Extreme value statistics in a gas of $2d$ charged particles

Large deviation theory in statistical physics

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1. Introduction
   - Extreme value statistics for i.i.d. random variables
   - Matrices of real eigenvalues
   - Matrices of complex eigenvalues
2. Results
3. Derivation of the Ginibre results
   - Determinantal process
   - Analysis of the CDF
   - Matching
4. Conclusion

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EVS for i.i.d.

For i.i.d. random variables $(x_1, \cdots, x_N)$ drawn from a PDF $p(x)$, only 3 classes of universality for the Extreme Value Statistics:

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- $p(x) \ll x^{-\alpha}\ , \ \ \forall\alpha$

$$Q_N(w) = \left[\int_0^w p(x)dx\right]^N \to G(a_N(w-b_N))\ ,\ \ G(x) = e^{-e^{-x}} \qquad (1)$$

Gumbel distribution

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- $p(x) \sim x^{-\alpha-1} \;, \;\; \alpha > 0$

$$Q_N(w) = \left[\int_0^w p(x)dx\right]^N \to F_\alpha(a_N(w-b_N)) \;, \;\; F_\alpha(x) = e^{-x^{-\alpha}} \tag{2}$$

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- $p(x) \sim (x^* - x)^\nu\ ,\ \ \nu \geq 0\ ,\ \ x \leq x^*$

$$Q_N(w) = \left[\int_0^w p(x)dx\right]^N \to W_\nu(a_N(w - b_N))\ ,\ \ W_\nu(x) = e^{-x^{\nu+1}} \qquad (3)$$

  Weibull distribution

Gaussian Unitary matrices

See Satya Majumdar's lecture on Random Matrix Theory

For M a matrix belonging in the Gaussian Unitary Ensemble (GUE) ,

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{12}^* & m_{22} & \cdots & m_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1N}^* & m_{2N}^* & \cdots & m_{NN} \end{pmatrix} \;,\quad m_{ij} = m_{ji}^* \sim \mathcal{N}(0, \frac{1}{\sqrt{2N}}) + i\mathcal{N}(0, \frac{1}{\sqrt{2N}}) \;,\; m_{ii} \sim \mathcal{N}(0, \frac{1}{\sqrt{N}})$$

the probability weight is $P(M) \propto e^{-\frac{1}{2}\operatorname{Tr}(M^\dagger M)}$.

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The joint PDF of the (real) eigenvalues is

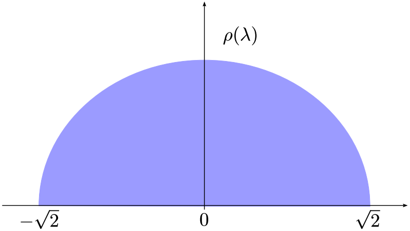
$$P(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum_k \lambda_k^2} = \frac{1}{Z_N} \exp\left(-N\sum_k \lambda_k^2 + \sum_{i\neq j} \ln|\lambda_i - \lambda_j|\right)$$

The variables are identically distributed but correlated!

Average density

Semi-circle law

$$\rho_N(\lambda) = \frac{1}{N}\langle \sum_k \delta(\lambda - \lambda_k)\rangle \;\to\; \rho(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$$



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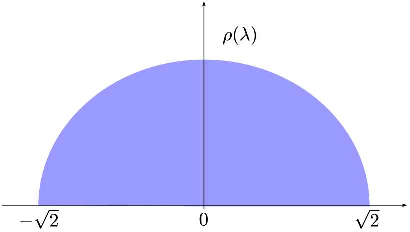


This must imply for EVS of $\lambda_{\max} = \max \lambda_k$, $\lambda_{\max} \to \sqrt{2}$.

