Quantum toroidal algebras: braid group actions and automorphisms (arXiv:2304.06773)

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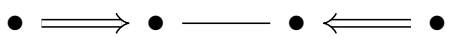
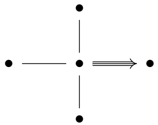
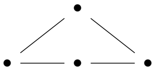
Kac-Moody Lie algebras

Generalised Cartan matrices $A = (a_{ij})_{i,j \in I}$ with

$\cdot$ all $a_{ii} = 2$ $\quad$ $\cdot$ $a_{ij} \leq 0$ if $i \neq j$ $\quad$ $\cdot$ $a_{ij} = 0$ iff $a_{ji} = 0$

Dynkin diagrams have vertex set I and $a_{ij}a_{ji}$ edges

$\cdot$ $i \to j$ if $a_{ij} < a_{ji}$ $\quad$ $\cdot$ $j \to i$ if $a_{ij} > a_{ji}$ $\quad$ $\cdot$ $i \leftrightarrow j$ if $a_{ij} = a_{ji}$

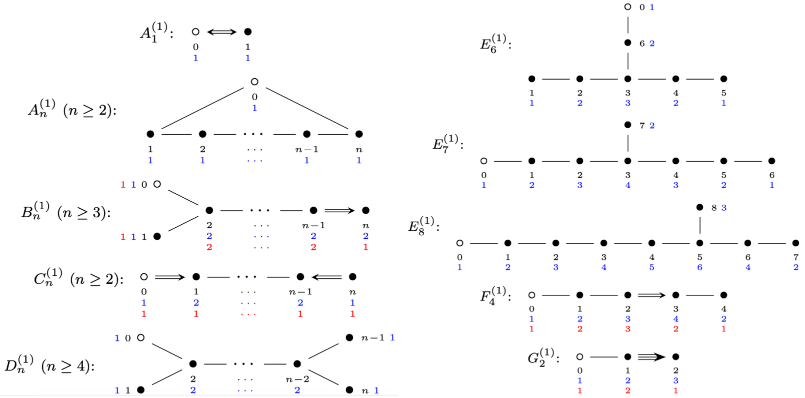
Kac-Moody algebras have generators e_i, f_i, $\pm h_i$ for each $i \in I$ and relations

- $[h_i, h_j] = 0$
- $[h_i, e_j] = a_{ij}e_j$, $[h_i, f_j] = -a_{ij}f_j$
- $[e_i, f_j] = \delta_{ij}h_i$
- Serre relations

Kac-Moody Lie algebras

type of $\mathfrak{g}$	finite	affine	indefinite
eigenvalues of A	positive	positive and zero	positive and negative

The affine case has $\ker(A) = \langle \delta \rangle$ for a unique positive vector $\delta = (a_0, \dots, a_n)$.



Loop realization of untwisted affine Lie algebras

finite dimensional simple Lie algebra $\mathfrak{g}$ of type X_n

$\downarrow$ regular rational maps $S^1 \to \mathfrak{g}$

loop Lie algebra $\mathfrak{g}[t, t^{-1}]$

$$[xt^m, yt^n] = [x, y]_{\mathfrak{g}}\, t^{n+m}$$

$\downarrow$ adjoin central c

$$\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$$

$$[xt^m, yt^n] = [x, y]_{\mathfrak{g}}\, t^{n+m} + m(x, y)\delta_{m+n,0}c$$

$\downarrow$ adjoin derivation d

$$\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

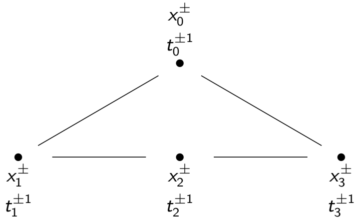
$$[d, xt^m] = mxt^m$$

$\cong$

affine Lie algebra $\hat{\mathfrak{g}}$ of type $X_n^{(1)}$

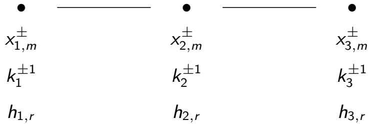
Drinfeld-Jimbo realization

$x_0^{\pm}$	$x_1^{\pm}$	$\cdots$	$x_n^{\pm}$
$t_0^{\pm 1}$	$t_1^{\pm 1}$	$\cdots$	$t_n^{\pm 1}$



