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UNIFORM-IN-TIME ERROR ESTIMATES FOR THE POSTPROCESSING GALERKIN METHOD APPLIED TO A DATA ASSIMILATION ALGORITHM*

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Abstract. We apply the postprocessing Galerkin method to a recently introduced continuous data assimilation (downscaling) algorithm for obtaining a numerical approximation of the solution of the two-dimensional Navier–Stokes equations corresponding to given measurements from a coarse spatial mesh. Under suitable conditions on the relaxation (nudging) parameter, the resolution of the coarse spatial mesh, and the resolution of the numerical scheme, we obtain uniform-in-time estimates for the error between the numerical approximation given by the postprocessing Galerkin method and the reference solution corresponding to the measurements. Our results are valid for a large class of interpolant operators, including low Fourier modes and local averages over finite volume elements. Notably, we use here the two-dimensional Navier–Stokes equations as a paradigm, but our results apply equally to other evolution equations, such as the Boussinesq system of Bénard convection and other oceanic and atmospheric circulation models.

Key words. data assimilation, downscaling, nudging, feedback control, two-dimensional Navier–Stokes equations, Galerkin method, postprocessing

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1. Introduction.

The purpose of data assimilation is to combine a forecast model with observational data in order to produce an accurate prediction of the future state of a physical system. Several methods have been developed with this goal. One of them is the *nudging* (or *Newtonian relaxation*) method, in which the original model is modified by adding an extra term which has the purpose of relaxing the coarse scales of the solution towards the spatially coarse observations (see, e.g., [4] and also [6, 31, 32, 41, 42, 48, 53] for a connection with feedback control theory). Notably, several authors have developed algorithms that combine the nudging approach with other data assimilation methods with the aim of addressing some of its difficulties, such as decreasing the effect of the relaxation term as the noise in the observations increases (see, e.g., [1, 49, 56]).

However, we notice that most of the related works consider rather simplified scenarios, e.g., in the context of finite-dimensional ordinary differential equations. Here, we follow the nudging type approach introduced in [5], which consists of a framework applicable to a large class of dissipative evolution equations, given as infinite-dimensional partial differential equations, and various types of observables from a finite coarse spatial mesh. Our goal is to consider a dimensionally reduced version of the nudging equation represented by a numerical approximation given by

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the postprocessing Galerkin method (PPGM) [27, 28] and to obtain an analytical estimate of the error between this numerical approximation and the full reference solution of the original (infinite-dimensional) model corresponding to the measurements.

In [5], the authors consider, as a paradigm, a theoretical model given by the two-dimensional (2D) incompressible Navier–Stokes equations and consider the simplifying assumptions of continuous-in-time and error-free measurements. Here, we also consider, for simplicity, the same setting. However, we remark that other works extended this approach to more general situations: continuous-in-time data with stochastic noise ([8]; see also [9, 39] for the context of the 3DVAR filtering method) and discrete-in-time data with systematic errors ([23]; see also [34]). Moreover, several works considered different reference theoretical models [2, 14, 16, 17, 18, 36, 45]. Notably, the works [14]–[18] dealt with the situation of incomplete data, i.e., observations of fewer components of the state variables.

We now briefly provide the ideas behind the algorithm used in this paper. The 2D incompressible Navier–Stokes equations are given by

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \quad (\mathbf{x},t)\in\Omega\times[t_0,\infty)\subset\mathbb{R}^2\times\mathbb{R}, \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2)(\mathbf{x}, t)$ and $p = p(\mathbf{x}, t)$ are the unknowns and represent the velocity vector field and the pressure, respectively, while $\nu > 0$ and $\mathbf{f} = \mathbf{f}(\mathbf{x})$ are given and denote the kinematic viscosity parameter and the density of volume body forces, respectively. We assume that $\mathbf{f}$ is time-independent for simplicity, but similar results are valid for a time-dependent $\mathbf{f}$, whose $L^2(\Omega)^2$-norm is uniformly bounded in time.

Our reference solution, whose exact value is unknown, is thus a solution $\mathbf{u}$ of (1.1). The given measurements, corresponding to $\mathbf{u}$, are observed from a coarse spatial mesh and are assumed, as mentioned before, to be continuous in time and error-free. We denote the operator used for interpolating these measurements in space by I_h, where h denotes the resolution of the coarse spatial mesh of the observed measurements. Thus, the interpolated measurements are represented by $I_h(\mathbf{u})$. Since the initial condition $\mathbf{u}(t_0)$ for $\mathbf{u}$ is missing, one cannot compute $\mathbf{u}$ by integrating (1.1) directly. The idea consists then in recovering the fine scales of $\mathbf{u}$ from the coarse scale measurements $I_h(\mathbf{u})$ through a downscaling algorithm.

In [5], this is done by seeking for a solution $\mathbf{v} = (v_1, v_2)$ of the following modified system for $(\mathbf{x}, t) \in \Omega \times [t_0, \infty)$:

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\tilde{p} = \mathbf{f} - \beta(I_h(\mathbf{v}) - I_h(\mathbf{u})), \quad \nabla\cdot\mathbf{v} = 0, \tag{1.2}$$

where the unknown $\tilde{p}$ is the pressure of the approximate flow $\mathbf{v}$; $\nu > 0$ and $\mathbf{f}$ are the same viscosity parameter and forcing term from (1.1), respectively; and β is the relaxation (nudging) parameter. The second term in the right-hand side of the first equation in (1.2) is called the feedback control term, and its role is to force (or nudge) the coarse spatial scales of the approximating solution $\mathbf{v}$ towards the coarse spatial scales of the reference solution $\mathbf{u}$, which is done by suitably tuning the relaxation parameter β. In [5, Theorems 1 and 2], the authors prove that, provided β is large enough and h is sufficiently small, both depending on given physical parameters, the approximate solution $\mathbf{v}$ of (1.2), corresponding to an arbitrary initial data $\mathbf{v}_0$, converges, exponentially in time, to the reference solution $\mathbf{u}$ of (1.1).

We rewrite system (1.2) in the following equivalent functional form:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \nu A\mathbf{v} + B(\mathbf{v},\mathbf{v}) = \mathbf{g} - \beta P_\sigma(I_h(\mathbf{v}) - I_h(\mathbf{u})), \tag{1.3}$$

where P_σ is the orthogonal projection of $(L^2(\Omega))^2$ onto the phase space H associated to (1.2), which is endowed with the norm of $(L^2(\Omega))^2$, $|\cdot|_{L^2}$; $A = -P_\sigma\Delta$ is the Stokes operator; $B(\mathbf{v},\mathbf{v}) = P_\sigma[(\mathbf{v}\cdot\nabla)\mathbf{v}]$ is a bilinear operator (see section 2 for more detailed definitions); and $\mathbf{g} = P_\sigma\mathbf{f}$. Since A is a positive and self-adjoint operator with compact inverse, the space H admits an orthonormal basis of eigenvectors $\{\mathbf{w}_i\}_{i\in\mathbb{N}}$ of A. Then, for each $N \in \mathbb{N}$, we can consider the finite-dimensional space $H_N = \operatorname{span}\{\mathbf{w}_1,\ldots,\mathbf{w}_N\} = P_N H$, with P_N denoting the orthogonal projection of H onto H_N. A numerical approximation of the solution $\mathbf{v}$ of (1.3) can be obtained by computing the Galerkin approximation $\mathbf{v}_N \in P_N H$, which satisfies the following system of ordinary differential equations:

$$\frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}t} + \nu A\mathbf{v}_N + P_N B(\mathbf{v}_N,\mathbf{v}_N) = P_N\mathbf{g} - \beta P_N P_\sigma(I_h(\mathbf{v}_N) - I_h(\mathbf{u})). \tag{1.4}$$

Notice that since $\mathbf{v}_N \in P_N H$, the error committed in approximating $\mathbf{v}$ by $\mathbf{v}_N$ must be greater than or equal to the error associated with the best approximation of $\mathbf{v}$ in $P_N H$, $P_N\mathbf{v}$, i.e.,

$$|\mathbf{v} - \mathbf{v}_N|_{L^2} \geq |\mathbf{v} - P_N\mathbf{v}|_{L^2} = |Q_N\mathbf{v}|_{L^2},$$

where $Q_N = I - P_N$.

The PPGM provides us with an efficient way of obtaining a better approximation of $\mathbf{v}$ than $\mathbf{v}_N$. The idea consists in complementing the finite-dimensional approximation $\mathbf{v}_N \in P_N H$ of $\mathbf{v}$ with a suitable part lying in the complement space $Q_N H$. Adapting the algorithm introduced in [27, 28] to our situation, we can summarize it in the following steps.

For obtaining an approximation of $\mathbf{v}$ at a certain time $T > t_0$,

(i) integrate (1.4) in time, over the time interval $[t_0, T]$, to obtain $\mathbf{v}_N$ and compute $\mathbf{v}_N(T)$;
(ii) obtain $\mathbf{q}_N$ satisfying $\nu A\mathbf{q}_N = Q_N[\mathbf{g} - B(\mathbf{v}_N(T),\mathbf{v}_N(T))]$;
(iii) compute the new approximation to $\mathbf{v}(T)$, and hence to $\mathbf{u}(T)$, given by $\mathbf{v}_N(T) + \mathbf{q}_N$.

The equation satisfied by $\mathbf{q}_N$ in step (ii) is inspired by the definition of the approximate inertial manifold introduced in [21], in which the authors obtain an approximation of $Q_N\mathbf{u}$, with $\mathbf{u}$ being a solution of (1.1), given by

$$Q_N\mathbf{u} \approx \Phi_1(P_N\mathbf{u}) := (\nu A)^{-1} Q_N[\mathbf{g} - B(P_N\mathbf{u}, P_N\mathbf{u})]. \tag{1.5}$$

The graph of the mapping $\Phi_1 : P_N H \to Q_N H$ is called an approximate inertial manifold. This approximation is obtained by applying the projection Q_N to equation (1.1) and, based on theoretical arguments, neglecting all lower-order terms, i.e., the time derivative of $Q_N\mathbf{u}$ and the nonlinear terms involving $Q_N\mathbf{u}$, in comparison to the remaining terms. Since our idea is to ultimately obtain an approximation of $\mathbf{u}$, it is natural to consider as an approximation of $Q_N\mathbf{v}$ the same type of approximation used for $Q_N\mathbf{u}$ in (1.5), in which $P_N\mathbf{u}$ is replaced by $\mathbf{v}_N$, given that this is the approximation of $P_N\mathbf{u}$ that we consider.

Our results show that the PPGM yields a better convergence rate than the standard Galerkin method, as also obtained in [28] (see also [43] for an explanation on why the PPGM is actually the correct leading order approximating scheme and not the standard Galerkin method as commonly believed). However, an important difference in our results is that our error estimate is uniform in time, while in [27, 28] it grows exponentially in time. This remarkable difference is due to the fact that system (1.4) has a stabilizing mechanism imposed by the feedback control term, which kills the instabilities in the large (coarse) spatial scales caused by the nonlinear term. Consequently, as proved in [5, Theorems 1 and 2], under suitable conditions on the parameters β and h, the solutions of (1.2), corresponding to arbitrary initial data, all converge to the same reference solution $\mathbf{u}$. This shows that, with appropriate conditions on β and h, system (1.2) is globally asymptotically stable. Hence, the Galerkin approximation $\mathbf{v}_N$ of $\mathbf{v}$ converges to $\mathbf{v}$ uniformly in time, as N tends to infinity. Such stabilizing effect was also observed by the numerical computations performed in [3, 30], which also showed that the required conditions on the parameters β and h are remarkably less strict than suggested by the analytical results in [5].

It is worth mentioning that the introduction of the PPGM was preceded by another spectral method also derived from the standard Galerkin approach and inspired by the idea of approximate inertial manifold, known as the nonlinear Galerkin method (see, e.g., [13, 19, 35, 44] and references therein). This latter method presents a similar convergence rate to the PPGM, but it has the disadvantage of being a lot more computationally expensive and thus, in practice, less efficient (cf. [27, 28, 33, 43]).

This paper is organized as follows. In section 2, we provide a summary of the necessary background concerning the 2D incompressible Navier–Stokes equations. Section 3 contains the main results of this paper. The purpose is to show a uniform-in-time estimate of the error committed when applying the PPGM described in (i)–(iii) above to system (1.2) in order to obtain an approximation of the reference solution $\mathbf{u}$ satisfying (1.1) (Theorems 3.8 and 3.12). We divide the presentation into two subsections: subsection 3.1 deals with the case of an interpolant operator given by a low Fourier modes projector, while subsection 3.2 deals with a more general class of interpolant operators satisfying suitable properties, for which an example is given by the operator defined as local averages over finite volume elements, in the case of periodic boundary conditions. Finally, in the Appendix, we show for completeness that such an example of interpolant operator verifies the properties considered in subsection 3.2.

Readers seeking more elaborate details of the proofs given here may also want to consult the arXiv version of this paper [47].

2. Preliminaries. In this section, we briefly recall the necessary background on the 2D incompressible Navier–Stokes equations (1.1). For further details, see, e.g., [12, 20, 50, 52].

Consider a spatial domain $\Omega \subset \mathbb{R}^2$ and a time interval $[t_0, \infty) \subset \mathbb{R}$. We assume, for simplicity, that the forcing $\mathbf{f}$ is time-independent and lies in the space $L^2(\Omega)^2$. We remark, however, that similar results are also valid in the case $\mathbf{f} \in L^\infty([t_0, \infty); L^2(\Omega)^2)$.

We consider two types of boundary conditions for system (1.1): periodic or no-slip Dirichlet. In the periodic case, we consider the fundamental domain $\Omega = (0, L) \times (0, L)$. Moreover, we assume that the velocity field and the pressure are periodic with period L in each spatial direction x_i, $i = 1, 2$, and that $\mathbf{f}$ has zero spatial average, i.e.,

$$\int_\Omega \mathbf{f}(\mathbf{x}) \mathrm{d}\mathbf{x} = 0.$$

In the no-slip Dirichlet case, we consider Ω as a bounded subset of $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$ and assume that $\mathbf{u} = 0$ on $\partial\Omega$.

The definition of the space of test functions, denoted here by $\mathcal{V}$, depends on the type of boundary condition being considered. In the periodic case, $\mathcal{V}$ is defined as the set of all L-periodic trigonometric polynomials from $\mathbb{R}^2$ to $\mathbb{R}^2$ that are divergence free and have zero spatial average. In the no-slip Dirichlet case, we define $\mathcal{V}$ as the family of C^∞ vector fields with values in $\mathbb{R}^2$ that are divergence free and compactly supported in Ω.

We denote by H the closure of $\mathcal{V}$ with respect to the norm in $L^2(\Omega)^2$ and by V the closure of $\mathcal{V}$ under the $H^1(\Omega)^2$ Sobolev norm. Following the notation from [20], we denote the inner products in H and V by $(\cdot,\cdot)_{L^2}$ and $(\!(\cdot,\cdot)\!)_{H^1}$, respectively. They are defined as

$$(\mathbf{u},\mathbf{v})_{L^2} = \int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\mathrm{d}\mathbf{x}, \quad \forall \mathbf{u},\mathbf{v}\in H,$$

$$(\!(\mathbf{u},\mathbf{v})\!)_{H^1} = \int_\Omega \sum_{i=1}^2 \frac{\partial \mathbf{u}}{\partial \mathbf{x}_i}\cdot\frac{\partial \mathbf{v}}{\partial x_i}\mathrm{d}\mathbf{x}, \quad \forall \mathbf{u},\mathbf{v}\in V,$$

and the associated norms are given by $|\mathbf{u}|_{L^2} = (\mathbf{u},\mathbf{u})_{L^2}^{1/2}$, $\|\mathbf{u}\|_{H_1} = (\!(\mathbf{u},\mathbf{u})\!)_{H^1}^{1/2}$.

