Quasilinear Elliptic Problems in a Domain with Imperfect Interface and L^1 data

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Outline

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Statement of the Problem

We consider the following quasilinear elliptic problem in a fixed two-component domain with an imperfect interface and L^1-data:

$$\begin{cases} -\mathrm{div}(B(x,u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\mathrm{div}(B(x,u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x,u_1)\nabla u_1)\nu_1 = (B(x,u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x,u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases} \tag{P}$$

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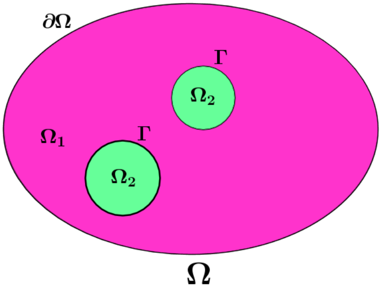
We prescribe

- a continuous flux on the interface; and
- a jump of the solution on the interface which is proportional to the flux.

The Two-component Domain Ω



- Ω is a connected bounded open set in $\mathbb{R}^N$ with its boundary $\partial\Omega$
- Ω_2 is an open set such that $\overline{\Omega_2} \subset \Omega$ with a Lipschitz boundary Γ
- $\Omega_1 = \Omega \setminus \overline{\Omega_2}$

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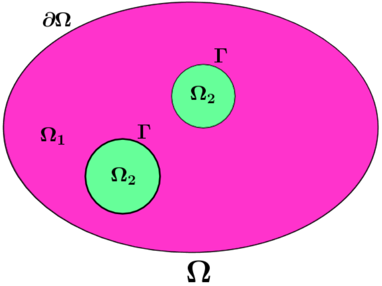

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• If we have a function u defined on $\Omega \setminus \Gamma$, then we denote $u_i = u\big|_{\Omega_i}$ the restriction of u in Ω_i.

Assumptions

We make the following assumptions:

(A1) $f \in L^1(\Omega)$

(A2) The function h satisfies

$$h \in L^\infty(\Gamma) \quad \text{and} \quad 0 < h_0 < h(y) \text{ a.e. on } \Gamma,$$

for some $h_0 \in \mathbb{R}^+$.

(A3) The matrix field B is a Carathéodory function with the following properties:

- (A3.1) $B(x,t)\xi \cdot \xi \geq \alpha|\xi|^2$, a.e. $x \in \Omega$, for all $t \in \mathbb{R}$, for any $\xi \in \mathbb{R}^N$, and
- (A3.2) for any $k > 0$, $B(x,t) \in L^\infty(\Omega \times (-k,k))^{N \times N}$.

The space V

The space that we will be working with is the following:

Definition

Let V_1 be the space defined by

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

Define $V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\}$, equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \sigma^2 \|v_1 - v_2\|_{L^2(\Gamma)}^2, \tag{1}$$

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for some positive constant σ.

Proposition [Monsurró (2013)]

The norm given in (1) is equivalent to the norm of $V_1 \times H^1(\Omega_2)$, that is, there exist 2 positive constants c_1, c_2 such that

$$c_1 \|v\|_V \leq \|v\|_{V_1 \times H^1(\Omega_2)} \leq c_2 \|v\|_V, \quad \forall v \in V.$$

Main Goal

Our main goal is to do the homogenization process on the following quasilinear elliptic problem on a two-component domain with the use of the periodic unfolding method (PUM).

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon) = f & \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}(A^\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon) = f & \text{in } \Omega_2^\varepsilon, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = (A^\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = -\varepsilon^\gamma h(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{cases}$$

Outline

Variational Case on a fixed domain

The case when the given data f is L^2 is presented in

- Beltran (MS Thesis, 2014), *"Homogenization of a quasilinear elliptic problem in a two-component domain with an imperfect interface"*

Homogenization on a Two-Component Domain using PUM

Linear case:

- Donato, Le Nguyen, Tardieu (2011): linear heat flux on the interface
- Donato, Le Nguyen (2015): nonlinear heat flux on the interface

Quasilinear case:

- Beltran (MS Thesis, 2014): first steps of the homogenization process
- Borja (MS Thesis, 2015): continued and finished the homogenization process

Renormalized solutions, L^1 data, fixed domain

Renormalized solutions are a convenient framework to deal with elliptic equation with L^1. It provides

- existence
- stability
- uniqueness results (under more restrictive conditions on the operators).