Drinfeld new realization

$\vdots$			$\vdots$	
$x_{1,1}^{\pm}$	$h_{1,1}$	$\cdots$	$x_{n,1}^{\pm}$	$h_{n,1}$
$x_{1,0}^{\pm}$	$k_1^{\pm 1}$	$\cdots$	$x_{n,0}^{\pm}$	$k_n^{\pm 1}$
$x_{1,-1}^{\pm}$	$h_{1,-1}$	$\cdots$	$x_{n,-1}^{\pm}$	$h_{n,-1}$
$\vdots$		$C^{\pm 1}$	$\vdots$	



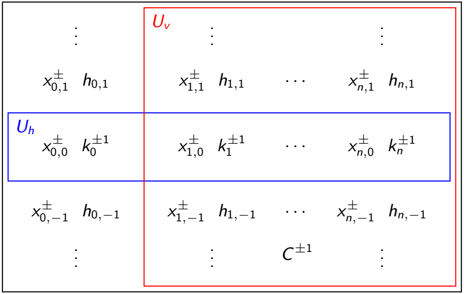
Central elements: C corresponds to $t_\delta = t_0^{a_0} \ldots t_n^{a_n}$.

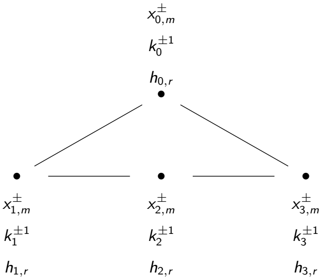
Quantum affinization

$$\boxed{\begin{array}{ccc} x_1^{\pm} & \cdots & x_n^{\pm} \\ t_1^{\pm 1} & \cdots & t_n^{\pm 1} \end{array}} \xrightarrow{\text{quantum affinization}} \boxed{\begin{array}{cccccc} \vdots & & & \vdots & \\ x_{1,1}^{\pm} & h_{1,1} & \cdots & x_{n,1}^{\pm} & h_{n,1} \\ \hline x_{1,0}^{\pm} & k_1^{\pm 1} & \cdots & x_{n,0}^{\pm} & k_n^{\pm 1} \\ \hline x_{1,-1}^{\pm} & h_{1,-1} & \cdots & x_{n,-1}^{\pm} & h_{n,-1} \\ \vdots & & C^{\pm 1} & \vdots & \end{array}}$$

- This process can be applied to any Drinfeld-Jimbo quantum group.
- Applying it to $U_q(\mathfrak{g})$ gives the Drinfeld new presentation of $U_q(\hat{\mathfrak{g}})$.
- What happens if we apply it to $U_q(\hat{\mathfrak{g}})$ in its Drinfeld-Jimbo presentation?

The quantum toroidal algebra is the quantum affinization of $U_q(\hat{\mathfrak{g}})$.





- $U_q(\mathfrak{g}_{\mathrm{tor}})$ contains vertical and a horizontal quantum affine subalgebras.
- These subalgebras U_v and U_h generate the entire algebra.
- Each contains a central element: C and $k_\delta = k_0^{a_0} \dots k_n^{a_n}$.

Why study quantum toroidal algebras?