The fact that $\|\cdot\|_{H^1}$ defines a norm in V is justified via the Poincaré inequality, given by

$$\lambda_1^{1/2}|\mathbf{u}|_{L^2} \leq \|\mathbf{u}\|_{H^1} \quad \forall \mathbf{u}\in V, \tag{2.1}$$

where λ_1 is the first eigenvalue of the Stokes operator, defined in (2.3) below.

Given $R>0$, we denote by $B_H(R)$ and $B_V(R)$ the closed balls centered at 0 with radius R, with respect to the norms in H and V, respectively.

We also consider the dual spaces of H and V, denoted by H' and V', respectively. After identifying H with its dual, we obtain $V\subseteq H\subseteq V'$, with the injections being continuous and compact and each space dense in the following one. Moreover, we denote the duality product between V and V' by $\langle\cdot,\cdot\rangle_{V',V}$.

Let P_σ be the Leray–Helmholtz projector, i.e., the orthogonal projection of $L^2(\Omega)^2$ onto H. Applying P_σ to system (1.1), we obtain its following equivalent functional formulation:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{g} \quad \text{in } V', \tag{2.2}$$

where $\mathbf{g} = P_\sigma\mathbf{f}\in H$, $B: V\times V\to V'$ is the bilinear operator defined as the continuous extension of the operator given by

$$B(\mathbf{u},\mathbf{v}) = P_\sigma((\mathbf{u}\cdot\nabla)\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v}\in\mathcal{V},$$

and $A: D(A)\subseteq V\to V'$ is the Stokes operator, defined as the continuous extension of

$$A\mathbf{u} = -P_\sigma\Delta\mathbf{u} \quad \forall \mathbf{u}\in\mathcal{V}, \tag{2.3}$$

with the domain of A, $D(A)$, given by $V\cap H^2(\Omega)^2$.

The Stokes operator is a positive and self-adjoint operator with compact inverse. Therefore, it admits an orthonormal basis of eigenvectors $\{\mathbf{w}_m\}_{m\in\mathbb{N}}$ associated with a nondecreasing sequence of positive eigenvalues $\{\lambda_m\}_{m\in\mathbb{N}}$, with $\lambda_m\to\infty$ as $m\to\infty$.

We also consider, for each $N\in\mathbb{N}$, the low modes projector P_N, which is defined as the orthogonal projector of H onto the subspace $H_N = \mathrm{span}\{\mathbf{w}_1,\ldots,\mathbf{w}_N\}$. Moreover, we denote $Q_N = I - P_N$.

The bilinear operator B satisfies the following property:

$$\langle B(\mathbf{u}_1,\mathbf{u}_2),\mathbf{u}_3\rangle_{V',V} = -\langle B(\mathbf{u}_1,\mathbf{u}_3),\mathbf{u}_2\rangle_{V',V} \quad \forall \mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\in V. \tag{2.4}$$

Recall the Brézis–Gallouet inequality [10, 22], given by

$$\|\mathbf{u}\|_{L^\infty} \le c_B \|\mathbf{u}\|_{H^1} \left[1 + \log\left(\frac{|A\mathbf{u}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{u}\|_{H^1}}\right)\right]^{1/2} \quad \forall \mathbf{u} \in D(A), \tag{2.5}$$

where c_B is a nondimensional (scale invariant) constant and $\|\cdot\|_{L^\infty}$ denotes the usual norm in $L^\infty(\Omega)^2$.

We now recall some inequalities satisfied by the bilinear term B. Using the Brézis–Gallouet inequality (2.5), we obtain that, for every $\mathbf{u}_1 \in D(A)$ with $\mathbf{u}_1 \neq 0$ and every $\mathbf{u}_2 \in V$ and $\mathbf{u}_3 \in H$,

$$|(B(\mathbf{u}_1,\mathbf{u}_2),\mathbf{u}_3)_{L^2}| \le c_B \|\mathbf{u}_1\|_{H^1}\|\mathbf{u}_2\|_{H^1}|\mathbf{u}_3|_{L^2} \left[1 + \log\left(\frac{|A\mathbf{u}_1|_{L^2}}{\lambda_1^{1/2}\|\mathbf{u}_1\|_{H^1}}\right)\right]^{1/2}. \tag{2.6}$$

We also recall the following logarithmic inequalities from [54].

For every $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3 \in V$, with $\mathbf{u}_3 \neq 0$,

$$|(B(\mathbf{u}_1,\mathbf{u}_2),\mathbf{u}_3)_{L^2}| \le c_T \|\mathbf{u}_1\|_{H^1}\|\mathbf{u}_2\|_{H^1}|\mathbf{u}_3|_{L^2} \left[1 + \log\left(\frac{\|\mathbf{u}_3\|_{H^1}}{\lambda_1^{1/2}|\mathbf{u}_3|_{L^2}}\right)\right]^{1/2}. \tag{2.7}$$

For every $\mathbf{u}_1 \in V$ and every $\mathbf{u}_2,\mathbf{u}_3 \in D(A)$, with $\mathbf{u}_2 \neq 0$,

$$|(B(\mathbf{u}_1,\mathbf{u}_2),A\mathbf{u}_3)_{L^2}| \le c_T \|\mathbf{u}_1\|_{H^1}\|\mathbf{u}_2\|_{H^1}|A\mathbf{u}_3|_{L^2} \left[1 + \log\left(\frac{|A\mathbf{u}_2|_{L^2}}{\lambda_1^{1/2}\|\mathbf{u}_2\|_{H^1}}\right)\right]^{1/2}. \tag{2.8}$$

Also, for every $\mathbf{u}_1,\mathbf{u}_2 \in V$, we have

$$B(\mathbf{u}_1,\mathbf{u}_1) - B(\mathbf{u}_2,\mathbf{u}_2) = B\left(\mathbf{u}_1 - \mathbf{u}_2, \frac{\mathbf{u}_1+\mathbf{u}_2}{2}\right) + B\left(\frac{\mathbf{u}_1+\mathbf{u}_2}{2}, \mathbf{u}_1 - \mathbf{u}_2\right).$$

Then it follows from the result in [12, Proposition 6.1] that, for every $\alpha > 1/2$ and $\mathbf{u}_1,\mathbf{u}_2 \in V$,

$$|A^{-\alpha}(B(\mathbf{u}_1,\mathbf{u}_1) - B(\mathbf{u}_2,\mathbf{u}_2))| \le c_\alpha |\Omega|^{\alpha-\frac{1}{2}}\|\mathbf{u}_1+\mathbf{u}_2\|_{H^1}|\mathbf{u}_1-\mathbf{u}_2|_{L^2}, \tag{2.9}$$

where $|\Omega|$ denotes the area of Ω and c_α is a positive constant depending on α through the Sobolev constants from the Sobolev embeddings of $H^{2\alpha}(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$ and of $H^s(\mathbb{R}^2)$ into $L^q(\mathbb{R}^2)$, with $1 > s > (2-2\alpha)$, and $q = 2/(1-s)$. Thus, $c_\alpha \to \infty$ as $\alpha \to \frac{1}{2}^+$.

In this paper, we denote by c a positive absolute constant or a nondimensional positive constant depending on Ω, whose value may change from line to line, while the capital letter C denotes a dimensional constant, depending on the physical parameters, such as ν, λ_1 and $|\mathbf{g}|_{L^2}$.

Finally, we recall some results concerning uniform bounds, with respect to the norms in H and V, for the solutions of (1.1). It is well known that, given $\mathbf{u}_0 \in H$, there exists a unique weak solution of (1.1) satisfying $\mathbf{u}(t_0) = \mathbf{u}_0$ and $\mathbf{u} \in \mathcal{C}([t_0,\infty);H) \cap L^2_{\text{loc}}(t_0,\infty;V)$, with $\mathrm{d}\mathbf{u}/\mathrm{d}t \in L^2_{\text{loc}}(t_0,\infty;V')$. From now on, whenever we refer to a solution of (1.1), we mean a solution in this sense.

The proof of the next proposition can be found in any of the references listed above [12, 20, 50, 52]. Recall the definition of the Grashof number, which is the nondimensional quantity given by

$$G = \frac{|\mathbf{g}|_{L^2}}{\nu^2 \lambda_1}.$$

PROPOSITION 2.1. *Let* $\mathbf{u}_0 \in H$*, and let* $\mathbf{u}$ *be a solution of* (1.1) *satisfying* $\mathbf{u}(t_0) = \mathbf{u}_0$*. Then there exists* $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, |\mathbf{u}_0|_{L^2}) \geq t_0$ *such that the following hold:*

(i) *In the case of periodic boundary conditions,*

$$|\mathbf{u}(t)|_{L^2} \leq 2\nu G, \quad \|\mathbf{u}(t)\|_{H^1} \leq 2\nu \lambda_1^{1/2} G, \quad \forall t \geq T. \tag{2.10}$$

(ii) *In the case of no-slip boundary conditions,*

$$|\mathbf{u}(t)|_{L^2} \leq 2\nu G, \quad \|\mathbf{u}(t)\|_{H^1} \leq c\nu \lambda_1^{1/2} G \,\mathrm{e}^{\frac{G^4}{2}}, \quad \forall t \geq T. \tag{2.11}$$

In order to simplify the notation, we write the uniform bounds in the H and V norms from Proposition 2.1 by using constants M_0 and M_1, respectively, i.e.,

$$|\mathbf{u}(t)|_{L^2} \leq M_0, \quad \|\mathbf{u}(t)\|_{H^1} \leq M_1, \quad \forall t \geq T. \tag{2.12}$$

Notice that the value of M_1 changes according to the boundary condition being considered.

The following theorem follows immediately from the result proved in [21, Theorem 1.1] (see also [55]).

THEOREM 2.2. *Let* $\mathbf{u}_0 \in H$*, and let* $\mathbf{u}$ *be a solution of* (1.1) *satisfying* $\mathbf{u}(t_0) = \mathbf{u}_0$*. Then there exists* $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, |\mathbf{u}_0|_{L^2}) \geq t_0$ *such that*

$$|Q_N \mathbf{u}(t)|_{L^2} \leq C_0 \frac{L_N}{\lambda_{N+1}}, \quad \forall t \geq T, \quad \forall N \in \mathbb{N} \tag{2.13}$$

$$\|Q_N \mathbf{u}(t)\|_{H^1} \leq C_1 \frac{L_N}{\lambda_{N+1}^{1/2}}, \quad \forall t \geq T, \quad \forall N \in \mathbb{N} \tag{2.14}$$

where

$$L_N = \left[1 + \log\left(\frac{\lambda_N}{\lambda_1}\right)\right]^{1/2} \tag{2.15}$$

$$C_0 = c\left(\frac{|Q_N \mathbf{g}|_{L^2} + M_1^2}{\nu}\right) \tag{2.16}$$

$$C_1 = c\left(\frac{|Q_N \mathbf{g}|_{L^2} + M_1^2}{\nu} + \frac{M_0 M_1^2}{\nu^2}\right) \tag{2.17}$$

and M_0 *and* M_1 *are as given in* (2.12).

The next theorem was proved in [21, Theorem 2.1], and it provides uniform-in-time estimates, in the H and V norms, for the distance between a solution $\mathbf{u}$ of (1.1) and its vertical projection on the graph of the mapping Φ_1, given in (1.5).

THEOREM 2.3. *Let* $\mathbf{u}_0 \in H$ *and let* $\mathbf{u}$ *be a solution of* (1.1) *satisfying* $\mathbf{u}(t_0) = \mathbf{u}_0$. *Then there exists* $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, |\mathbf{u}_0|_{L^2}) \geq t_0$ *such that*

$$|\Phi_1(P_N\mathbf{u}(t)) - Q_N\mathbf{u}(t)|_{L^2} \leq C\frac{L_N}{\lambda_{N+1}^{3/2}}, \quad \forall t \geq T, \quad \forall N \in \mathbb{N}, \tag{2.18}$$

and

$$\|\Phi_1(P_N\mathbf{u}(t)) - Q_N\mathbf{u}(t)\|_{H^1} \leq C\frac{L_N}{\lambda_{N+1}}, \quad \forall t \geq T, \quad \forall N \in \mathbb{N}, \tag{2.19}$$

where C *is a constant depending on* ν, λ_1 *and* $|\mathbf{g}|_{L^2}$ *but independent of* N.

Remark 2.4. In the results of section 3, we will assume that the reference solution of (1.1) has evolved long enough so that the uniform bounds from Proposition 2.1, Theorem 2.2, and Theorem 2.3 are always valid; i.e., for simplicity, we assume that $T = t_0$. Notice that, in particular, the uniform bounds from Proposition 2.1, Theorem 2.2, and Theorem 2.3 are valid for any trajectory $\mathbf{u} = \mathbf{u}(t)$ lying in the global attractor of (1.1) for every $t \in \mathbb{R}$.

3. Main results.

The purpose of this section is to establish analytical estimates of the error that occurs when using the PPGM applied to the data assimilation algorithm (1.2) in order to obtain an approximation of the reference solution $\mathbf{u}$, which satisfies the 2D Navier–Stokes equations (1.1). This means we want to establish an estimate of the difference $[(\mathbf{v}_N + \Phi_1(\mathbf{v}_N)) - \mathbf{u}]$ in some appropriate norm, where $\mathbf{v}_N$ denotes the Galerkin approximation of $\mathbf{v}$, the solution of (1.2), in P_NH. This is done here for the norms in the spaces H and V.

We start by giving some of the main ideas behind our results. From now on, we reserve the letter $N \in \mathbb{N}$ for the number of modes in the Galerkin approximation of (1.2), and we adopt the following notation for the low and high modes of the reference solution $\mathbf{u}$: $\mathbf{p} = P_N\mathbf{u}$ and $\mathbf{q} = Q_N\mathbf{u}$, respectively. Moreover, we assume that $\mathbf{u}$ satisfies the bounds from (2.12)–(2.19) for every $t \geq t_0$.

First, we rewrite the error in implementing the PPGM as

$$(\mathbf{v}_N + \Phi_1(\mathbf{v}_N)) - \mathbf{u} = (\mathbf{v}_N - \mathbf{p}) + (\Phi_1(\mathbf{p}) - \mathbf{q}) + (\Phi_1(\mathbf{v}_N) - \Phi_1(\mathbf{p})). \tag{3.1}$$

Theorem 2.3 provides estimates, in the L^2 and H^1 norms, of the second term in the right-hand side of (3.1). Moreover, it is not difficult to see that the restriction of Φ_1 to the set $P_NB_V(R)$, for any $R > 0$, is a Lipschitz continuous mapping with respect to the norms in both H and V (see, e.g., [13, Appendix]). More specifically, we have

$$|\Phi_1(\mathbf{p}_1) - \Phi_1(\mathbf{p}_2)|_{L^2} \leq l|\mathbf{p}_1 - \mathbf{p}_2|_{L^2} \quad \forall \mathbf{p}_1, \mathbf{p}_2 \in P_NB_V(R) \tag{3.2}$$

and

$$\|\Phi_1(\mathbf{p}_1) - \Phi_1(\mathbf{p}_2)\|_{H^1} \leq l\|\mathbf{p}_1 - \mathbf{p}_2\|_{H^1} \quad \forall \mathbf{p}_1, \mathbf{p}_2 \in P_NB_V(R), \tag{3.3}$$

where $l = C\lambda_{N+1}^{-1/4}$, with C being a constant depending on ν, λ_1 and R.