Fluctuations $|\lambda_{\max} - \sqrt{2}| \sim w_N \ll \sqrt{2}$

$$\int_{\sqrt{2}-w_N}^{\sqrt{2}} \rho(\lambda) d\lambda \propto w_N^{\frac{3}{2}} \sim N^{-1}\ , \quad w_N \sim N^{-\frac{2}{3}}$$

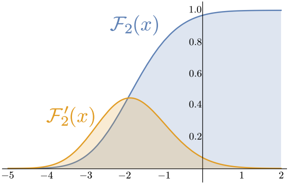
Extreme Value Statistics in GUE

The CDF, $\Pr(\lambda_{\max} \le x)$ is described in its typical regime $|\lambda_{\max} - \sqrt{2}| \sim w_N$ by the Tracy-Widom distribution

$$\boxed{\Pr(\lambda_{\max} \le x) = \mathcal{F}_2\left(\sqrt{2}N^{2/3}(x-\sqrt{2})\right)}\,.$$

This distribution has two tails of the form

$$\mathcal{F}_2(x) \sim \begin{cases} e^{-\frac{|x|^3}{12}}\,, & x \to -\infty \\ 1 - e^{-\frac{4\,x^{3/2}}{3}}\,, & x \to +\infty\,. \end{cases}$$



Large deviations

This CDF $\Pr(\lambda_{\max} \leq x)$ takes large deviation forms on both sides

$$\Pr(\lambda_{\max} \leq x) \sim \begin{cases} e^{-N^2\Phi_-(x)} \ , \text{ for } \ 0 < (\sqrt{2} - x) = O(1) \\ \mathcal{F}_2\left(\sqrt{2}N^{2/3}(x - \sqrt{2})\right) \ , \text{ for } \ x - \sqrt{2} = O(N^{-2/3}) \\ 1 - e^{-N\Phi_+(x)} \ , \text{ for } \ 0 < (x - \sqrt{2}) = O(1) \ . \end{cases}$$

Majumdar and Schehr, J. Stat. Mech. P01012 (2014)

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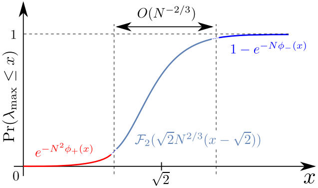
The tails behave as

$$\Phi_-(x) \sim \frac{\sqrt{2}}{6}(\sqrt{2}-x)^3\ , \ \ x \to \sqrt{2}_-\ ,$$

$$\Phi_+(x) \sim \frac{2^{11/4}}{3}(x-\sqrt{2})^{\frac{3}{2}}\ , \ \ x \to \sqrt{2}_+\ ,$$

allowing a smooth matching with the central part.



Obtaining the LD rate functions

To compute the CDF $\Pr(\lambda_{\max} \leq w)$, one imposes a wall at w. The LD functions can be evaluated as the change in the energy with and without the wall

$$E_N(\lambda_1, \cdots, \lambda_N) = N \sum_k \lambda_k^2 - \sum_{i \neq j} \ln |\lambda_i - \lambda_j|$$

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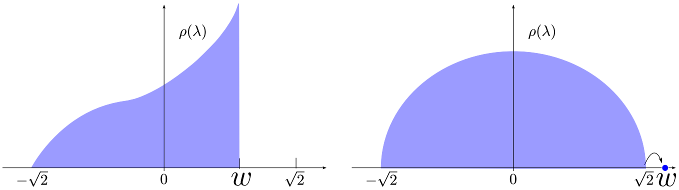

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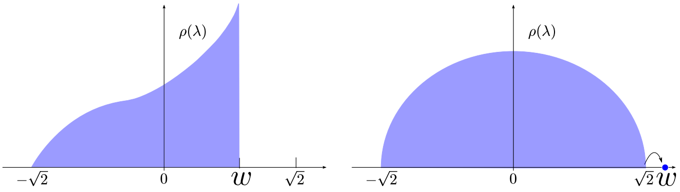

- For $w < \sqrt{2}$, all eigenvalues move $\Delta E \sim N^2$, different $\rho(\lambda)$
- For $w > \sqrt{2}$, only one eigenvalue moves $\Delta E \sim N$, same $\rho(\lambda)$

Case of $\Phi_-(w)$

To evaluate $\Phi_-(w)$, we use

$$E_N(\lambda_1, \cdots, \lambda_N) \;\to\; N^2 E[\rho, w] \;,$$

with $E[\rho, w] = \int_0^w d\lambda \rho(\lambda)\lambda^2 - \iint_0^w d\lambda d\lambda' \rho(\lambda)\rho(\lambda') \ln|\lambda - \lambda'| + \mu(w)\left(\int_0^w d\lambda \rho(\lambda) - 1\right) .$

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The functional is then minimize with respect to $\rho(\lambda)$,

$$\left. \frac{\delta E[\rho, w]}{\delta \rho(\lambda)} \right|_{\rho = \rho_w} = 0 = \lambda^2 - 2 \int_0^w d\lambda' \rho_w(\lambda') \ln|\lambda - \lambda'| + \mu(w)$$