$\vdots$		U_v $\vdots$			$\vdots$	
$x_{0,1}^{\pm}$	$h_{0,1}$	$x_{1,1}^{\pm}$	$h_{1,1}$	$\cdots$	$x_{n,1}^{\pm}$	$h_{n,1}$
U_h $x_{0,0}^{\pm}$	$k_0^{\pm 1}$	$x_{1,0}^{\pm}$	$k_1^{\pm 1}$	$\cdots$	$x_{n,0}^{\pm}$	$k_n^{\pm 1}$
$x_{0,-1}^{\pm}$	$h_{0,-1}$	$x_{1,-1}^{\pm}$	$h_{1,-1}$	$\cdots$	$x_{n,-1}^{\pm}$	$h_{n,-1}$
$\vdots$		$\vdots$		$C^{\pm 1}$	$\vdots$	

- [Ginzburg-Kapranov-Vasserot '95] In the *ADE* case,

$$U_q(\mathfrak{g}_{\mathrm{tor}}) \curvearrowright \mathbb{C}[\text{some vector bundles on an algebraic surface}]$$

- [Nakajima '01] In the simply laced case,

$$U_q(\mathfrak{g}_{\mathrm{tor}}) \to \bigoplus_{\underline{v}} K^G(\mathcal{M}(\underline{v},\underline{w}) \times_{\mathcal{M}_0(\underline{v},\underline{w})} \mathcal{M}(\underline{v},\underline{w}))$$

- $U_q(\mathfrak{g}_{\mathrm{tor}})$ are the next class of quantum affinizations after $U_q(\hat{\mathfrak{g}})$
- Studying $U_q(\mathfrak{g}_{\mathrm{tor}})$ could lead to results for $U_q(\hat{\mathfrak{g}})$
- [Varagnolo-Vasserot '95] Schur-Weyl duality of $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$ with DAHA

Braid groups

Kac-Moody algebras have braid groups $\mathcal{B} = \langle T_i | i \in I, \underbrace{T_i T_j T_i \dots}_{a_{ij}a_{ji}+2} = \underbrace{T_j T_i T_j \dots}_{a_{ij}a_{ji}+2} \rangle$.

Let Ω be the outer automorphism group of the affine Dynkin diagram.

Extended affine braid group has Coxeter presentation $\dot{\mathcal{B}} = \Omega \ltimes \langle T_0, \dots, T_n \rangle$ with $\pi T_i \pi^{-1} = T_{\pi(i)}$.

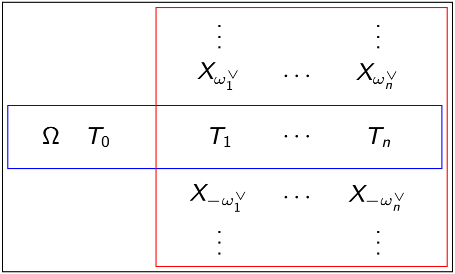
$\Omega \quad T_0$	$T_1 \ \cdots \ T_n$

Bernstein presentation of $\dot{\mathcal{B}}$ generated by finite braid group $\langle T_1, \dots, T_n \rangle$ and lattice $\{X_\beta : \beta \in P^\vee\}$.

(the $\omega_i^\vee$ are the fundamental coweights)

$\vdots$		$\vdots$
$X_{\omega_1^\vee}$	$\dots$	$X_{\omega_n^\vee}$
T_1	$\cdots$	T_n
$X_{-\omega_1^\vee}$	$\dots$	$X_{-\omega_n^\vee}$
$\vdots$		$\vdots$

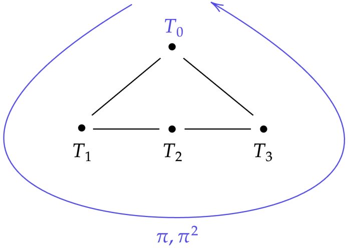
Extended double affine braid group $\ddot{\mathcal{B}}$



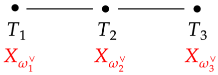
- $T_0, \ldots, T_n$ satisfy the braid relations
- $\pi T_i \pi^{-1} = T_{\pi(i)}$
- $\pi X_\beta \pi^{-1} = X_{\pi(\beta)}$
- $T_i X_\beta = X_\beta T_i$ if $(\beta, \alpha_i) = 0$,
- $T_i^{-1} X_\beta T_i^{-1} = X_{s_i(\beta)}$ if $(\beta, \alpha_i) = 1$