It follows from Propositions 3.2 and 3.9 below that, given a solution $\mathbf{u}$ of (1.1) satisfying (2.12)–(2.14), for every $t \geq t_0$, and given $\mathbf{v}_0 \in B_V(M_1)$, under suitable conditions on the parameters β and h, the solution $\mathbf{v}_N$ of (1.2), with $\mathbf{v}_N(t_0) = P_N\mathbf{v}_0$, satisfies $\mathbf{v}_N(t) \in B_V(3M_1)$ for all $t \geq t_0$. Thus, using (2.18), (2.19), and (3.2)–(3.3) with $R = 3M_1$, we obtain from (3.1) that, for every $t \geq t_0$,

$$|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)|_{L^2} \leq (1+l)|\mathbf{v}_N(t) - \mathbf{p}(t)|_{L^2} + C\frac{L_N}{\lambda_{N+1}^{3/2}} \tag{3.4}$$

and

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{H^1} \leq (1+l)\|\mathbf{v}_N(t) - \mathbf{p}(t)\|_{H^1} + C\frac{L_N}{\lambda_{N+1}}. \tag{3.5}$$

Moreover, we also have

$$\|\mathbf{v}_N(t) - \mathbf{p}(t)\|_{H^1} \leq \lambda_N^{1/2}|\mathbf{v}_N(t) - \mathbf{p}(t)|_{L^2}. \tag{3.6}$$

Thus, using also (3.6), we see from (3.4) and (3.5) that it suffices to obtain an estimate of $|\mathbf{v}_N(t) - \mathbf{p}(t)|_{L^2}$ in order to achieve our goal.

Applying P_N to (2.2), we see that $\mathbf{p} = P_N\mathbf{u}$ satisfies the equation

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} + \nu A\mathbf{p} + P_N B(\mathbf{p},\mathbf{p}) - P_N\mathbf{g} = -P_N G, \tag{3.7}$$

where

$$\begin{aligned} G(t) &= B(\mathbf{u}(t),\mathbf{u}(t)) - B(\mathbf{p}(t),\mathbf{p}(t)) \\ &= B(\mathbf{p}(t),\mathbf{q}(t)) + B(\mathbf{q}(t),\mathbf{p}(t)) + B(\mathbf{q}(t),\mathbf{q}(t)). \end{aligned} \tag{3.8}$$

Now, denoting $\mathbf{w} = \mathbf{v}_N - \mathbf{p}$ and taking the difference between (3.7) and (1.4), we obtain that

$$\begin{aligned} \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \nu A\mathbf{w} + \beta\mathbf{w} + P_N[B(\mathbf{v}_N,\mathbf{v}_N) - B(\mathbf{p},\mathbf{p})] &= P_N G - \beta P_N P_\sigma[I_h(\mathbf{w}) - \mathbf{w}] \\ &\quad + \beta P_N P_\sigma I_h(\mathbf{q}). \end{aligned} \tag{3.9}$$

The terms $\nu A\mathbf{w}$ and $\beta\mathbf{w}$ represent the dissipative terms in (3.9), which act on stabilizing $\mathbf{w}$. The term $A\mathbf{w}$ has a stronger effect than $\beta\mathbf{w}$ on the high modes of $\mathbf{w}$ for small values of ν, while $\beta\mathbf{w}$ has a stronger effect than $\nu A\mathbf{w}$ on the low modes of $\mathbf{w}$.

The estimate of $|\mathbf{w}|_{L^2}$ follows by applying Duhamel's (variation of constants) formula to (3.9) and by using the smoothing effect of the operator $\mathrm{e}^{-(s-t)(\nu AP_N + \beta P_N)}$, with the finite-dimensionality of the operator P_N also playing a crucial role. Moreover, the estimates make use of suitable properties of the interpolant operator I_h.

We consider two types of interpolant operators I_h, treated in two different sections. In the first one, section 3.1, we consider I_h as a low Fourier modes projector, i.e., $I_h = P_K$, $K \in \mathbb{N}$. In this case, we notice that we can commute P_N with $I_h = P_K$, and thus the last term on the right-hand side of (3.9) is zero, which simplifies the analysis.

In section 3.2, we consider a more general class of interpolant operators, satisfying suitable properties (see properties (P1)–(P3) in section 3.2, below), which are, in particular, satisfied by the example of a low Fourier modes projector considered in section 3.1. Another particular example of such class of interpolant operators is given by local averages over finite volume elements, which is illustrated in the Appendix in the case of periodic boundary conditions. In this latter example, this approach can be viewed as a hybrid method in the sense that observations are acquired through a finite elements method, while the approximate model is numerically solved through a spectral method, the PPGM.

The proof of the estimate for $|\mathbf{v}_N - \mathbf{p}|_{L^2}$, in both cases, follows similar ideas to the proof given in [28, Theorem 2], where, for a given initial condition $\mathbf{u}(t_0) = \mathbf{u}_0$, an estimate was obtained for $|\mathbf{u}_N - \mathbf{p}|_{L^2}$, with $\mathbf{u}_N$ being the Galerkin approximation of $\mathbf{u}$ satisfying $\mathbf{u}_N(t_0) = P_N\mathbf{u}_0$. We remark, however, that an advantage of our result is that the estimate for $|\mathbf{v}_N - \mathbf{p}|_{L^2}$ is uniform in time (see Theorems 3.5 and 3.10), while the estimate for $|\mathbf{u}_N - \mathbf{p}|_{L^2}$ given in [28, Theorem 2] grows exponentially in time. This important difference is justified by the presence of the additional dissipative term $\beta\mathbf{w}$ in (3.9), which helps to stabilize the large scales of $\mathbf{w}$ when the parameter β is suitably chosen. More specifically, β needs to be chosen large enough in order to stabilize the large spatial scales of $\mathbf{w}$ but not too large so as not to destabilize the small spatial scales of $\mathbf{w}$ as well, which are dissipated by $\nu A\mathbf{w}$, for small values of ν. For this reason, we need, roughly, $\beta \leq c\nu/h^2$.

Using the previous ideas, we prove in Theorems 3.5 and 3.10 below that, for sufficiently large t,

$$|\mathbf{v}_N(t) - \mathbf{p}(t)|_{L^2} \leq O(L_N^4 \lambda_{N+1}^{-3/2}) \tag{3.10}$$

in the case of an interpolant operator given by a low Fourier modes projector and

$$|\mathbf{v}_N(t) - \mathbf{p}(t)|_{L^2} \leq O(L_N \lambda_{N+1}^{-5/4}) \tag{3.11}$$

in the general interpolant operator case.

Thus, from (3.4) and (3.5), it follows that, for t large enough,

$$|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)|_{L^2} \leq O(L_N^4 \lambda_{N+1}^{-3/2}) \tag{3.12}$$

and

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{H^1} \leq O(L_N^4 \lambda_{N+1}^{-1}) \tag{3.13}$$

in the case of an interpolant operator given by a low Fourier modes projector (cf. Theorem 3.8) and

$$|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)|_{L^2} \leq O(L_N \lambda_{N+1}^{-5/4}) \tag{3.14}$$

and

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{H^1} \leq O(L_N \lambda_{N+1}^{-3/4}) \tag{3.15}$$

in the general interpolant operator case (cf. Theorem 3.12 below).

On the other hand, from (2.13), (2.14), (3.6), (3.10), and (3.11), we obtain that the error between the Galerkin approximation $\mathbf{v}_N$ of $\mathbf{v}$ and the reference solution $\mathbf{u}$ satisfies, for t large enough,

$$|\mathbf{v}_N(t) - \mathbf{u}(t)|_{L^2} \leq |\mathbf{v}_N(t) - \mathbf{p}(t)|_{L^2} + |\mathbf{q}(t)|_{L^2} \leq O(L_N \lambda_{N+1}^{-1}) \tag{3.16}$$

$$\|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{H^1} \leq \|\mathbf{v}_N(t) - \mathbf{p}(t)\|_{H^1} + \|\mathbf{q}(t)\|_{H^1} \leq O(L_N \lambda_{N+1}^{-1/2}) \tag{3.17}$$

in both cases of interpolant operators (cf. Corollaries 3.7 and 3.11 below).

Comparing (3.12) and (3.14) with (3.16) and (3.13) and (3.15) with (3.17), we see that, as mentioned in section 1, the PPGM indeed yields a better convergence rate than the standard Galerkin method. Notably, this improved rate is achieved due to essentially three facts: first, by using the Lipschitz property of Φ_1 (cf. (3.2), (3.3)); second, by exploring the fact that the error in the low modes, $|\mathbf{v}_N - \mathbf{p}|_{L^2}$, is much smaller than the error committed in the high modes in the Galerkin approximation, $|\mathbf{q}|_{L^2}$ (cf. (3.10), (3.11), and (2.13)); and, finally, by complementing the finite-dimensional approximation $\mathbf{v}_N \in P_N H$ with a suitable approximation of the high modes, given by $\Phi_1(\mathbf{v}_N) \in Q_N H$, which, due to the previous two facts, yields a better approximation to $\mathbf{q}$ than 0.

Remark 3.1. We notice that the convergence rates with respect to N in (3.14)–(3.15), obtained for the error committed when implementing the PPGM to (1.2) in the general interpolant operator case, is not as good as the rate in (3.12)–(3.13) for the case of an interpolant operator given by a low Fourier modes projector. In general terms, as pointed out before, this is due to the fact that the former case concerns a hybrid method, where the observations are acquired through, e.g., a finite elements method, while the approximate model (1.2) is discretized in space through a spectral method, the PPGM. On the mathematical side, this is represented by the possible lack of commutativity between the operators $P_\sigma I_h$ and A, an issue that does not occur in the case of an interpolant operator given by a low Fourier modes projector and which introduces additional error to the estimates.

3.1. The case of an interpolant operator given by a low Fourier modes projector. We consider an interpolant operator given by the orthogonal projection on low modes of the Fourier domain, i.e., $I_h = P_K$ for some $K \in \mathbb{N}$. The data assimilation algorithm (1.3) is given in this particular case by

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \nu A\mathbf{v} + B(\mathbf{v},\mathbf{v}) = \mathbf{g} - \beta P_K(\mathbf{v} - \mathbf{u}). \tag{3.18}$$

For every $N \in \mathbb{N}$ with $N \geq K$, we consider the Galerkin approximation system of (3.18) in the space $P_N H$, given by

$$\frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}t} + \nu A\mathbf{v}_N + P_N B(\mathbf{v}_N,\mathbf{v}_N) = P_N\mathbf{g} - \beta P_K(\mathbf{v}_N - \mathbf{p}), \tag{3.19}$$

with the initial condition $\mathbf{v}_N(t_0) = P_N\mathbf{v}_0$, where $\mathbf{v}_0$ is chosen in a suitable space but arbitrarily. We assume either periodic or no-slip Dirichlet boundary conditions.

The condition $N \geq K$ is assumed here for simplicity purposes. Nevertheless, it is a natural assumption since one would expect to have the resolution of the numerical method to be greater than or equal to the resolution associated to the observations.

The following result provides a first uniform in time bound of the finite-dimensional difference $\mathbf{v}_N - \mathbf{p}$ in the H^1 norm under suitable conditions on β and K. Since we assume that the reference solution $\mathbf{u}$ satisfies the bounds from (2.12)–(2.14), for every $t \geq t_0$, we also have in particular that $\mathbf{p}$ is uniformly bounded in V. Thus, as a consequence of the following proposition, we obtain that $\mathbf{v}_N$ is also uniformly bounded in V, provided β and K satisfy the appropriate conditions.

In the statement below, we consider an auxiliary parameter $m \in \mathbb{N}$ that is used for one of the lower bounds needed for β. More specifically, we choose β such that, in particular, $\beta \geq \nu\lambda_m$. This auxiliary parameter plays a more important role in the proof of Theorem 3.5 below, but we also use it here in order to be consistent.

PROPOSITION 3.2. *Let* $\mathbf{u}$ *be a solution of* (1.1) *satisfying* (2.12)–(2.14) *for every* $t \geq t_0$. *Let* $\mathbf{v}_0 \in B_V(M_1)$, *with* M_1 *as in* (2.12). *For every* $N \in \mathbb{N}$, *let* $\mathbf{v}_N$ *be the unique solution of* (3.19) *satisfying* $\mathbf{v}_N(t_0) = P_N\mathbf{v}_0$. *Consider* $m \in \mathbb{N}$ *large enough such that*

$$\lambda_m \geq \max\left\{\frac{\lambda_1 \mathrm{e}}{2}, c\frac{C_1}{\nu}L_m^2, c\left(\frac{C_1^2}{\nu M_1}\right)^{2/3} L_m^2\right\}. \tag{3.20}$$

If $\beta > 0$ *and* $K \in \mathbb{N}$ *are large enough such that*

$$\beta \geq \max\left\{\nu\lambda_m, c\frac{M_1^2}{\nu}\left[1 + \log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right)\right]\right\} \tag{3.21}$$

and

$$\lambda_{K+1} \geq \frac{2\beta}{\nu}, \tag{3.22}$$

then, for every $N \geq K$,

$$\sup_{t\geq t_0} \|\mathbf{v}_N(t) - \mathbf{p}(t)\|_{H^1} \leq 2M_1. \tag{3.23}$$

Proof. Projecting (1.1) onto $P_N H$, we have

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} + \nu A\mathbf{p} + P_N B(\mathbf{u}, \mathbf{u}) = P_N\mathbf{g}. \tag{3.24}$$

Denote $\mathbf{w} = \mathbf{v}_N - \mathbf{p}$. Subtracting (3.24) from (3.19), we obtain that

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \nu A\mathbf{w} + P_N[B(\mathbf{v}_N, \mathbf{v}_N) - B(\mathbf{u}, \mathbf{u})] = -\beta P_K\mathbf{w}. \tag{3.25}$$

Notice that

$$\begin{aligned} B(\mathbf{v}_N, \mathbf{v}_N) - B(\mathbf{u}, \mathbf{u}) &= B(\mathbf{v}_N, \mathbf{v}_N) - B(\mathbf{p} + \mathbf{q}, \mathbf{p} + \mathbf{q}) \\ &= B(\mathbf{v}_N, \mathbf{v}_N) - B(\mathbf{p}, \mathbf{p}) - B(\mathbf{p}, \mathbf{q}) - B(\mathbf{q}, \mathbf{p}) - B(\mathbf{q}, \mathbf{q}) \\ &= B(\mathbf{w}, \mathbf{p}) + B(\mathbf{p}, \mathbf{w}) + B(\mathbf{w}, \mathbf{w}) - B(\mathbf{p}, \mathbf{q}) - B(\mathbf{q}, \mathbf{p}) - B(\mathbf{q}, \mathbf{q}). \end{aligned} \tag{3.26}$$

Thus, from (3.25) and (3.26), we have

$$\begin{aligned} \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \nu A\mathbf{w} &= -\beta P_K\mathbf{w} \\ &\quad - P_N[B(\mathbf{w}, \mathbf{p}) + B(\mathbf{p}, \mathbf{w}) + B(\mathbf{w}, \mathbf{w}) - B(\mathbf{p}, \mathbf{q}) - B(\mathbf{q}, \mathbf{p}) - B(\mathbf{q}, \mathbf{q})]. \end{aligned} \tag{3.27}$$

Taking the inner product in L^2 of (3.27) with $A\mathbf{w}$ yields

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{H^1}^2 + \nu|A\mathbf{w}|_{L^2}^2 &= -\beta\|P_K\mathbf{w}\|_{H^1}^2 - (B(\mathbf{w}, \mathbf{p}), A\mathbf{w})_{L^2} - (B(\mathbf{p}, \mathbf{w}), A\mathbf{w})_{L^2} \\ &\quad - (B(\mathbf{w}, \mathbf{w}), A\mathbf{w})_{L^2} + (B(\mathbf{p}, \mathbf{q}), A\mathbf{w})_{L^2} + (B(\mathbf{q}, \mathbf{p}), A\mathbf{w})_{L^2} + (B(\mathbf{q}, \mathbf{q}), A\mathbf{w})_{L^2}. \end{aligned} \tag{3.28}$$

Now we estimate the terms in the right-hand side of (3.28).