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Then, we evaluate (with $\rho_\infty(\lambda) = \frac{1}{\pi}\sqrt{2 - \lambda^2}$)

$$\Phi_-(w) = E[\rho_w, w] - E[\rho_\infty, \infty]$$

Case of Φ_+

To evaluate $\Phi_+(w)$, we compute the energy for one charge in the semi-cirle

$$E_{in}(w) = w^2 - \int \rho(\lambda) \ln |\lambda - w| d\lambda \ , \ \ -\sqrt{2} < w < \sqrt{2}$$

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$$E_{in}(w) = w^2 - \int \rho(\lambda) \ln|\lambda - w| d\lambda \;, \;\; -\sqrt{2} < w < \sqrt{2}$$

and the energy out of the semi-cirle

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$$E_{out}(w) = w^2 - \int \rho(\lambda) \ln|\lambda - w| d\lambda \ , \ \ w > \sqrt{2}$$

and the LD rate function is just the energy difference

$$\Phi_+(w) = E_{out}(w) - E_{in}(w)$$

Different classes of universality

If we consider the joint PDF of eigenvalues

$$P(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \exp\left(-\frac{\beta}{2}\left[\sum_k \underbrace{\lambda_k^2}_{\text{Potential } v(\lambda)} - \sum_{i \neq j} \underbrace{\ln|\lambda_i - \lambda_j|}_{\text{Interaction}}\right]\right)$$

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Majumdar and Schehr, J. Stat. Mech. P01012 (2014)

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- Changing the interaction, everything is different

  See for instance Dahr et al. Phys. Rev. Lett. 119, 060601 (2017) (talk by A. Kundu last week)

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- Changing the dimension?

Complex Ginibre matrices

G is a Complex Ginbre matrices for

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Joint PDF of (complex) eigenvalues

$$P_{\text{joint}}(z_1, \cdots, z_N) = \frac{1}{Z_N} \prod_{i<j} |z_i - z_j|^2 \, e^{-N \sum_k |z_k|^2} = \frac{1}{Z_N} \exp\left(\underbrace{-N \sum_k |z_k|^2}_{\text{Potential}} + \underbrace{\sum_{i \neq j} \ln |z_i - z_j|}_{\text{Interaction}} \right)$$

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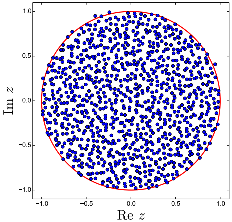
Only difference with GUE: eigenvalues live in $2d$ space

Average density

Girko's law

$$\rho_N(z) = \frac{1}{N}\langle\sum_k \delta(z - z_k)\rangle \;\to\; \rho(z) = \frac{\Theta(1 - |z|)}{\pi}$$

Girko, Theory Probab. Appl. 29, 694 (1984)



For the EVS of $r_{\max} = \max_k |z_k|$, $r_{\max} \to 1$.

Extreme value statistics

We are interested in the CDF $Q_N(w) = \mathrm{Prob.}\left[r_{\max} \leq w\right]$ with $r_{\max} = \max_{1 \leq i \leq N} |z_i|$

$$Q_N(w) = \int_{|z| \leq w} dz_1 dz_1^* \cdots \int_{|z| \leq w} dz_N dz_N^* P_{\mathrm{joint}}(z_1, \cdots, z_N) \; .$$

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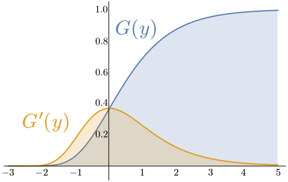
$$Q_N(w) = \int_{|z|\le w} dz_1 dz_1^* \cdots \int_{|z|\le w} dz_N dz_N^* P_{\mathrm{joint}}(z_1, \cdots, z_N)\ .$$

In the typical regime $|w-1| \ll 1$, the distribution is a Gumbel as for EVS of i.i.d. variables

$$Q_N(w) \sim G\left(a_N(w-b_N)\right)\ , \quad \text{with} \quad \boxed{G(y) = \exp(-\exp(-y))}$$

$a_N \sim \sqrt{4Nc_N}$, $b_N - 1 \sim \sqrt{c_N/(4N)}$ and $c_N = \ln N - 2\ln\ln N - \ln 2\pi$.