Braid groups in type A_3

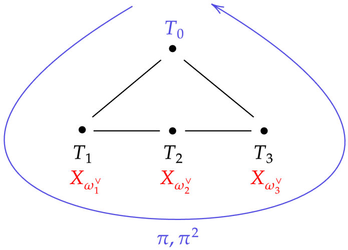
Coxeter presentation of $\dot{\mathfrak{B}}$



Bernstein presentation of $\dot{\mathfrak{B}}$



Extended double affine braid group $\ddot{\mathfrak{B}}$



Actions of braid groups on quantum algebras – the affine level

Theorem (Lusztig, Beck) The extended affine braid group $\dot{\mathcal{B}}$ acts on the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ in all untwisted types.

$$
\begin{array}{|cc|ccc|}
\hline
\Omega & T_0 & T_1 & \cdots & T_n \\
\hline
\end{array}
\quad \curvearrowright \quad
\begin{array}{|c|ccc|}
\hline
x^{\pm}_{0,0} & x^{\pm}_{1,0} & \cdots & x^{\pm}_{n,0} \\
t_0^{\pm 1} & t_1^{\pm 1} & \cdots & t_n^{\pm 1} \\
\hline
\end{array}
$$

- T_i intertwines generators at vertices $j \sim i$ with those at i
- $\pi \in \Omega$ permutes the generators around: $\pi(x_i^{\pm}) = x^{\pm}_{\pi(i)}$ and $\pi(t_i) = t_{\pi(i)}$

$$
\begin{array}{|ccc|}
\hline
\vdots & & \vdots \\
X_{\omega_1^\vee} & \cdots & X_{\omega_n^\vee} \\
\hline
T_1 & \cdots & T_n \\
\hline
X_{-\omega_1^\vee} & \cdots & X_{-\omega_n^\vee} \\
\vdots & & \vdots \\
\hline
\end{array}
\quad \curvearrowright \quad
\begin{array}{|ccccc|}
\hline
\vdots & & & \vdots & \\
x^{\pm}_{1,1} \; h_{1,1} & & \cdots & x^{\pm}_{n,1} \; h_{n,1} & \\
\hline
x^{\pm}_{1,0} \; k_1^{\pm 1} & & \cdots & x^{\pm}_{n,0} \; k_n^{\pm 1} & \\
\hline
x^{\pm}_{1,-1} \; h_{1,-1} & & \cdots & x^{\pm}_{n,-1} \; h_{n,-1} & \\
\vdots & & C^{\pm 1} & \vdots & \\
\hline
\end{array}
$$

- T_i intertwines generators at vertices $j \sim i$ with those at i
- $X_{\omega_i^\vee}$ shifts generators at vertex i 'up and down' (ie. $x^{\pm}_{i,m} \to x^{\pm}_{i,m\mp 1}$)

Actions of braid groups on quantum algebras – the toroidal level

Theorem (L. '23) The extended double affine braid group $\ddot{\mathcal{B}}$ acts on the quantum toroidal algebra $U_q(\mathfrak{g}_{\mathrm{tor}})$ in all types (other than $A_1^{(1)}$ and $A_2^{(2)}$).

		$\vdots$		$\vdots$
		$X_{\omega_1^\vee}$	$\cdots$	$X_{\omega_n^\vee}$
Ω	T_0	T_1	$\cdots$	T_n
		$X_{-\omega_1^\vee}$	$\cdots$	$X_{-\omega_n^\vee}$
		$\vdots$		$\vdots$