Using (2.6) and (2.12), we obtain that

$$|(B(\mathbf{w},\mathbf{p}),A\mathbf{w})_{L^2}| \leq c_B M_1 \|\mathbf{w}\|_{H^1} |A\mathbf{w}|_{L^2} \left[1 + \log\left(\frac{|A\mathbf{w}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}}\right)\right]^{1/2} \tag{3.29}$$

$$|(B(\mathbf{w},\mathbf{w}),A\mathbf{w})_{L^2}| \leq c_B \|\mathbf{w}\|_{H^1}^2 |A\mathbf{w}|_{L^2} \left[1 + \log\left(\frac{|A\mathbf{w}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}}\right)\right]^{1/2}. \tag{3.30}$$

Thanks to (2.6), (2.12), and (2.14), we have

$$\begin{aligned} |(B(\mathbf{p},\mathbf{q}),A\mathbf{w})_{L^2}| &\leq c_B \|\mathbf{p}\|_{H^1}\|\mathbf{q}\|_{H^1} |A\mathbf{w}|_{L^2} \left[1 + \log\left(\frac{|A\mathbf{p}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{p}\|_{H^1}}\right)\right]^{1/2} \\ &\leq c_B M_1 C_1 \frac{L_N^2}{\lambda_{N+1}^{1/2}} |A\mathbf{w}|_{L^2} \leq \frac{\nu}{12}|A\mathbf{w}|_{L^2}^2 + c\frac{C_1^2}{\nu}\frac{L_N^4}{\lambda_{N+1}} M_1^2. \end{aligned} \tag{3.31}$$

From (2.7) and (2.14), it follows that

$$\begin{aligned} |(B(\mathbf{q},\mathbf{q}),A\mathbf{w})_{L^2}| &\leq c_T \|\mathbf{q}\|_{H^1}^2 |A\mathbf{w}|_{L^2} \left[1 + \log\left(\frac{|A^{3/2}\mathbf{w}|_{L^2}}{\lambda_1^{1/2}|A\mathbf{w}|_{L^2}}\right)\right]^{1/2} \\ &\leq c_T C_1^2 \frac{L_N^3}{\lambda_{N+1}} |A\mathbf{w}|_{L^2} \leq \frac{\nu}{12}|A\mathbf{w}|_{L^2}^2 + c\frac{C_1^4}{\nu}\frac{L_N^6}{\lambda_{N+1}^2}. \end{aligned} \tag{3.32}$$

From (2.8) and (2.12), we obtain that

$$|(B(\mathbf{p},\mathbf{w}),A\mathbf{w})_{L^2}| \leq c_T M_1 \|\mathbf{w}\|_{H^1} |A\mathbf{w}|_{L^2} \left[1 + \log\left(\frac{|A\mathbf{w}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}}\right)\right]^{1/2}. \tag{3.33}$$

Moreover, (2.8) and (2.14) imply

$$\begin{aligned} |(B(\mathbf{q},\mathbf{p}),A\mathbf{w})_{L^2}| &\leq c_T \|\mathbf{q}\|_{H^1}\|\mathbf{p}\|_{H^1} |A\mathbf{w}|_{L^2} \left[1 + \log\left(\frac{|A\mathbf{p}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{p}\|_{H^1}}\right)\right]^{1/2} \\ &\leq c_T C_1 \frac{L_N^2}{\lambda_{N+1}^{1/2}} M_1 |A\mathbf{w}|_{L^2} \leq \frac{\nu}{12}|A\mathbf{w}|_{L^2}^2 + c_T^2\frac{C_1^2}{\nu}\frac{L_N^4}{\lambda_{N+1}} M_1^2. \end{aligned} \tag{3.34}$$

Also, observe that

$$\begin{aligned} -\beta\|P_K\mathbf{w}\|_{H^1}^2 &= -\beta\|\mathbf{w}\|_{H^1}^2 + \beta\|Q_K\mathbf{w}\|_{H^1}^2 \\ &\leq -\beta\|\mathbf{w}\|_{H^1}^2 + \frac{\beta}{\lambda_{K+1}}|A\mathbf{w}|_{L^2}^2 \leq -\beta\|\mathbf{w}\|_{H^1}^2 + \frac{\nu}{2}|A\mathbf{w}|_{L^2}^2, \end{aligned} \tag{3.35}$$

where in the last inequality we used hypothesis (3.22).

Plugging estimates (3.29)–(3.35) into (3.28), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{H^1}^2 + \frac{\nu}{2}|A\mathbf{w}|_{L^2}^2 \le -\beta\|\mathbf{w}\|_{H^1}^2 + cM_1\|\mathbf{w}\|_{H^1}|A\mathbf{w}|_{L^2}\left[1+\log\left(\frac{|A\mathbf{w}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}}\right)\right]^{1/2} + c\|\mathbf{w}\|_{H^1}^2|A\mathbf{w}|_{L^2}\left[1+\log\left(\frac{|A\mathbf{w}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}}\right)\right]^{1/2} + c\frac{C_1^4}{\nu}\frac{L_N^6}{\lambda_{N+1}^2} + c\frac{C_1^2}{\nu}\frac{L_N^4}{\lambda_{N+1}}M_1^2. \tag{3.36}$$

Since $\mathbf{v}_N \in \mathcal{C}([t_0,\infty);V)$ [5, Theorem 5] and

$$\|\mathbf{w}(t_0)\|_{H^1} \le \|P_N\mathbf{v}_0\|_{H^1} + \|\mathbf{p}(t_0)\|_{H^1} \le 2M_1,$$

then there exists $\tau \in (t_0,\infty)$ such that

$$\|\mathbf{w}(t)\|_{H^1} \le 3M_1, \quad \forall t \in [t_0,\tau].$$

Define

$$\tilde{t} = \sup\left\{\tau \in (t_0,\infty) : \max_{t\in[t_0,\tau]}\|\mathbf{w}(t)\|_{H^1} \le 3M_1\right\}. \tag{3.37}$$

Suppose that $\tilde{t} < \infty$.

Then, from (3.36), we obtain that, for all $t \in [t_0,\tilde{t}]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{H^1}^2 + \frac{\nu}{2}|A\mathbf{w}|_{L^2}^2 \le -\beta\|\mathbf{w}\|_{H^1}^2 + cM_1\|\mathbf{w}\|_{H^1}|A\mathbf{w}|_{L^2}\left[1+\log\left(\frac{|A\mathbf{w}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}}\right)\right]^{1/2} + c\frac{C_1^4}{\nu}\frac{L_N^6}{\lambda_{N+1}^2} + c\frac{C_1^2}{\nu}\frac{L_N^4}{\lambda_{N+1}}M_1^2. \tag{3.38}$$

Observe that

$$\frac{\nu}{4}|A\mathbf{w}|_{L^2}^2 - cM_1\|\mathbf{w}\|_{H^1}|A\mathbf{w}|_{L^2}\left(1+\log\left(\frac{|A\mathbf{w}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}}\right)\right)^{1/2} + \frac{\beta}{2}\|\mathbf{w}\|_{H^1}^2 = \frac{\nu\lambda_1}{4}\|\mathbf{w}\|_{H^1}^2\left[\frac{|A\mathbf{w}|_{L^2}^2}{\lambda_1\|\mathbf{w}\|_{H^1}^2} - c\frac{M_1}{\nu\lambda_1^{1/2}}\frac{|A\mathbf{w}|_{L^2}}{\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}}\left(1+\log\left(\frac{|A\mathbf{w}|_{L^2}^2}{\lambda_1\|\mathbf{w}\|_{H^1}^2}\right)\right)^{1/2} + \frac{2\beta}{\nu\lambda_1}\right]. \tag{3.39}$$

Define

$$\phi(r) = r^2 - \rho r(1+\log(r^2))^{1/2} + M, \quad r \ge 1, \tag{3.40}$$

where

$$\rho = c\frac{M_1}{\nu\lambda_1^{1/2}}, \quad M = \frac{2\beta}{\nu\lambda_1}. \tag{3.41}$$

Notice that

$$\phi(r) = \frac{r(\widetilde{\phi}(r^2)+M) + \rho(1+\log(r^2))^{1/2}}{r+\rho(1+\log(r^2))^{1/2}}, \tag{3.42}$$

where $\widetilde{\phi}(r) = r - \rho^2(1 + \log r)$.

One easily verifies that

$$\min_{r \geq 1} \widetilde{\phi}(r) \geq -\rho^2 \log(\rho^2). \tag{3.43}$$

Thus, from (3.42) and (3.43), it follows that if

$$M \geq \rho^2 \log(\rho^2), \tag{3.44}$$

then

$$\phi(r) \geq 0 \quad \forall r \geq 1. \tag{3.45}$$

Now, by the definition of ρ and M in (3.41), we see that (3.44) follows from hypothesis (3.21) on β.

Using (3.45) with $r = |A\mathbf{w}|_{L^2}/(\lambda_1^{1/2}\|\mathbf{w}\|_{H^1}) \geq 1$, we conclude that the right-hand side of (3.39) is nonnegative. Thus, from (3.38), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{H^1}^2 + \frac{\nu}{4}|A\mathbf{w}|_{L^2}^2 \leq -\frac{\beta}{2}\|\mathbf{w}\|_{H^1}^2 + c\frac{C_1^4}{\nu}\frac{L_N^6}{\lambda_{N+1}^2} + c\frac{C_1^2}{\nu}\frac{L_N^4}{\lambda_{N+1}}M_1^2. \tag{3.46}$$

Ignoring the second term on the left-hand side of (3.46) and integrating from t_0 to $t \in [t_0, \tilde{t}]$, we obtain that

$$\begin{aligned} \|\mathbf{w}(t)\|_{H^1}^2 \leq {} & \|\mathbf{w}(t_0)\|_{H^1}^2 \,\mathrm{e}^{-\frac{\beta}{2}(t-t_0)} \\ & + \frac{c}{\beta}\left[\frac{C_1^4}{\nu}\frac{L_N^6}{\lambda_{N+1}^2} + \frac{C_1^2}{\nu}\frac{L_N^4}{\lambda_{N+1}}M_1^2\right](1 - \mathrm{e}^{-\frac{\beta}{2}(t-t_0)}). \end{aligned} \tag{3.47}$$

Notice that the functions $f_1(x) = (1 + \log x)^3/x^2$ and $f_2(x) = (1 + \log x)^2/x$ are both decreasing for $x \geq \mathrm{e}$. Since $N \geq K$ and, by hypotheses (3.20), (3.21), and (3.22), we have

$$\frac{\lambda_{N+1}}{\lambda_1} \geq \frac{\lambda_{K+1}}{\lambda_1} \geq \frac{2\beta}{\nu\lambda_1} \geq \frac{2\lambda_m}{\lambda_1} \geq \mathrm{e},$$

it then follows that

$$\frac{L_N^6}{\lambda_{N+1}^2} \leq c\frac{L_m^6}{\lambda_m^2} \text{ and } \frac{L_N^4}{\lambda_{N+1}} \leq c\frac{L_m^4}{\lambda_m}. \tag{3.48}$$

Using the estimates from (3.48) into (3.47) and using hypothesis (3.20) with a suitable absolute constant c, we obtain that

$$\|\mathbf{w}(t)\|_{H^1}^2 \leq \|\mathbf{w}(t_0)\|_{H^1}^2 \,\mathrm{e}^{-\frac{\beta}{2}(t-t_0)} + 4M_1^2(1 - \mathrm{e}^{-\frac{\beta}{2}(t-t_0)}) \leq 4M_1^2, \quad \forall t \in [t_0, \tilde{t}]. \tag{3.49}$$

Thus,

$$\|\mathbf{w}(t)\|_{H^1} \leq 2M_1 \quad \forall t \in [t_0, \tilde{t}]. \tag{3.50}$$

In particular, $\|\mathbf{w}(\tilde{t})\|_{H^1} \leq 2M_1$, which, by the definition of $\tilde{t}$ and the fact that $\mathbf{w} \in \mathcal{C}([t_0, \infty); V)$, contradicts the assumption that $\tilde{t} < \infty$. Therefore, the above argument implies that $\|\mathbf{w}(t)\|_{H^1} \leq 2M_1$ for every $t \geq t_0$. ◻

Next, we present a technical lemma.

LEMMA 3.3. *Assume that* $y : [t_0, \infty) \to [0, \infty)$ *is a continuous function satisfying*

$$y(s) \le a\,\mathrm{e}^{-b(s-t)}\,y(t) + \gamma \sup_{t \le \tau \le s} y(\tau) + \varepsilon \quad \forall s \ge t \ge t_0 \tag{3.51}$$

with $\varepsilon \ge 0$, $a \ge 0$, $b > 0$ *and* $\gamma \in (0,1)$ *such that*

$$\theta = a\left(\mathrm{e}^{-\frac{b}{\nu\lambda_1}} + \frac{\gamma}{1-\gamma}\right) < 1. \tag{3.52}$$

Then

$$y(t) \le a\frac{\theta^{(t-t_0)\nu\lambda_1 - 1}}{1-\gamma}y(t_0) + \left(\frac{a}{(1-\theta)(1-\gamma)} + 1\right)\frac{\varepsilon}{1-\gamma} \quad \forall t \ge t_0. \tag{3.53}$$

Proof. Taking the sup on both sides of (3.51) over $s \in [t, t+(\nu\lambda_1)^{-1}]$, it follows that

$$\sup_{t \le s \le t+(\nu\lambda_1)^{-1}} y(s) \le a y(t) + \gamma \sup_{t \le \tau \le t+(\nu\lambda_1)^{-1}} y(\tau) + \varepsilon.$$

Thus,

$$\sup_{t \le \tau \le t+(\nu\lambda_1)^{-1}} y(\tau) \le \frac{a}{1-\gamma}y(t) + \frac{\varepsilon}{1-\gamma}. \tag{3.54}$$

Using (3.54) in (3.51) with $s = t + (\nu\lambda_1)^{-1}$, $t \ge t_0$, yields

$$y(t+(\nu\lambda_1)^{-1}) \le \theta y(t) + \frac{\varepsilon}{1-\gamma} \tag{3.55}$$

with θ as defined in (3.52).