Rider, J. Phys. A 36(12), 3401, (2003).



Large deviations

This CDF $Q_N(w)$ takes large deviation forms on both sides

$$Q_N(w) \sim \begin{cases} e^{-N^2\Psi_-(w)} \;, \text{ for } 0 < (1-w) = O(1) \\ G(a_N(w-b_N)) \;, \text{ for } (w-b_N) = O(a_N^{-1}) \\ 1 - e^{-N\Psi_+(w)} \;, \text{ for } 0 < (w-b_N) = O(1) \;. \end{cases}$$

Cunden et al., J. Stat. Mech. 053303 (2017).

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The rate functions are

$$\Psi_-(w) = \frac{1}{4}(4w^2 - w^4 - 4\ln w - 3) \text{ , for } 0 < w < 1$$

$$\Psi_+(w) = w^2 - 2\ln w - 1 \text{ , for } w > 1$$

To obtain $\Psi_\pm$, one can use the same method as for $\Phi_\pm$ (GUE case)

Right matching

If one wants to check the right matching:

- From the inside, $G'(x) \to exp(-x)$, $x \to \infty$

$$Q'_N(w) \sim \exp\left[-\sqrt{4Nc_N}(w - b_N)\right] \ , \ \ 0 < w - b_N \gg a_N$$

Right matching

If one wants to check the right matching:

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$$Q_N'(w) \sim \exp\left[-\sqrt{4Nc_N}(w - b_N)\right] \ , \ \ 0 < w - b_N \gg a_N$$

- From the outside, $\Psi_+(w) \to \Psi_+(b_N) + (w - b_N)\Psi_+'(b_N)$, $w \to b_{N+}$

$$N\Psi_+'(1 + \sqrt{\frac{c_N}{4N}}) = 2N(1 + \sqrt{\frac{c_N}{4N}} - (1 + \sqrt{\frac{c_N}{4N}})^{-1}) \sim \sqrt{4Nc_N}$$

$$Q_N'(w) \sim \exp\left[-\sqrt{4Nc_N}(w - b_N)\right] \ , \ \ |w - b_N| \ll 1$$

Left matching

- From the inside, $G(y) \to exp(-\exp(-y))$, $y \to -\infty$

$$Q_N(w) \sim \exp\left[-\exp(-a_N(w - b_N))\right] \ , \ \ 0 < b_N - w \gg a_N$$

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Left matching

- From the inside, $G(y) \to exp(-\exp(-y))$, $y \to -\infty$

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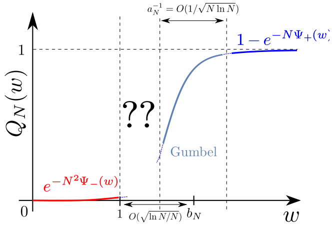
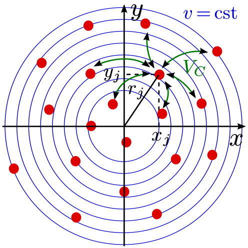


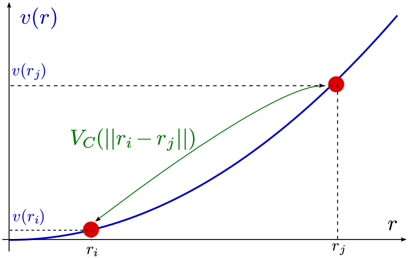
$2d$ One Component Plasma

Gas of charged particles in $2d$

Competition between

- Coulomb repulsion $V_C(|r_i - r_j| = r) = -\ln r$
- External symmetric potential $v(r)$





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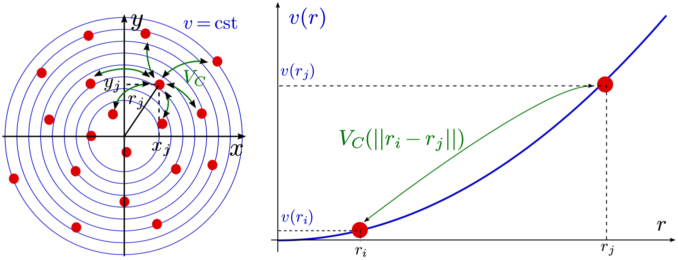