This was introduced first by DiPerna and Lions (1989) for first order and Boltzmann equations. Then further developed for elliptic PDEs by:

- Murat (1993-1994, L^1 data)
- Dal Maso, Murat, Orsina, Prignet (1999, bounded Radon measure).

Homogenization, L^1 data, PUM

The following paper considered an L^1 data in homogenization using PUM:

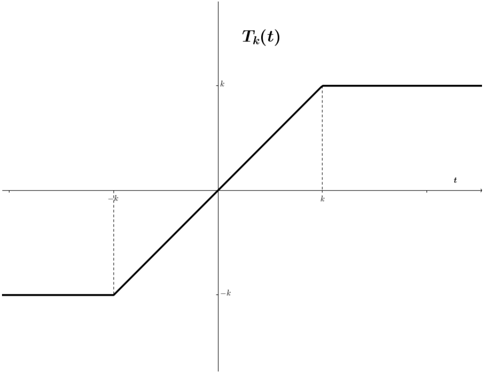
- Donato, Guibé, Oropeza (2017), *"Homogenization of quasilinear elliptic problems with nonlinear Robin conditions and L^1 data"*

Outline

Truncation function T_k

Define the function $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$T_k(t) = \begin{cases} -k, & \text{if } t \leq -k \\ t, & \text{if } -k \leq t \leq k \\ k, & \text{if } t \geq k. \end{cases}$$



Definition of gradient and trace of u

Proposition

Let $u = (u_1, u_2) : \Omega \setminus \Gamma \longrightarrow \mathbb{R}$ be a measurable function such that $T_k(u) \in V$ for every $k > 0$. For $i = 1, 2$,

1. [Benilan, et al. (1995)] There exists a unique measurable function $v_i : \Omega_i \longrightarrow \mathbb{R}^N$ such that

$$\nabla T_k(u_i) = v_i \chi_{\{|u_i| < k\}} \quad \text{a.e. in } \Omega_i,$$

   where $\chi_{\{|u_i|<k\}}$ denotes the characteristic function of $\{x \in \Omega_i : |u_i(x)| < k\}$. We define v_i as the gradient of u_i and write $v_i = \nabla u_i$.

2 If

$$\sup_{k\geq 1} \frac{1}{k} \| T_k(u) \|_V^2 < \infty,$$

then there exists a unique measurable function $w_i : \Gamma \longrightarrow \mathbb{R}$, for $i = 1, 2$, such that

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma,$$

where $\gamma_i : H^1(\Omega_i) \longrightarrow L^2(\Gamma)$ is the trace operator. We define the function w_i as the trace of u_i on Γ and set $\gamma_i(u_i) = w_i$.

Definition of Renormalized Solution of (P)

Definition

Let $u : \Omega \setminus \Gamma \longrightarrow \mathbb{R}$ be a measurable function. Then u is a renormalized solution of (P) if

$$T_k(u) \in V \tag{2a}$$

and

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \tag{2b}$$

for any $k > 0$;

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{|u|<n\}} B(x,u)\nabla u \cdot \nabla u \, dx = 0; \tag{3a}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) \, d\sigma = 0; \tag{3b}$$

and for any $S \in C^1(\mathbb{R})$ (or equivalently for any $S \in W^{1,\infty}(\mathbb{R})$) with compact support, u satisfies

$$\int_{\Omega_1} S(u_1)B(x,u_1)\nabla u_1 \cdot \nabla v_1 \, dx + \int_{\Omega_1} S'(u_1)B(x,u_1)\nabla u_1 \cdot \nabla u_1 \, v_1 \, dx$$
$$\int_{\Omega_2} S(u_2)B(x,u_2)\nabla u_2 \cdot \nabla v_2 \, dx + \int_{\Omega_2} S'(u_2)B(x,u_2)\nabla u_2 \cdot \nabla u_2 \, v_2 \, dx \tag{4}$$
$$+ \int_\Gamma h(x)(u_1 - u_2)(v_1 S(u_1) - v_2 S(u_2)) \, d\sigma = \int_\Omega f v S(u) \, dx$$

for all $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$.

Remarks

- Conditions (2a) (the regularity of the truncate) and (3a) (the decay of the "truncated energy") are standard in the framework of renormalized solutions.