For each $n \in \mathbb{N}$, let $t_n = t_0 + n(\nu\lambda_1)^{-1}$. Since (3.55) is valid for every $t \ge t_0$, in particular,

$$y(t_n) = y(t_{n-1} + (\nu\lambda_1)^{-1}) \le \theta y(t_{n-1}) + \frac{\varepsilon}{1-\gamma} \quad \forall n \in \mathbb{N}. \tag{3.56}$$

Hence, by induction, one has

$$y(t_n) \le \theta^n y(t_0) + \frac{\varepsilon}{(1-\theta)(1-\gamma)} \quad \forall n \in \mathbb{N}. \tag{3.57}$$

Using (3.57) in (3.54) with $t = t_n$, it follows that

$$\sup_{t_n \le s \le t_{n+1}} y(s) \le a\frac{\theta^n}{1-\gamma}y(t_0) + \left(\frac{a}{(1-\theta)(1-\gamma)} + 1\right)\frac{\varepsilon}{1-\gamma}. \tag{3.58}$$

Notice that, for every $t \in [t_n, t_{n+1}]$,

$$n = (t_{n+1} - t_0)\nu\lambda_1 - 1 \ge (t - t_0)\nu\lambda_1 - 1. \tag{3.59}$$

Since $\theta \in [0,1)$, by hypothesis (3.52), it then follows from (3.58) and (3.59) that, for every $t \in [t_n, t_{n+1}]$,

$$y(t) \le \sup_{t_n \le s \le t_{n+1}} y(s) \le a\frac{\theta^{(t-t_0)\nu\lambda_1 - 1}}{1-\gamma}y(t_0) + \left(\frac{a}{(1-\theta)(1-\gamma)} + 1\right)\frac{\varepsilon}{1-\gamma}. \tag{3.60}$$

Since (3.60) is valid for any $n \in \mathbb{N}$, (3.53) follows. ∎

The following proposition is a direct consequence of the result proved in [28, Lemma 1] (see also [55]).

Proposition 3.4. *Let* $\mathbf{u}$ *be a solution of* (1.1) *satisfying* (2.12)–(2.14) *for every* $t \geq t_0$. *Then the following inequalities hold*

$$|A^{-1}P_N B(\mathbf{p},\mathbf{q})|_{L^2},\ |A^{-1}P_N B(\mathbf{q},\mathbf{p})|_{L^2} \leq cM_1 L_N \|\mathbf{q}\|_{V'}, \tag{3.61}$$

$$|A^{-1}P_N B(\mathbf{q},\mathbf{q})|_{L^2} \leq cL_N |\mathbf{q}|_{L^2}^2, \tag{3.62}$$

with L_N *as defined in* (2.15).

Using the results of Lemma 3.3 and Propositions 3.2 and 3.4, we can now obtain a uniform-in-time estimate for $|\mathbf{v}_N(t)-\mathbf{p}(t)|_{L^2}$. The proof below follows similar ideas to the proof of [28, Theorem 2]. We use the notation $\|\cdot\|_{\mathcal{L}(X)}$ to denote the operator norm in the space $\mathcal{L}(X)$, the space of bounded linear operators on a Hilbert space X.

Theorem 3.5. *Let* $\mathbf{u}$ *be a solution of* (1.1) *satisfying* (2.12)–(2.14) *for every* $t \geq t_0$. *Let* $\mathbf{v}_0 \in B_V(M_1)$, *with* M_1 *as in* (2.12). *For every* $N \in \mathbb{N}$, *let* $\mathbf{v}_N$ *be the unique solution of* (3.19) *satisfying* $\mathbf{v}_N(t_0) = P_N\mathbf{v}_0$. *Fix* $\alpha \in (1/2, 1)$, *and consider* $m \in \mathbb{N}$ *large enough such that*

$$\lambda_m \geq \max\left\{\frac{\lambda_1 \mathrm{e}}{2}, c\frac{C_1}{\nu}L_m^2, c\left(\frac{C_1^2}{\nu M_1}\right)^{2/3} L_m^2, \left[cc_\alpha\left(1+\frac{\mathrm{e}^{-\alpha}}{1-\alpha}\right)\frac{|\Omega|^{\alpha-\frac{1}{2}}M_1}{\nu}\right]^{\frac{1}{1-\alpha}}\right\}, \tag{3.63}$$

where c_α *is the constant from* (2.9).

If $\beta > 0$ *and* $K \in \mathbb{N}$ *are large enough such that*

$$\beta \geq \max\left\{\nu\lambda_m, c\frac{M_1^2}{\nu}\left[1+\log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right)\right]\right\} \tag{3.64}$$

and

$$\lambda_{K+1} \geq \frac{2\beta}{\nu}, \tag{3.65}$$

then there exists $\theta = \theta(\beta) \in [0,1)$ *and a constant* $C = C(\nu, \lambda_1, |\mathbf{g}|_{L^2})$ *such that, for every* $N \geq K$,

$$|\mathbf{v}_N(t)-\mathbf{p}(t)|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1 - 1}|\mathbf{v}_N(t_0)-\mathbf{p}(t_0)|_{L^2} + C\frac{L_N^4}{\lambda_{N+1}^{3/2}}. \tag{3.66}$$

Proof. Denote $\mathbf{w} = \mathbf{v}_N - \mathbf{p}$. Subtracting (3.24) from (3.19) yields

$$\begin{aligned}\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \nu A\mathbf{w} &= -P_N[B(\mathbf{v}_N,\mathbf{v}_N) - B(\mathbf{u},\mathbf{u})] - \beta P_K\mathbf{w} \\ &= -P_N[B(\mathbf{v}_N,\mathbf{v}_N) - B(\mathbf{p},\mathbf{p})] + P_N G - \beta P_K \mathbf{w},\end{aligned} \tag{3.67}$$

where

$$G(t) = B(\mathbf{u}(t),\mathbf{u}(t)) - B(\mathbf{p}(t),\mathbf{p}(t)) \quad \forall t \geq t_0. \tag{3.68}$$

Using that $P_N\mathbf{w} = \mathbf{w}$, we can also rewrite (3.67) as

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + [\nu A P_N + \beta P_K]\mathbf{w} = -P_N[B(\mathbf{v}_N, \mathbf{v}_N) - B(\mathbf{p}, \mathbf{p})] + P_N G. \tag{3.69}$$

Using Duhamel's formula, it follows that, for every $s \geq t \geq t_0$,

$$\begin{aligned} |\mathbf{w}(s)|_{L^2} &\leq |\,\mathrm{e}^{-(s-t)(\nu A P_N + \beta P_K)}\,\mathbf{w}(t)|_{L^2} \\ &+ \int_t^s \left|\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\, P_N[B(\mathbf{v}_N(\tau), \mathbf{v}_N(\tau)) - B(\mathbf{p}(\tau), \mathbf{p}(\tau))]\right|_{L^2} \mathrm{d}\tau \\ &+ \int_t^s \left|\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\, P_N G(\tau)\right|_{L^2} \mathrm{d}\tau. \end{aligned} \tag{3.70}$$

We now estimate each term on the right-hand side of (3.70).

Notice that, for every $s \geq t \geq t_0$,

$$\begin{aligned} &|\,\mathrm{e}^{-(s-t)(\nu A P_N + \beta P_K)}\,\mathbf{w}(t)|_{L^2} \leq \\ &\leq (\|\,\mathrm{e}^{-(s-t)(\nu A P_K + \beta P_K)}\,\|_{\mathcal{L}(P_K H)} + \|\,\mathrm{e}^{-(s-t)\nu A P_N Q_K}\,\|_{\mathcal{L}(P_N Q_K H)})|\mathbf{w}(t)|_{L^2} \\ &= \left[\left(\max_{1\leq j\leq K} \mathrm{e}^{-(s-t)(\nu\lambda_j + \beta)}\right) + \left(\max_{K+1\leq j\leq N} \mathrm{e}^{-(s-t)\nu\lambda_j}\right)\right] |\mathbf{w}(t)|_{L^2} \\ &= \left(\mathrm{e}^{-(s-t)(\nu\lambda_1 + \beta)} + \mathrm{e}^{-(s-t)(\nu\lambda_{K+1})}\right) |\mathbf{w}(t)|_{L^2} \\ &\leq 2\,\mathrm{e}^{-(s-t)\beta}\,|\mathbf{w}(t)|_{L^2}, \end{aligned} \tag{3.71}$$

where in the last inequality we used that $\nu\lambda_{K+1} \geq 2\beta$ from hypothesis (3.65).

Using (2.9), we obtain that

$$\begin{aligned} &\left|\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\, P_N[B(\mathbf{v}_N(\tau), \mathbf{v}_N(\tau)) - B(\mathbf{p}(\tau), \mathbf{p}(\tau))]\right|_{L^2} \\ &= \frac{1}{\nu^\alpha}\left|\nu^\alpha A^\alpha\,\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\, A^{-\alpha} P_N[B(\mathbf{v}_N(\tau), \mathbf{v}_N(\tau)) - B(\mathbf{p}(\tau), \mathbf{p}(\tau))]\right|_{L^2} \\ &\leq c_\alpha \frac{|\Omega|^{\alpha - \frac{1}{2}}}{\nu^\alpha} \|\nu^\alpha A^\alpha\,\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\,\|_{\mathcal{L}(P_N H)} \|\mathbf{v}_N(\tau) + \mathbf{p}(\tau)\|_{H^1} |\mathbf{w}(\tau)|_{L^2}. \end{aligned} \tag{3.72}$$

By Proposition 3.2, we have that

$$\|\mathbf{v}_N(\tau) + \mathbf{p}(\tau)\|_{H^1} \leq 3M_1, \quad \forall \tau \geq t_0, \quad \forall N \geq K. \tag{3.73}$$

It then follows from (3.72) that

$$\begin{aligned} &\left|\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\, P_N[B(\mathbf{v}_N(\tau), \mathbf{v}_N(\tau)) - B(\mathbf{p}(\tau), \mathbf{p}(\tau))]\right|_{L^2} \\ &\leq 3c_\alpha \frac{|\Omega|^{\alpha - \frac{1}{2}} M_1}{\nu^\alpha} \|\nu^\alpha A^\alpha\,\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\,\|_{\mathcal{L}(P_N H)} |\mathbf{w}(\tau)|_{L^2}. \end{aligned} \tag{3.74}$$

Notice that, by using hypotheses (3.64) and (3.65), we have

$$\frac{\nu\lambda_{K+1}}{2} \geq \beta \geq \nu\lambda_m, \tag{3.75}$$

which implies in particular that $K \geq m$.

Now, we write

$$\mathrm{e}^{-(s-\tau)(\nu AP_N+\beta P_K)} = \mathrm{e}^{-(s-\tau)(\nu AP_m+\beta P_m)} + \mathrm{e}^{-(s-\tau)(\nu AP_KQ_m+\beta P_KQ_m)} + \mathrm{e}^{-(s-\tau)\nu AP_NQ_K}. \tag{3.76}$$

Therefore,

$$\begin{aligned}
&\int_t^s \left|\mathrm{e}^{-(s-\tau)(\nu AP_N+\beta P_K)} P_N[B(\mathbf{v}_N(\tau),\mathbf{v}_N(\tau)) - B(\mathbf{p}(\tau),\mathbf{p}(\tau))]\right|_{L^2} \mathrm{d}\tau \\
&\le 3c_\alpha \frac{|\Omega|^{\alpha-\frac{1}{2}}M_1}{\nu^\alpha}\left(\sup_{t\le\tau\le s}|\mathbf{w}(\tau)|_{L^2}\right)\int_t^s \|\nu^\alpha A^\alpha \mathrm{e}^{-(s-\tau)(\nu AP_N+\beta P_K)}\|_{\mathcal{L}(P_NH)}\mathrm{d}\tau \\
&\le 3c_\alpha \frac{|\Omega|^{\alpha-\frac{1}{2}}M_1}{\nu^\alpha}\left(\sup_{t\le\tau\le s}|\mathbf{w}(\tau)|_{L^2}\right)\left(\int_t^s \|\nu^\alpha A^\alpha \mathrm{e}^{-(\xi-t)(\nu AP_m+\beta P_m)}\|_{\mathcal{L}(P_mH)}\mathrm{d}\xi + \right. \\
&\qquad + \int_t^s \|\nu^\alpha A^\alpha \mathrm{e}^{-(\xi-t)(\nu AP_KQ_m+\beta P_KQ_m)}\|_{\mathcal{L}(P_KH)}\mathrm{d}\xi \\
&\qquad \left. + \int_t^s \|\nu^\alpha A^\alpha \mathrm{e}^{-(\xi-t)\nu AP_NQ_K}\|_{\mathcal{L}(P_NQ_KH)}\mathrm{d}\xi\right),
\end{aligned} \tag{3.77}$$

where in the second inequality we used (3.76) and applied the change of variables $\xi = s - \tau + t$.

Notice that

$$\begin{aligned}
\|\nu^\alpha A^\alpha \mathrm{e}^{-(\xi-t)(\nu AP_m+\beta P_m)}\|_{\mathcal{L}(P_mH)} &= \max_{1\le j\le m}(\nu\lambda_j)^\alpha \mathrm{e}^{-(\xi-t)(\nu\lambda_j+\beta)} \\
&\le \mathrm{e}^{-(\xi-t)\beta} \max_{\nu\lambda_1\le x\le\nu\lambda_m} x^\alpha \mathrm{e}^{-(\xi-t)x} \\
&= \mathrm{e}^{-(\xi-t)\beta}\cdot\begin{cases}
(\nu\lambda_m)^\alpha \mathrm{e}^{-(\xi-t)\nu\lambda_m}, & \text{if } \xi < t + \frac{\alpha}{\nu\lambda_m}, \\
\dfrac{\alpha^\alpha}{(\xi-t)^\alpha}\mathrm{e}^{-\alpha}, & \text{if } t + \frac{\alpha}{\nu\lambda_m} \le \xi \le t + \frac{\alpha}{\nu\lambda_1}, \\
(\nu\lambda_1)^\alpha \mathrm{e}^{-(\xi-t)\nu\lambda_1}, & \text{if } \xi > t + \frac{\alpha}{\nu\lambda_1}.
\end{cases}
\end{aligned} \tag{3.78}$$

Let us decompose $[t,s]$ as the union of the intervals

$$I_1 = \left[t, t+\frac{\alpha}{\nu\lambda_m}\right]\cap[t,s], \quad I_2 = \left[t+\frac{\alpha}{\nu\lambda_m}, t+\frac{\alpha}{\nu\lambda_1}\right]\cap[t,s], \quad I_3 = \left[t+\frac{\alpha}{\nu\lambda_1},\infty\right)\cap[t,s]. \tag{3.79}$$