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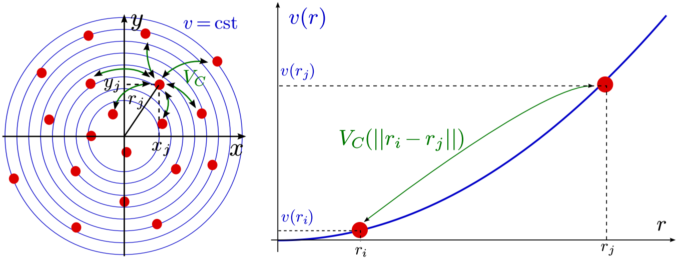

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For $\mathbf{r} = (x, y) \leftrightarrow z = x + iy = re^{i\theta}$

For $v(r) = r^2$ this gives Ginibre

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There is an intermediate deviation regime for the CDF $Q_N(w)$ close to the edge $r_{\rm edge} = 1$ on a scale $|w - r_{\rm edge}| \sim w_N = (2N)^{-1/2}$

$$\boxed{Q_N(w) \approx \exp\left[-\frac{r_{\rm edge}}{w_N}\phi_I\left(\frac{w - r_{\rm edge}}{w_N}\right)\right]}$$

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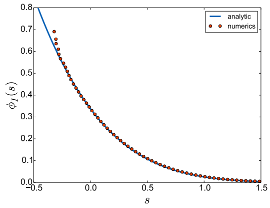

Figure: Numerics by diagonalization of 10^6 complex Ginibre matrices of rank $N = 200$

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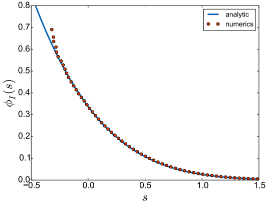

Figure: Numerics by diagonalization of 10^6 complex Ginibre matrices of rank $N = 200$

Universal for all symmetric potentials $v(r) \gg \ln r^2$ for $r \to \infty$.
In this case $r_{\rm edge} v'(r_{\rm edge}) = 2$ and $w_N = [2\pi N \rho(r_{\rm edge})]^{-1/2}$

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Orthogonal polynomials

We introduce orthogonal polynomials $\pi_k(z) = z^{k-1}$ for $k = 1, \cdots, N$ which verify

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Computing the following quantity

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Chafaï and Péché, J. Stat. Phys., 156(2), 368-383, (2014)

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This system of charges (or eigenvalues) is determinantal

CDF of $r_{\max}$

Using Cauchy-Binet-Andreiev the CDF $Q_N(w)$ is now obtained as

$$Q_N(w) = \int_{|z|\leq w} dz_1 dz_1^* \cdots \int_{|z|\leq w} dz_N dz_N^* \frac{1}{N!} \left| \det_{1\leq i,j\leq N} \frac{N^{j/2}\pi_j(z_i)}{\sqrt{\pi\Gamma(j)}} e^{-\frac{N|z_i|^2}{2}} \right|^2$$

$$Q_N(w) = \det_{1\leq i,j\leq N} \left(\frac{2N^j}{\Gamma(j)} \int_0^w r^{2j-1} e^{-Nr^2} dr \delta_{i,j} \right) = \prod_{k=1}^{N} \left[\frac{\gamma(k, Nw^2)}{\Gamma(k)} \right]$$

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$r_{\max}$ is the maximum of independent but non-identically distributed random variables x_k such that $q_k(w) = \Pr[x_k \leq w] = \frac{\gamma(k,Nw^2)}{\Gamma(k)}$

Regime of interest

The average density can be obtained for finite N as

$$\rho_N(r) = \frac{1}{N}\langle\sum_{k=1}^{N}\delta(r - z_k)\rangle = \frac{\Gamma(N, Nr^2)}{\pi\Gamma(N)}$$

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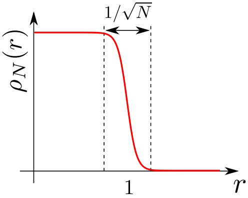
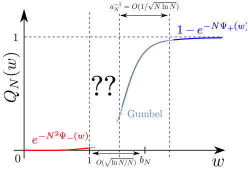
- For infinite N, the average density follows the Girko's law $\boxed{\rho_{\mathrm{b}}(r) = \frac{\Theta(1-r)}{\pi}}$
- For large but finite N, there is a scaling form on a scale $w_N = (2N)^{-1/2}$

$$\boxed{\rho_N(r) \to \frac{1}{\pi}\tilde{\rho}\left(\frac{r - r_{\mathrm{edge}}}{w_N}\right)} \quad \text{with} \quad \boxed{\tilde{\rho}(u) = \frac{1}{2}\,\mathrm{erfc}(u)}$$

Forrester and Honner, J. Phys. A 32, 2961 (1999)

This scale matches the unknown regime for the CDF!