$\curvearrowright$

$\vdots$		U_v $\vdots$			$\vdots$	
$x_{0,1}^\pm$	$h_{0,1}$	$x_{1,1}^\pm$	$h_{1,1}$	$\cdots$	$x_{n,1}^\pm$	$h_{n,1}$
U_h $x_{0,0}^\pm$	$k_0^{\pm1}$	$x_{1,0}^\pm$	$k_1^{\pm1}$	$\cdots$	$x_{n,0}^\pm$	$k_n^{\pm1}$
$x_{0,-1}^\pm$	$h_{0,-1}$	$x_{1,-1}^\pm$	$h_{1,-1}$	$\cdots$	$x_{n,-1}^\pm$	$h_{n,-1}$
$\vdots$		$\vdots$		$C^{\pm1}$	$\vdots$	

- Horizontal preserves horizontal, vertical preserves vertical
- T_i intertwines generators at vertices $j \sim i$ with those at i
- $\pi \in \Omega$ permutes generators around: $\pi(x_{i,m}^\pm) = \pm x_{\pi(i),m}^\pm$, $\pi(k_i) = k_{\pi(i)}$, etc.
- $X_{\omega_i^\vee}$ shifts generators at vertices i and 0 'up and down' their columns

U_v (columns $1, \dots, n$), U_h (row 0)

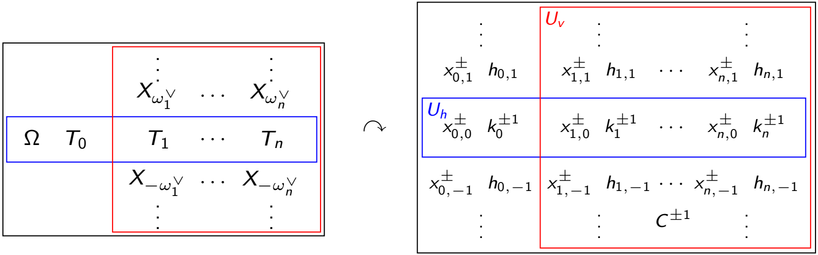
$\vdots$		$\vdots$			$\vdots$	
$x_{0,1}^{\pm}$	$h_{0,1}$	$x_{1,1}^{\pm}$	$h_{1,1}$	$\cdots$	$x_{n,1}^{\pm}$	$h_{n,1}$
$x_{0,0}^{\pm}$	$k_0^{\pm 1}$	$x_{1,0}^{\pm}$	$k_1^{\pm 1}$	$\cdots$	$x_{n,0}^{\pm}$	$k_n^{\pm 1}$
$x_{0,-1}^{\pm}$	$h_{0,-1}$	$x_{1,-1}^{\pm}$	$h_{1,-1}$	$\cdots$	$x_{n,-1}^{\pm}$	$h_{n,-1}$
$\vdots$		$\vdots$		$C^{\pm 1}$	$\vdots$	

(Miki '99) obtained an automorphism of $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$ exchanging U_v and U_h

(Miki '00) used this to...

- classify some irreducible highest weight representations by Drinfeld polynomials
- study R-matrices on their tensor products
- relate known representations (vertex and Fock space)

<u>Idea:</u> find an involution $\mathfrak{t}$ of $\ddot{\mathcal{B}}$ exchanging $\mathcal{B}_v$ and $\mathcal{B}_h$, and pass it across the action to obtain an automorphism of $U_q(\mathfrak{g}_{\mathrm{tor}})$ exchanging U_v and U_h.



Each generator equals $b \cdot z$ for some $b \in \ddot{\mathcal{B}}$ and $z \in U_v \cap U_h$.

<u>Theorem (L. '23)</u> $b \cdot z \mapsto \mathfrak{t}(b) \cdot z$ extends to an automorphism Φ of $U_q(\mathfrak{g}_{\mathrm{tor}})$.

<u>Proposition</u> For all $b \in \ddot{\mathcal{B}}$ we have $\Phi \circ b = \mathfrak{t}(b) \circ \Phi$.

Future directions

1. Extend our (anti-)automorphisms to non-simply laced types

2. Obtain quantum algebra analogues of other braid group phenomena

3. Use our automorphism to study the representation theory of $U_q(\mathfrak{g}_{\mathrm{tor}})$