We then have

$$\begin{aligned}
&\int_{I_1}\|\nu^\alpha A^\alpha \mathrm{e}^{-(\xi-t)(\nu AP_m+\beta P_m)}\|_{\mathcal{L}(P_mH)}\mathrm{d}\xi \le \\
&\quad\le \int_t^{t+\frac{\alpha}{\nu\lambda_m}}(\nu\lambda_m)^\alpha \mathrm{e}^{-(\xi-t)(\nu\lambda_m+\beta)}\,\mathrm{d}\xi = \frac{(\nu\lambda_m)^\alpha}{\nu\lambda_m+\beta}(1-\mathrm{e}^{-\alpha}\,\mathrm{e}^{-\frac{\alpha\beta}{\nu\lambda_m}})
\end{aligned} \tag{3.80}$$

$$
\begin{aligned}
(3.81)\quad & \int_{I_2} \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)(\nu A P_m + \beta P_m)}\|_{\mathcal{L}(P_m H)} \mathrm{d}\xi \le \\
& \le \int_{t+\frac{\alpha}{\nu\lambda_m}}^{t+\frac{\alpha}{\nu\lambda_1}} \frac{\alpha^\alpha}{(\xi-t)^\alpha}\,\mathrm{e}^{-\alpha}\,\mathrm{e}^{-(\xi-t)\beta}\,\mathrm{d}\xi \le (\nu\lambda_m)^\alpha\,\mathrm{e}^{-\alpha}\int_{t+\frac{\alpha}{\nu\lambda_m}}^{t+\frac{\alpha}{\nu\lambda_1}} \mathrm{e}^{-(\xi-t)\beta}\,\mathrm{d}\xi \\
& = \frac{(\nu\lambda_m)^\alpha\,\mathrm{e}^{-\alpha}}{\beta}(\mathrm{e}^{-\frac{\alpha\beta}{\nu\lambda_m}} - \mathrm{e}^{-\frac{\alpha\beta}{\nu\lambda_1}})
\end{aligned}
$$

and

$$
\begin{aligned}
(3.82)\quad & \int_{I_3} \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)(\nu A P_m + \beta P_m)}\|_{\mathcal{L}(P_m H)} \mathrm{d}\xi \le \int_{t+\frac{\alpha}{\nu\lambda_1}}^{\infty} (\nu\lambda_1)^\alpha\,\mathrm{e}^{-(\xi-t)(\nu\lambda_1+\beta)}\,\mathrm{d}\xi \\
& = \frac{(\nu\lambda_1)^\alpha}{\nu\lambda_1+\beta}\,\mathrm{e}^{-\alpha}\,\mathrm{e}^{-\frac{\alpha\beta}{\nu\lambda_1}}.
\end{aligned}
$$

Notice that the estimate in (3.82) is smaller than the absolute value of the negative term in (3.81). Thus, from (3.80)–(3.82), it follows that

$$
\begin{aligned}
(3.83)\quad & \int_t^s \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)(\nu A P_m + \beta P_m)}\|_{\mathcal{L}(P_m H)} \mathrm{d}\xi \le \\
& \le \frac{(\nu\lambda_m)^\alpha}{\nu\lambda_m+\beta}(1-\mathrm{e}^{-\alpha}\,\mathrm{e}^{-\frac{\alpha\beta}{\nu\lambda_m}}) + \frac{(\nu\lambda_m)^\alpha}{\beta}\,\mathrm{e}^{-\alpha}\,\mathrm{e}^{-\frac{\alpha\beta}{\nu\lambda_m}} \le \frac{(\nu\lambda_m)^\alpha}{\beta}.
\end{aligned}
$$

Now, similarly as in (3.78), we have that

$$
\begin{aligned}
(3.84)\quad & \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)(\nu A P_K Q_m + \beta P_K Q_m)}\|_{\mathcal{L}(P_K Q_m H)} \le \\
& \le \mathrm{e}^{-(\xi-t)\beta}\cdot \begin{cases} (\nu\lambda_K)^\alpha\,\mathrm{e}^{-(\xi-t)\nu\lambda_K}, & \text{if } \xi < t+\frac{\alpha}{\nu\lambda_K}, \\ \dfrac{\alpha^\alpha}{(\xi-t)^\alpha}\,\mathrm{e}^{-\alpha}, & \text{if } t+\frac{\alpha}{\nu\lambda_K} \le \xi \le t+\frac{\alpha}{\nu\lambda_{m+1}}, \\ (\nu\lambda_{m+1})^\alpha\,\mathrm{e}^{-(\xi-t)\nu\lambda_{m+1}}, & \text{if } \xi > t+\frac{\alpha}{\nu\lambda_{m+1}}. \end{cases}
\end{aligned}
$$

We decompose $[t,s]$ as the union of the intervals

$$
\begin{aligned}
(3.85)\quad & J_1 = \left[t, t+\frac{\alpha}{\nu\lambda_K}\right]\cap[t,s], \quad J_2 = \left[t+\frac{\alpha}{\nu\lambda_K}, t+\frac{\alpha}{\nu\lambda_{m+1}}\right]\cap[t,s], \\
& J_3 = \left[t+\frac{\alpha}{\nu\lambda_{m+1}}, \infty\right)\cap[t,s].
\end{aligned}
$$

We have

$$
\begin{aligned}
(3.86)\quad & \int_{J_1} \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)(\nu A P_K Q_m + \beta P_K Q_m)}\|_{\mathcal{L}(P_K Q_m H)} \mathrm{d}\xi \le \\
& \le \int_t^{t+\frac{\alpha}{\nu\lambda_K}} (\nu\lambda_K)^\alpha\,\mathrm{e}^{-(\xi-t)\nu\lambda_K}\,\mathrm{d}\xi = \frac{1-\mathrm{e}^{-\alpha}}{(\nu\lambda_K)^{1-\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
(3.87)\quad & \int_{J_2} \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)(\nu A P_K Q_m + \beta P_K Q_m)}\|_{\mathcal{L}(P_K Q_m H)} \mathrm{d}\xi \le \\
& \le \int_{t+\frac{\alpha}{\nu\lambda_K}}^{t+\frac{\alpha}{\nu\lambda_{m+1}}} \frac{\alpha^\alpha}{(\xi-t)^\alpha}\,\mathrm{e}^{-\alpha}\,\mathrm{d}\xi = \frac{\alpha\,\mathrm{e}^{-\alpha}}{1-\alpha}\left(\frac{1}{(\nu\lambda_{m+1})^{1-\alpha}} - \frac{1}{(\nu\lambda_K)^{1-\alpha}}\right)
\end{aligned}
$$

and

$$\int_{J_3} \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)(\nu A P_K Q_m + \beta P_K Q_m)}\|_{\mathcal{L}(P_K Q_m H)} \mathrm{d}\xi \leq \tag{3.88}$$
$$\leq \int_{t+\frac{\alpha}{\nu\lambda_{m+1}}}^{\infty} (\nu\lambda_{m+1})^\alpha \,\mathrm{e}^{-(\xi-t)\nu\lambda_{m+1}} \,\mathrm{d}\xi = \frac{\mathrm{e}^{-\alpha}}{(\nu\lambda_{m+1})^{1-\alpha}}.$$

Thus, summing up (3.86)–(3.88), we obtain

$$\int_t^s \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)(\nu A P_K Q_m + \beta P_K Q_m)}\|_{\mathcal{L}(P_K Q_m H)} \mathrm{d}\xi \leq \tag{3.89}$$
$$\leq \left(1 - \frac{\mathrm{e}^{-\alpha}}{1-\alpha}\right) \frac{1}{(\nu\lambda_K)^{1-\alpha}} + \frac{\mathrm{e}^{-\alpha}}{1-\alpha} \frac{1}{(\nu\lambda_{m+1})^{1-\alpha}} < \left(\frac{\mathrm{e}^{-\alpha}}{1-\alpha}\right) \frac{1}{(\nu\lambda_{m+1})^{1-\alpha}},$$

where in the last inequality we used the fact that

$$1 - \frac{\mathrm{e}^{-\alpha}}{1-\alpha} < 0 \quad \forall \alpha > 0. \tag{3.90}$$

Moreover, analogously to (3.84)–(3.89), one obtains that

$$\int_t^s \|\nu^\alpha A^\alpha \,\mathrm{e}^{-(\xi-t)\nu A P_N Q_K}\|_{\mathcal{L}(P_N Q_K H)} \mathrm{d}\xi < \left(\frac{\mathrm{e}^{-\alpha}}{1-\alpha}\right) \frac{1}{(\nu\lambda_{K+1})^{1-\alpha}}. \tag{3.91}$$

Now, let us estimate the third term on the right-hand side of (3.70).

Notice that

$$\int_t^s \left|\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)} P_N G(\tau)\right|_{L^2} \mathrm{d}\tau = \tag{3.92}$$
$$= \frac{1}{\nu} \int_t^s \left|\nu A \,\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)} A^{-1} P_N G(\tau)\right|_{L^2} \mathrm{d}\tau \leq$$
$$\leq \frac{1}{\nu} \int_t^s \|\nu A \,\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\|_{\mathcal{L}(P_N H)} |A^{-1} P_N G(\tau)|_{L^2} \mathrm{d}\tau.$$

By Proposition 3.4 and (2.13), it follows that

$$|A^{-1} P_N G|_{L^2} \leq c M_1 L_N \|\mathbf{q}\|_{V'} + c L_N |\mathbf{q}|_{L^2}^2 \leq c M_1 L_N \frac{|\mathbf{q}|_{L^2}}{\lambda_{N+1}^{1/2}} + c L_N |\mathbf{q}|_{L^2}^2 \leq c C_N, \tag{3.93}$$

where

$$C_N = C_0 \frac{L_N^2}{\lambda_{N+1}^{3/2}} \left(M_1 + C_0 \frac{L_N}{\lambda_{N+1}^{1/2}}\right), \tag{3.94}$$

with C_0 as defined in (2.16).

Now, similarly as in (3.78)–(3.83), one obtains that

$$\int_t^s \|\nu A \,\mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)}\|_{\mathcal{L}(P_N H)} \mathrm{d}\tau \tag{3.95}$$
$$= \int_t^s \|\nu A \,\mathrm{e}^{-(\xi-t)(\nu A P_N + \beta P_K)}\|_{\mathcal{L}(P_N H)} \mathrm{d}\xi$$
$$\leq \frac{\nu\lambda_N}{\nu\lambda_N + \beta}(1 - \mathrm{e}^{-1} \,\mathrm{e}^{-\frac{\beta}{\nu\lambda_N}}) + \log\left(\frac{\lambda_N}{\lambda_1}\right) \mathrm{e}^{-1} \,\mathrm{e}^{-\frac{\beta}{\nu\lambda_N}} + \frac{\nu\lambda_1}{\nu\lambda_1 + \beta} \mathrm{e}^{-1} \,\mathrm{e}^{-\frac{\beta}{\nu\lambda_1}}$$
$$\leq 1 + \log\left(\frac{\lambda_N}{\lambda_1}\right) = L_N^2.$$

Hence, from (3.92), (3.93), and (3.95), we have

$$\int_t^s \left| \mathrm{e}^{-(s-\tau)(\nu A P_N + \beta P_K)} \, P_N G(\tau) \right|_{L^2} \mathrm{d}\tau \leq c \frac{C_N L_N^2}{\nu}, \tag{3.96}$$

with C_N as defined in (3.94).

Now, plugging estimates (3.71), (3.77), (3.83), (3.89), (3.91), and (3.96) into (3.70), we obtain that, for all $s \geq t \geq t_0$,

$$\begin{aligned} |\mathbf{w}(s)|_{L^2} \leq\ & 2\,\mathrm{e}^{-(s-t)\beta}\,|\mathbf{w}(t)|_{L^2} + \\ & + 3c_\alpha \frac{|\Omega|^{\alpha-\frac{1}{2}} M_1}{\nu^\alpha} \left(\sup_{t \leq \tau \leq s} |\mathbf{w}(\tau)|_{L^2} \right) \left[\frac{(\nu\lambda_m)^\alpha}{\beta} + \frac{\mathrm{e}^{-\alpha}}{1-\alpha} \frac{1}{(\nu\lambda_{m+1})^{1-\alpha}} + \right. \\ & \left. + \frac{\mathrm{e}^{-\alpha}}{1-\alpha} \frac{1}{(\nu\lambda_{K+1})^{1-\alpha}} \right] + c\frac{C_N L_N^2}{\nu}. \end{aligned} \tag{3.97}$$

Since $K \geq m$ (cf. (3.75)), we have that

$$\begin{aligned} \frac{(\nu\lambda_m)^\alpha}{\beta} + \frac{\mathrm{e}^{-\alpha}}{1-\alpha} \frac{1}{(\nu\lambda_{m+1})^{1-\alpha}} + \frac{\mathrm{e}^{-\alpha}}{1-\alpha} \frac{1}{(\nu\lambda_{K+1})^{1-\alpha}} \leq & \\ \left(1 + 2\frac{\mathrm{e}^{-\alpha}}{1-\alpha} \right) \frac{1}{(\nu\lambda_m)^{1-\alpha}}. & \end{aligned} \tag{3.98}$$

Hence, from (3.97), we obtain that

$$\begin{aligned} |\mathbf{w}(s)|_{L^2} \leq\ & 2\,\mathrm{e}^{-(s-t)\beta}\,|\mathbf{w}(t)|_{L^2} + \\ & + cc_\alpha \left(1 + \frac{\mathrm{e}^{-\alpha}}{1-\alpha} \right) \frac{|\Omega|^{\alpha-\frac{1}{2}} M_1}{\nu\lambda_m^{1-\alpha}} \sup_{t \leq \tau \leq s} |\mathbf{w}(\tau)|_{L^2} + c\frac{C_N L_N^2}{\nu}. \end{aligned} \tag{3.99}$$

Let

$$\gamma = cc_\alpha \left(1 + \frac{\mathrm{e}^{-\alpha}}{1-\alpha} \right) \frac{|\Omega|^{\alpha-\frac{1}{2}} M_1}{\nu\lambda_m^{1-\alpha}} \tag{3.100}$$

and

$$\theta = 2\left(\mathrm{e}^{-\frac{\beta}{\nu\lambda_1}} + \frac{\gamma}{1-\gamma} \right).$$

Using hypothesis (3.63) with a suitable absolute constant c and also hypothesis (3.64), we obtain that $\gamma < 1$ and $\theta < 1$. Therefore, (3.66) follows from (3.99) and Lemma 3.3 with $y{=}|\mathbf{w}(\cdot)|_{L^2}$, $a{=}2$, $b{=}\beta$, γ given in (3.100), and $\varepsilon{=}cC_N L_N^2/\nu$. ∎

Remark 3.6. We notice that, by using an explicit form of the constant c_α from (2.9) (see, e.g., [7, 40, 46]), one could obtain an optimal choice of α by minimizing the coefficient of $\sup_{t\leq\tau\leq s}|\mathbf{w}(\tau)|_{L^2}$ in (3.99) with respect to α. Thus, in this case, the values of γ, θ, and the condition (3.63) on λ_m would be given explicitly in terms of this optimal value of α. However, we chose not to deal with these technical details here.

With the result of Theorem 3.5, we can obtain an estimate for the error committed when applying the standard Galerkin method to (1.2) in order to obtain an approximation of the reference solution $\mathbf{u}$ of (1.1). The proof follows as in (3.16)–(3.17).

Corollary 3.7. *Assume the hypotheses from Theorem* 3.5. *Then there exists* $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, N) \geq t_0$ *such that, for every* $N \geq K$,

$$\sup_{t\geq T} |\mathbf{v}_N(t) - \mathbf{u}(t)|_{L^2} \leq C\frac{L_N}{\lambda_{N+1}} \tag{3.101}$$

and

$$\sup_{t\geq T} \|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{H^1} \leq C\frac{L_N}{\lambda_{N+1}^{1/2}}, \tag{3.102}$$

where C *is a constant depending on* ν, λ_1 *and* $|\mathbf{g}|_{L^2}$ *but independent of* N.