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- Conditions (2a) and (2b) allow one to give a sense of all the terms in (4). The main difficulty we encountered was giving a meaning to the integral on the boundary in (4).
- Conditions (3a) and (3b) which are crucial to obtain uniqueness results (a paper is in progress) and also to recover that formally, for any $k > 0$, $T_k(u)$ is an admissible function in (P).

Existence Theorem

Theorem

Suppose the assumptions (A1)-(A3) hold. Then there exists a renormalized solution to (P) in the sense of the previous definition.

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Idea of Proof:

Step 1: Introducing the approximate problem and showing the existence of solution of the approximate problem

Let $\varepsilon > 0$. Suppose $\{f^\varepsilon\} \subset L^2(\Omega)$ such that

$$f^\varepsilon \longrightarrow f \text{ strongly in } L^1(\Omega)$$

as $\varepsilon \to 0$.
Define $B_\varepsilon(x,t) = B(x, T_{1/\varepsilon}(t))$.

Sketch of the proof

We now consider the following approximate problem

$$\begin{cases} -\mathrm{div}(B_\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon) = f^\varepsilon & \text{in } \Omega_1, \\ -\mathrm{div}(B_\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon) = f^\varepsilon & \text{in } \Omega_2, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (B_\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = (B_\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1 & \text{on } \Gamma, \\ (B_\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = -h(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma, \end{cases} \tag{P_ε}$$

The variational formulation of problem (P_ε) is the following

$$\begin{cases} \text{Find } u^\varepsilon \in V \text{ such that } \forall \varphi \in V \\ \displaystyle\int_{\Omega_1} B_\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon \cdot \nabla\varphi_1\,dx + \int_{\Omega_2} B_\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon \cdot \nabla\varphi_2\,dx \\ \qquad\qquad \displaystyle + \int_\Gamma h(x)(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1 - \varphi_2)\,d\sigma = \int_\Omega f^\varepsilon \varphi\,dx. \end{cases} \tag{5}$$

The proof of the existence of a solution, via the Schauder's Fixed Point Theorem, for (5) is quite standard.

Step 2: Extracting subsequences and examining convergences

Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a solution to the approximate problem (P_ε).
We show that the sequence $\{T_k(u^\varepsilon)\}$ is bounded in V for every $k > 0$.
We then extract a subsequence (still denoted by ε), which satisfies the following:

$$\begin{cases} u_i^\varepsilon \longrightarrow u_i & \text{a.e. in } \Omega \\ T_k(u_i^\varepsilon) \longrightarrow T_k(u_i) & \text{strongly in } L^2(\Omega_i), \text{ a.e. in } \Omega_i, \\ \gamma_i(u_i^\varepsilon) \longrightarrow \gamma_i(u_i) & \text{a.e. on } \Gamma, \\ \gamma_i(T_k(u_i^\varepsilon)) \longrightarrow \gamma_i(T_k(u_i)) & \text{strongly in } L^2(\Gamma), \text{ a.e. in } \Gamma \\ \nabla T_k(u_i^\varepsilon) \rightharpoonup \nabla T_k(u_i) & \text{weakly in } (L^2(\Omega_i))^N. \end{cases}$$

From the first four convergences, conditions (2) can be shown.

Step 3: Showing conditions (3)**.**

By lower semicontinuity of the weak convergence, Fatou's Lemma, and the nonnegativity of

$$\int_{\Omega\setminus\Gamma} B(x, T_n(u^\varepsilon))\nabla T_n(u^\varepsilon)\nabla T_n(u^\varepsilon)\,dx \quad \text{and} \quad \int_\Gamma (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon))\,d\sigma,$$

it suffices to show that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\Big(\int_{\Omega\setminus\Gamma} B(x, T_n(u^\varepsilon))\nabla T_n(u^\varepsilon)\nabla T_n(u^\varepsilon)\,dx + \int_\Gamma (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon))\,d\sigma\Big) = 0. \tag{6}$$

Using $\frac{1}{n}T_n(u^\varepsilon)$ as a test function in (5) and passing to the limit first as $\varepsilon \longrightarrow 0$ and then $n \longrightarrow \infty$ will give us (6).

Step 4: Show that u satisfies (4).

Let $S \in C^1(\mathbb{R})$ with compact support. We use

$$\psi = vS(u)S_n(u^\varepsilon) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$$

as a test function in (5).

We pass to the limit first as $\varepsilon \longrightarrow 0$ and then as $n \longrightarrow \infty$ to have (4). □

Thank you very much for your attention.