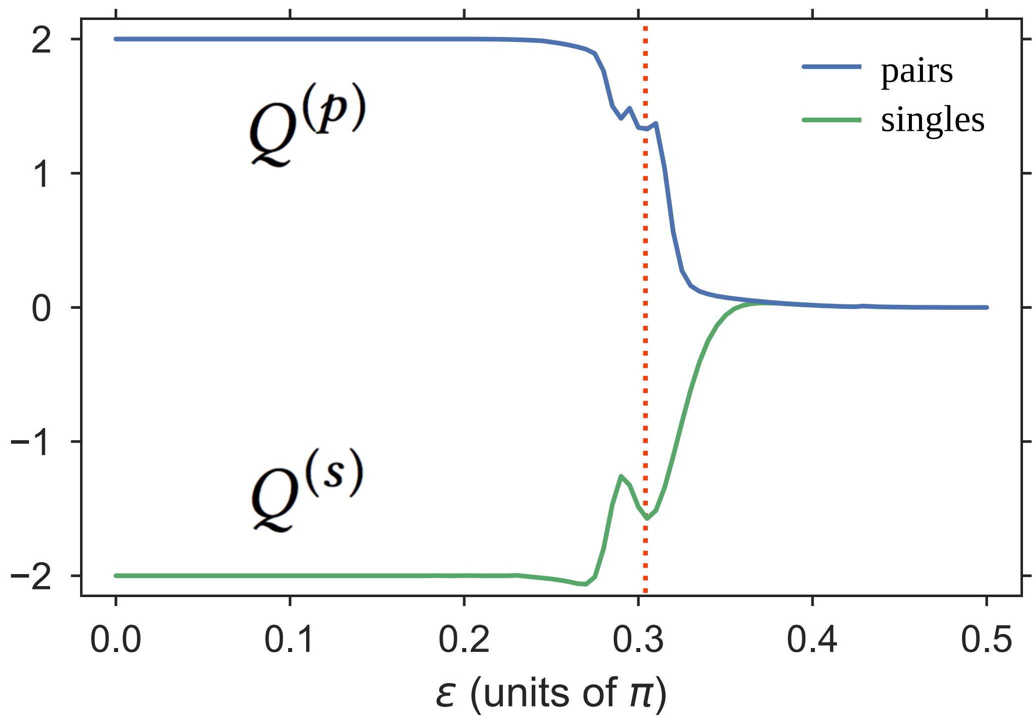


Topological pumping

- filling the pairs (p) or singles (s) bands gives many-body states with topological response to cycle that encircles λ_c

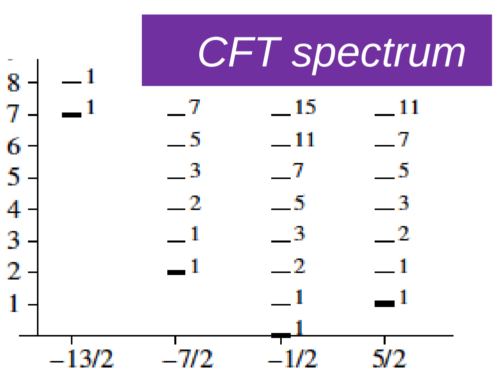
$$Q^{(p)} = 2C_p = +2, \qquad Q^{(s)} = C_s = -2$$

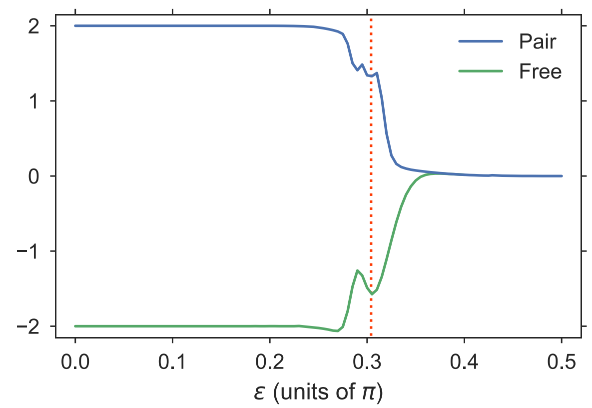




wrap up

- lattice model – field theory correspondence particularly rich for the 1D M_k models
- superfrustrated phases not well understood – deep questions both about the math and the physics
- interplay of susy with (non-equilibrium) dynamics





thanks to susy friends and collaborators …

Paul Fendley,

Jan de Boer, Bernard Nienhuis,

Hendrik van Eerten,

Liza Huijse, Jim Halverson,

Jiri Vala, Nial Moran, Dhagash Mehta,

Bela Bauer, Erez Berg, Matthias Troyer,

Thessa Fokkema,

Jan de Gier, Gyorgy Feher, Sasha Garbali,

Ruben La, **Sergey Shadrin**,

Bart van Voorden,

Hosho Katsura