Finally, we now state the result about the error associated with the PPGM applied to (3.18) relative to the reference solution $\mathbf{u}$. Compared to the result from Corollary 3.7, the estimates show that the PPGM has a better convergence rate than the standard Galerkin method. The proof follows immediately from the result of Theorem 3.5 and (3.4)–(3.6).

Theorem 3.8. *Assume the hypotheses from Theorem* 3.5, *with* $\mathbf{u}$ *satisfying, in addition,* (2.18) *and* (2.19) *for every* $t \geq t_0$. *Then there exists* $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, N) \geq t_0$ *such that, for every* $N \geq K$,

$$\sup_{t\geq T} |[\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))] - \mathbf{u}(t)|_{L^2} \leq C\frac{L_N^4}{\lambda_{N+1}^{3/2}} \tag{3.103}$$

and

$$\sup_{t\geq T} \|[\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))] - \mathbf{u}(t)\|_{H^1} \leq C\frac{L_N^4}{\lambda_{N+1}}, \tag{3.104}$$

where C *is a constant depending on* ν, λ_1 *and* $|\mathbf{g}|_{L^2}$ *but independent of* N.

3.2. A general class of interpolant operators. We now consider the class of linear interpolant operators $I_h : L^2(\Omega)^2 \to L^2(\Omega)^2$ satisfying the following properties:

P1. There exists a positive constant c_0 such that

$$|\varphi - I_h(\varphi)|_{L^2} \leq c_0 h \|\varphi\|_{H^1} \quad \forall \varphi \in H^1(\Omega)^2. \tag{3.105}$$

P2. There exists a positive constant c_{-1} such that

$$\|\varphi - I_h(\varphi)\|_{H^{-1}} \leq c_{-1} h |\varphi|_{L^2} \quad \forall \varphi \in L^2(\Omega)^2. \tag{3.106}$$

P3. There exists a positive constant $\widetilde{c_0}$ such that

$$|I_h(\mathbf{q})|_{L^2} \leq \widetilde{c_0} \frac{|\Omega|^{3/4}}{h^2 \lambda_{N+1}^{1/4}} |\mathbf{q}|_{L^2} \quad \forall \mathbf{q} \in Q_N H. \tag{3.107}$$

As one easily verifies, the example of interpolant operator given by the low Fourier modes projector P_K, $N \geq K$, considered in subsection 3.1, satisfies properties (P1)–(P3). In particular, property (P3) is immediately verified since $I_h(\mathbf{q}) = P_K \mathbf{q} = 0$. Indeed, the only reason for assuming property (P3) is that, as will be clearer in the proof of Theorem 3.10, we do not assume $P_\sigma I_h$ to commute with A, a property that P_K satisfies. This is the key difference between the proofs of Theorems 3.5 and 3.10.

A more physically interesting example of operator I_h satisfying properties (P1)–(P3) is given by local averages over finite volume elements. For illustrational purposes, this is proved in the Appendix.

The next results follow a similar outline from the ones in subsection 3.1. We again assume either periodic or no-slip Dirichlet boundary conditions. As before, we start by obtaining a uniform estimate of the V norm of $\mathbf{v}_N - \mathbf{p}$.

PROPOSITION 3.9. *Let* $\mathbf{u}$ *be a solution of* (1.1) *satisfying* (2.12)–(2.14) *for every* $t \geq t_0$. *Let* $\mathbf{v}_0 \in B_V(M_1)$, *with* M_1 *as in* (2.12). *For every* $N \in \mathbb{N}$, *let* $\mathbf{v}_N$ *be the unique solution of* (3.19) *satisfying* $\mathbf{v}_N(t_0) = P_N\mathbf{v}_0$. *Consider* $m \in \mathbb{N}$ *large enough such that*

$$\lambda_m \geq \max\left\{\frac{\lambda_1 \mathrm{e}}{2}, c\frac{C_1}{\nu}L_m^2, c\left(\frac{C_1^2}{\nu M_1}\right)^{2/3} L_m^2, c\left(\frac{C_1}{M_1}\right)^2 L_m^2\right\}. \tag{3.108}$$

If $\beta > 0$ *is large enough such that*

$$\beta \geq \max\left\{\nu\lambda_m, c\frac{M_1^2}{\nu}\left[1 + \log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right)\right]\right\} \tag{3.109}$$

and if h *is small enough such that*

$$h \leq \frac{1}{c_0}\left(\frac{\nu}{\beta}\right)^{1/2}, \tag{3.110}$$

where c_0 *is the constant from* (3.105), *then, for every* $N \geq m$, *we have*

$$\sup_{t\geq t_0} \|\mathbf{v}_N(t) - \mathbf{p}(t)\|_{H^1} \leq 2M_1. \tag{3.111}$$

Proof. Denote $\mathbf{w} = \mathbf{v}_N - \mathbf{p}$. Subtracting (3.24) from (1.4) and using (3.26), we obtain that

$$\begin{aligned}\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \nu A\mathbf{w} &= -\beta\mathbf{w} - \beta P_N P_\sigma[I_h(\mathbf{w}) - \mathbf{w}] + \beta P_N P_\sigma[I_h(\mathbf{q}) - \mathbf{q}] \\ &\quad - P_N[B(\mathbf{w},\mathbf{p}) + B(\mathbf{p},\mathbf{w}) + B(\mathbf{w},\mathbf{w}) - B(\mathbf{p},\mathbf{q}) - B(\mathbf{q},\mathbf{p}) - B(\mathbf{q},\mathbf{q})].\end{aligned} \tag{3.112}$$

Taking the inner product in L^2 of (3.112) with $A\mathbf{w}$ yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{H^1}^2 + \nu|A\mathbf{w}|_{L^2}^2 &= -(B(\mathbf{w},\mathbf{p}), A\mathbf{w})_{L^2} - (B(\mathbf{p},\mathbf{w}), A\mathbf{w})_{L^2} \\ - (B(\mathbf{w},\mathbf{w}), A\mathbf{w})_{L^2} &+ (B(\mathbf{p},\mathbf{q}), A\mathbf{w})_{L^2} + (B(\mathbf{q},\mathbf{p}), A\mathbf{w})_{L^2} + (B(\mathbf{q},\mathbf{q}), A\mathbf{w})_{L^2} \\ &- \beta\|\mathbf{w}\|_{H^1}^2 + \beta(\mathbf{w} - I_h(\mathbf{w}), A\mathbf{w})_{L^2} - \beta(\mathbf{q} - I_h(\mathbf{q}), A\mathbf{w})_{L^2}.\end{aligned} \tag{3.113}$$

Using property (P1) of I_h, we have

$$\begin{aligned}|\beta(\mathbf{w} - I_h\mathbf{w}, A\mathbf{w})_{L^2}| &\leq c_0\beta h\|\mathbf{w}\|_{H^1}|A\mathbf{w}|_{L^2} \\ &\leq \frac{\beta}{2}\|\mathbf{w}\|_{H^1}^2 + \frac{c_0^2\beta h^2}{2}|A\mathbf{w}|_{L^2}^2 \leq \frac{\beta}{2}\|\mathbf{w}\|_{H^1}^2 + \frac{\nu}{2}|A\mathbf{w}|_{L^2}^2,\end{aligned} \tag{3.114}$$

where in the last inequality we used hypothesis (3.110).

Now, using property (P1) of I_h and (2.14), we have

$$(3.115)\quad |\beta(\mathbf{q} - I_h(\mathbf{q}), A\mathbf{w})_{L^2}| \leq c_0\beta h\|\mathbf{q}\|_{H^1}|A\mathbf{w}|_{L^2} \leq \beta\|\mathbf{q}\|_{H^1}^2 + \frac{c_0^2\beta h^2}{4}|A\mathbf{w}|_{L^2}^2 \leq \beta C_1^2\frac{L_N^2}{\lambda_{N+1}} + \frac{\nu}{4}|A\mathbf{w}|_{L^2}^2.$$

The remaining of the proof follows analogously as in the proof of Proposition 3.2, but now using estimates (3.114) and (3.115). ∎

Using the results of Proposition 3.9, Lemma 3.3, and Proposition 3.4, we can now obtain a uniform in time estimate of $|\mathbf{v}_N - \mathbf{p}|_{L^2}$.

THEOREM 3.10. *Let $\mathbf{u}$ be a solution of* (1.1) *satisfying* (2.12)–(2.14) *for every $t \geq t_0$. Let $\mathbf{v}_0 \in B_V(M_1)$, with M_1 as in* (2.12)*. For every $N \in \mathbb{N}$, let $\mathbf{v}_N$ be the unique solution of* (3.19) *satisfying $\mathbf{v}_N(t_0) = P_N\mathbf{v}_0$. Fix $\alpha \in (1/2, 1)$, and consider $m \in \mathbb{N}$ large enough such that*

$$(3.116)\quad \lambda_m \geq \max\left\{\frac{\lambda_1\,\mathrm{e}}{2}, c\frac{C_1}{\nu}L_m^2, c\left(\frac{C_1^2}{\nu M_1}\right)^{2/3}L_m^2, c\left(\frac{C_1}{M_1}\right)^2 L_m^2, \left[cc_\alpha\left(1 + \frac{\mathrm{e}^{-\alpha}}{1-\alpha}\right)\frac{|\Omega|^{\alpha-\frac{1}{2}}M_1}{\nu}\right]^{\frac{1}{1-\alpha}}\right\},$$

where c_α is the constant from (2.9).

If $\beta > 0$ is large enough such that

$$(3.117)\qquad \beta \geq \max\left\{\nu\lambda_m, c\frac{M_1^2}{\nu}\left[1 + \log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right)\right]\right\}$$

and if $h \geq 0$ is small enough such that

$$(3.118)\qquad h \leq c\min\left\{\left(\frac{\nu}{\beta}\right)^{1/2}, \frac{\nu\lambda_m^{1/2}}{\beta}\right\},$$

then there exists $\theta = \theta(\beta) \in [0, 1)$ and a constant $C = C(\nu, \lambda_1, |\mathbf{g}|_{L^2}, 1/h^2)$ such that, for every $N \geq m$, we have

$$(3.119)\qquad |\mathbf{v}_N(t) - \mathbf{p}(t)|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1 - 1}|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)|_{L^2} + C\frac{L_N}{\lambda_{N+1}^{5/4}}.$$

Proof. We recall equation (3.9) satisfied by $\mathbf{w} = \mathbf{v}_N - \mathbf{p}$:

$$(3.120)\quad \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + [\nu AP_N + \beta P_N]\mathbf{w} = -P_N[B(\mathbf{v}_N, \mathbf{v}_N) - B(\mathbf{p}, \mathbf{p})] + P_N G - \beta P_N P_\sigma[I_h(\mathbf{w}) - \mathbf{w}] + \beta P_N P_\sigma I_h(\mathbf{q}),$$

where

$$(3.121)\qquad G(t) = B(\mathbf{u}(t), \mathbf{u}(t)) - B(\mathbf{p}(t), \mathbf{p}(t)) \quad \forall t \geq t_0.$$

Using Duhamel's formula, it follows that

$$
\begin{aligned}
(3.122)\quad |\mathbf{w}(t)|_{L^2} &\leq |\,\mathrm{e}^{-(t-t_0)(\nu AP_N+\beta P_N)}\,\mathbf{w}(t_0)|_{L^2} \\
&+ \int_{t_0}^{t} \Big|\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,P_N[B(\mathbf{v}_N(\tau),\mathbf{v}_N(\tau)) - B(\mathbf{p}(\tau),\mathbf{p}(\tau))]\Big|_{L^2}\,\mathrm{d}\tau \\
&+ \int_{t_0}^{t} \Big|\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,P_N G(\tau)\Big|_{L^2}\,\mathrm{d}\tau \\
&+ \beta\int_{t_0}^{t} \Big|\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,P_N P_\sigma[I_h(\mathbf{w}(\tau)) - \mathbf{w}(\tau)]\Big|_{L^2}\,\mathrm{d}\tau \\
&+ \beta\int_{t_0}^{t} \Big|\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,P_N P_\sigma I_h(\mathbf{q}(\tau))\Big|_{L^2}\,\mathrm{d}\tau.
\end{aligned}
$$

The estimates for the first three terms in the right-hand side of (3.122) now follow by writing

$$
(3.123)\qquad \mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)} = \mathrm{e}^{-(t-\tau)(\nu AP_m+\beta P_m)} + \mathrm{e}^{-(t-\tau)(\nu AP_N Q_m+\beta P_N Q_m)}
$$

and proceeding analogously as in the proof of Theorem 3.5.

In order to estimate the fourth term on the right-hand side of (3.122), we use property (P2) of I_h and obtain that

$$
\begin{aligned}
(3.124)\quad &\beta\int_{t_0}^{t} \Big|\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,P_N P_\sigma[I_h(\mathbf{w}(\tau)) - \mathbf{w}(\tau)]\Big|_{L^2}\,\mathrm{d}\tau \\
&\leq \frac{\beta}{\nu^{1/2}}\int_{t_0}^{t} \|\nu^{1/2}A^{1/2}\,\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,\|_{\mathcal{L}(P_N H)}|A^{-1/2}P_\sigma[I_h(\mathbf{w})-\mathbf{w}]|_{L^2}\mathrm{d}\tau \\
&\leq c_{-1}\frac{\beta h}{\nu^{1/2}}\sup_{\tau\geq t_0}|\mathbf{w}(\tau)|_{L^2}\int_{t_0}^{t} \|\nu^{1/2}A^{1/2}\,\mathrm{e}^{-(s-t_0)(\nu AP_N+\beta P_N)}\,\|_{\mathcal{L}(P_N H)}\mathrm{d}s \\
&\leq c_{-1}\frac{\beta h}{\nu^{1/2}}\left(\frac{(\nu\lambda_m)^{\frac{1}{2}}}{\beta} + 2\frac{\mathrm{e}^{-\frac{1}{2}}}{(\nu\lambda_{m+1})^{\frac{1}{2}}}\right)\sup_{\tau\geq t_0}|\mathbf{w}(\tau)|_{L^2},
\end{aligned}
$$

where in the last inequality we used again (3.123) and similar calculations from (3.78)–(3.83).

Finally, for the last term in the right-hand side of (3.122), we use property (P3) of I_h and (2.13) to obtain that

$$
\begin{aligned}
(3.125)\quad &\beta\int_{t_0}^{t} \Big|\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,P_N P_\sigma I_h(\mathbf{q}(\tau))\Big|_{L^2}\,\mathrm{d}\tau \leq \\
&\leq \widetilde{c}_0\frac{|\Omega|^{3/4}}{h^2\lambda_{N+1}^{1/4}}\beta\int_{t_0}^{t}\|\,\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,\|_{\mathcal{L}(P_N H)}|\mathbf{q}(\tau)|_{L^2}\mathrm{d}\tau \\
&\leq \widetilde{c}_0\frac{C_0|\Omega|^{3/4}\beta}{h^2}\frac{L_N}{\lambda_{N+1}^{5/4}}\int_{t_0}^{t}\|\,\mathrm{e}^{-(t-\tau)(\nu AP_N+\beta P_N)}\,\|_{\mathcal{L}(P_N H)}\mathrm{d}\tau \\
&= \widetilde{c}_0\frac{C_0|\Omega|^{3/4}\beta}{h^2}\frac{L_N}{\lambda_{N+1}^{5/4}}\int_{t_0}^{t}\max_{1\leq j\leq N}\,\mathrm{e}^{-(t-\tau)(\nu\lambda_j+\beta)}\,\mathrm{d}\tau \leq \widetilde{c}_0\frac{C_0|\Omega|^{3/4}}{h^2}\frac{L_N}{\lambda_{N+1}^{5/4}}.
\end{aligned}
$$

The remainder of the proof follows analogously as in the proof of Theorem 3.5. ∎

The result of Theorem 3.10 now yields, as in (3.16)–(3.17), an estimate of the error associated to the Galerkin approximation of (1.2) relative to the reference solution $\mathbf{u}$ of (1.1), in the general case of an interpolant operator satisfying properties (P1)–(P3).