Individual CDF

By setting $u = k/N$, the individual CDF q_k reads

$$q_{Nu}(w) = \frac{\int_0^w e^{-N(r^2 - 2u \ln r)} \frac{dr}{r}}{\int_0^\infty e^{-N(r^2 - 2u \ln r)} \frac{dr}{r}}$$

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This integral is evaluated by a saddle point approximation,

$$q_{Nu}(w) \approx \begin{cases} \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{2N}(\sqrt{u} - w)\right] \;, & w > \sqrt{u}\;, \\ \sqrt{\frac{u}{2\pi N}}\,\frac{e^{N\left[u - w^2 - \ln\left(\frac{u}{w^2}\right)\right]}}{u - w^2} \;, & w < \sqrt{u}\;. \end{cases}$$

Intermediate regime

The full CDF takes the following form

$$Q_N(w) \approx \exp\left[N\int_0^1 du \ln\frac{1}{2}\operatorname{erfc}\left[\sqrt{2N}(\sqrt{u}-w)\right] - \int_0^1 du\Theta(u-w^2)f(u,w)\right] .$$

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$$\phi_I(y) = -\int_y^\infty dp \ln\frac{1}{2}\operatorname{erfc}(-p)$$

Matching with the typical regime

For $p \gg 1$, $erfc(-p) \to 2 - e^{-p^2}/(\sqrt{\pi}p)$,

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Then we may use from the outside part of the distribution

$$y = \sqrt{2N}(w-1) = \sqrt{2N}(b_N - 1 + x/a_N) = \sqrt{c_N/2} + x/\sqrt{2c_N},$$

$$Q_N(x = a_N(w - b_N)) = \exp\left[-\sqrt{2N}\phi_I(\sqrt{c_N/2} + \frac{x}{\sqrt{2c_N}})\right] \approx \exp\left[-\sqrt{2N}\frac{e^{-\frac{c_N}{2}-x}}{2\sqrt{\pi}c_N}\right] = e^{-e^{-x}}$$

Matching with the left LD

For $0 < -p \gg 1$, $erfc(-p) \to e^{-|p|^2}/(\sqrt{\pi}|p|)\Theta(-p)$,

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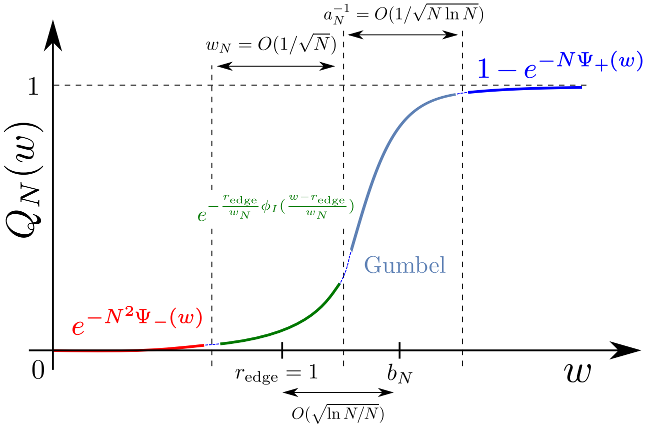
Ginibre results

All regimes match smoothly!



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The intermediate rate function ϕ_I is universal w.r.t. $v(r)$

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A similar result was observed for fermions in a hard box

See LACT et al. arXiv. 1706.03598 and talk by G. Schehr tomorrow

Other classes of universality

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  For $\alpha = 1$, this is realized by computing the jPDF of eigenvalues of $M = A^{-1}B$ with A, B independent Ginibre matrices

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- Joint PDF with hard edge $v_\nu(r) = -\nu/N \ln(1-r^2)$ with $\nu \geq 0$

  For any positive integer ν, this is realized computing the jPDF of a sub-block of a random unitary matrix.

  Zyczkowski and Sommers, J. Phys. A, 33, 2045 (2000)

Thank you for your attention!