Corollary 3.11. *Assume the hypotheses from Theorem* 3.10. *Then there exists* $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, N) \geq t_0$ *such that, for every* $N \geq m$,

$$\sup_{t\geq T} |\mathbf{v}_N(t) - \mathbf{u}(t)|_{L^2} \leq C \frac{L_N}{\lambda_{N+1}} \tag{3.126}$$

and

$$\sup_{t\geq T} \|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{H^1} \leq C \frac{L_N}{\lambda_{N+1}^{1/2}}, \tag{3.127}$$

where C *is a constant depending on* ν, λ_1, $|\mathbf{g}|_{L^2}$ *and* $1/h^2$ *but independent of* N.

Finally, we now obtain an estimate of the error committed when applying the PPGM to system (1.2), in order to obtain an approximation of the reference solution $\mathbf{u}$ of (1.1), in the case of an interpolant operator satisfying properties (P1)–(P3). The result shows that the convergence rate of the PPGM in this case, although not as good as the one obtained in Theorem 3.8, is still better than the convergence rate of the standard Galerkin method. The proof follows immediately from the result of Theorem 3.10 and (3.4)–(3.6).

Theorem 3.12. *Assume the hypotheses from Theorem* 3.10, *with* $\mathbf{u}$ *satisfying, in addition,* (2.18) *and* (2.19), *for every* $t \geq t_0$. *Then there exists* $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, N) \geq t_0$ *such that, for every* $N \geq m$,

$$\sup_{t\geq T} |[\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))] - \mathbf{u}(t)|_{L^2} \leq C \frac{L_N}{\lambda_{N+1}^{5/4}} \tag{3.128}$$

and

$$\sup_{t\geq T} \|[\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))] - \mathbf{u}(t)\|_{H^1} \leq C \frac{L_N}{\lambda_{N+1}^{3/4}}, \tag{3.129}$$

where C *is a constant depending on* ν, λ_1, $|\mathbf{g}|_{L^2}$ *and* $1/h^2$ *but independent of* N.

Remark 3.13. We emphasize that the main purpose of the postprocessing step applied to the Galerkin method is to improve the accuracy of the numerical approximation of $\mathbf{v}$, solution of (1.2), and thus $\mathbf{u}$, solution of (1.1). The fact that the numerical approximation of $\mathbf{v}$ given by the PPGM yields a uniform-in-time error estimate is actually due to the fact that the Galerkin approximation $\mathbf{v}_N$ of $\mathbf{v}$ yields a uniform-in-time error estimate. Indeed, the latter is valid for an even more general class of interpolant operators than the one considered in subsection 3.2, namely, the family of operators $I_h : H^1(\Omega)^2 \to L^2(\Omega)^2$, which are only required to satisfy property (P1), and also the family of operators $I_h : H^2(\Omega)^2 \to L^2(\Omega)^2$ satisfying (see [5])

$$\|\varphi - I_h(\varphi)\|_{H^2} \leq c_1 h \|\varphi\|_{H^1} + c_2 h^2 \|\varphi\|_{H^2} \quad \forall \varphi \in H^2(\Omega)^2,$$

where c_1 and c_2 are positive constants and $\|\cdot\|_{H^2}$ denotes the usual Sobolev norm of the space $H^2(\Omega)^2$. A physically relevant example of interpolant operator of this latter type is given by measurements at a finite set of nodal points in Ω.

It is not difficult to show that (using, in particular, similar ideas from the proof of Proposition 3.9), under the appropriate conditions on the parameters β and h and for both types of interpolant operators, there exists $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, N) \geq t_0$ large enough such that

$$\sup_{t\geq T} \|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{H^1} \leq C\frac{L_N^2}{\lambda_{N+1}^{1/2}},$$

where C is a constant depending on ν, λ_1 and $|\mathbf{g}|_{L^2}$ but independent of N. Moreover, for the former class of interpolant operators, one can also show that

$$\sup_{t\geq T} |\mathbf{v}_N(t) - \mathbf{u}(t)|_{L^2} \leq C\frac{L_N}{\lambda_{N+1}},$$

where, again, $T = T(\nu, \lambda_1, |\mathbf{g}|_{L^2}, N) \geq t_0$ and $C = C(\nu, \lambda_1, |\mathbf{g}|_{L^2})$.

Appendix A.

The aim of this section is to show that the example of interpolant operator given by local averages over finite volume elements (see, e.g., [26, 37, 38]), assuming periodic boundary conditions, satisfies properties (P1)–(P3) considered in subsection 3.2.

Let $\Omega = (0, L) \times (0, L) \subset \mathbb{R}^2$ be a basic domain of periodicity, and consider a partition of Ω into K squares with sides of length $h = L/\sqrt{K}$. Let

$$\Lambda = \{(j, l) \in \mathbb{N}^2 : 1 \leq j, l \leq \sqrt{K}\},$$

and, for every $\alpha = (j, l) \in \Lambda$, let Q_α be the volume element given by the square

$$Q_\alpha = [(j-1)h, jh) \times [(l-1)h, lh).$$

Consider the interpolant operator $I_h : L^2(\Omega)^2 \to L^2(\Omega)^2$ given by

$$I_h(\varphi) = \sum_{\alpha\in\Lambda} \overline{\varphi_\alpha}\chi_{Q_\alpha} \quad \forall \varphi \in L^2(\Omega)^2, \tag{A.1}$$

where $\overline{\varphi_\alpha}$ is the local average of φ over the volume element Q_α, i.e.,

$$\overline{\varphi_\alpha} = \frac{1}{|Q_\alpha|}\int_{Q_\alpha} \varphi(y)\mathrm{d}y. \tag{A.2}$$

The fact that I_h defined in (A.1) satisfies property (P1) follows from the calculations in [37, Appendix]. Thus, it only remains to verify properties (P2) and (P3). In fact, we show that this particular example of I_h sastisfies a stronger property than (P3) with respect to the $(L^\infty(\Omega))^2$-norm.

Notice that, in the present case, $|\Omega| = L^2$.

PROPOSITION A.1. *Let $I_h : L^2(\Omega)^2 \to L^2(\Omega)^2$ be the operator defined by* (A.1). *Then the following holds:*

(i) *There exists a positive constant c_{-1} such that*

$$\|\varphi - I_h(\varphi)\|_{H^{-1}} \leq c_{-1}h|\varphi|_{L^2} \quad \forall \varphi \in L^2(\Omega)^2. \tag{A.3}$$

(ii) *There exists a positive constant $\widetilde{c}_0$ such that*

$$\|I_h(\mathbf{q})\|_{L^\infty} \leq c\frac{L^{1/2}}{h^2\lambda_{N+1}^{1/4}}|\mathbf{q}|_{L^2} \quad \forall \mathbf{q} \in Q_N H. \tag{A.4}$$

Consequently,

$$|I_h(\mathbf{q})|_{L^2} \leq \widetilde{c}_0 \frac{L^{3/2}}{h^2 \lambda_{N+1}^{1/4}} |\mathbf{q}|_{L^2} \quad \forall \mathbf{q} \in Q_N H. \tag{A.5}$$

Proof. From its definition, it follows immediately that I_h is a symmetric operator, i.e.,

$$(I_h(\varphi), \psi) = (\varphi, I_h(\psi)) \quad \forall \psi \in L^2(\Omega)^2. \tag{A.6}$$

Thus, using (A.6) and property (P1), we obtain that

$$\begin{aligned} \|\varphi - I_h(\varphi)\|_{H^{-1}} &= \sup_{\substack{\psi \in H_0^1(\Omega)^2 \\ \|\psi\|_{H^1}=1}} |(\varphi - I_h(\varphi), \psi)| = \sup_{\substack{\psi \in H_0^1(\Omega)^2 \\ \|\psi\|_{H^1}=1}} |(\varphi, \psi - I_h(\psi))| \\ &\leq \sup_{\substack{\psi \in H_0^1(\Omega)^2 \\ \|\psi\|_{H^1}=1}} c_0 h |\varphi|_{L^2} \|\psi\|_{H^1} = c_0 h |\varphi|_{L^2}, \end{aligned} \tag{A.7}$$

which proves that (i) is satisfied with $c_{-1} = c_0$.

Now let us prove (ii). Let $\mathbf{q} \in Q_N H$, and consider its Fourier expansion, given by

$$\mathbf{q}(\mathbf{y}) = \sum_{|k| \geq \kappa_N} \hat{\mathbf{u}}_k \, \mathrm{e}^{2\pi i \frac{k}{L} \cdot \mathbf{y}} \quad \forall \mathbf{y} \in \Omega, \tag{A.8}$$

where

$$\kappa_N = \frac{L}{2\pi} \lambda_{N+1}^{1/2}. \tag{A.9}$$

From (A.8) and the definition of I_h in (A.1), we have that

$$I_h(\mathbf{q})(\mathbf{x}) = \sum_{\alpha \in \Lambda} \sum_{|k| \geq \kappa_N} \frac{1}{|Q_\alpha|} \hat{\mathbf{u}}_k \left(\int_{Q_\alpha} \mathrm{e}^{2\pi i \frac{k}{L} \cdot \mathbf{y}} \, \mathrm{d}\mathbf{y} \right) \chi_{Q_\alpha}(\mathbf{x}).$$

Thus,

$$|I_h(\mathbf{q})(\mathbf{x})| \leq \sum_{\alpha \in \Lambda} \sum_{|k| \geq \kappa_N} \frac{1}{|Q_\alpha|} |\hat{\mathbf{u}}_k| \left| \int_{Q_\alpha} \mathrm{e}^{2\pi i \frac{k}{L} \cdot \mathbf{y}} \, \mathrm{d}\mathbf{y} \right| \chi_{Q_\alpha}(\mathbf{x}) = (S_1 + S_2 + S_3)(\mathbf{x}), \tag{A.10}$$

where

$$S_1(\mathbf{x}) = \sum_{\alpha \in \Lambda} \sum_{\substack{|k| \geq \kappa_N \\ k_1 = 0}} \frac{1}{|Q_\alpha|} |\hat{\mathbf{u}}_k| \left| \int_{Q_\alpha} \mathrm{e}^{2\pi i \frac{k}{L} \cdot \mathbf{y}} \, \mathrm{d}\mathbf{y} \right| \chi_{Q_\alpha}(\mathbf{x}),$$

$$S_2(\mathbf{x}) = \sum_{\alpha \in \Lambda} \sum_{\substack{|k| \geq \kappa_N \\ k_2 = 0}} \frac{1}{|Q_\alpha|} |\hat{\mathbf{u}}_k| \left| \int_{Q_\alpha} \mathrm{e}^{2\pi i \frac{k}{L} \cdot \mathbf{y}} \, \mathrm{d}\mathbf{y} \right| \chi_{Q_\alpha}(\mathbf{x}),$$

$$S_3(\mathbf{x}) = \sum_{\alpha \in \Lambda} \sum_{\substack{|k| \geq \kappa_N \\ k_1 \neq 0, k_2 \neq 0}} \frac{1}{|Q_\alpha|} |\hat{\mathbf{u}}_k| \left| \int_{Q_\alpha} \mathrm{e}^{2\pi i \frac{k}{L} \cdot \mathbf{y}} \, \mathrm{d}\mathbf{y} \right| \chi_{Q_\alpha}(\mathbf{x}).$$

Notice that

$$S_1(\mathbf{x}) = \sum_{\alpha\in\Lambda} \sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} \frac{1}{|Q_\alpha|} |\hat{\mathbf{u}}_k| \left| \int_{(l-1)h}^{lh} \int_{(j-1)h}^{jh} \mathrm{e}^{2\pi i \frac{k_2}{L} y_2}\, \mathrm{d}y_1 \mathrm{d}y_2 \right| \chi_{Q_\alpha}(\mathbf{x}) \tag{A.11}$$

$$= \sum_{\alpha\in\Lambda} \sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} \frac{1}{h^2} |\hat{\mathbf{u}}_k| h \left| \frac{L}{2\pi i k_2} \mathrm{e}^{2\pi i \frac{k_2}{L} lh} (1 - \mathrm{e}^{-2\pi i \frac{k_2}{L} h}) \right| \chi_{Q_\alpha}(\mathbf{x})$$

$$\leq \frac{L}{\pi h} \sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} |\hat{\mathbf{u}}_k| \frac{1}{|k_2|} \left(\sum_{\alpha\in\Lambda} \chi_{Q_\alpha}(\mathbf{x}) \right) = \frac{L}{\pi h} \sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} |\hat{\mathbf{u}}_k| \frac{1}{|k_2|}$$

$$\leq \frac{L}{\pi h} \left(\sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} |\hat{\mathbf{u}}_k|^2 \right)^{1/2} \left(\sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} \frac{1}{|k_2|^2} \right)^{1/2} \leq \frac{L}{\pi h} \frac{|\mathbf{q}|_{L^2}}{|\Omega|^{1/2}} \left(\sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} \frac{1}{|k_2|^2} \right)^{1/2}$$

$$= \frac{1}{\pi h} |\mathbf{q}|_{L^2} \left(\sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} \frac{1}{|k_2|^2} \right)^{1/2} \leq \frac{c}{h} |\mathbf{q}|_{L^2} \frac{1}{\kappa_N^{1/2}} \leq \frac{c}{hL^{1/2}} |\mathbf{q}|_{L^2} \frac{1}{\lambda_{N+1}^{1/4}}.$$

Analogously,

$$S_2(\mathbf{x}) \leq \frac{c}{hL^{1/2}} |\mathbf{q}|_{L^2} \frac{1}{\lambda_{N+1}^{1/4}}. \tag{A.12}$$

Moreover,

$$S_3(\mathbf{x}) = \sum_{\alpha\in\Lambda} \sum_{\substack{|k|\geq\kappa_N\\ k_1=0}} \frac{1}{|Q_\alpha|} |\hat{\mathbf{u}}_k| \left| \int_{(l-1)h}^{lh} \int_{(j-1)h}^{jh} \mathrm{e}^{2\pi i \frac{k_1}{L} y_1}\, \mathrm{e}^{2\pi i \frac{k_2}{L} y_2}\, \mathrm{d}y_1 \mathrm{d}y_2 \right| \chi_{Q_\alpha}(\mathbf{x}) \tag{A.13}$$

$$\leq \frac{L^2}{\pi^2 h^2} \sum_{\substack{|k|\geq\kappa_N\\ k_1\neq 0, k_2\neq 0}} |\hat{\mathbf{u}}_k| \frac{1}{|k_1||k_2|} \leq \frac{L^2}{\pi^2 h^2} \frac{|\mathbf{q}|_{L^2}}{|\Omega|^{1/2}} \left[\sum_{\substack{|k|\geq\kappa_N\\ k_1\neq 0, k_2\neq 0}} \frac{1}{k_1^2 k_2^2} \right]^{1/2}$$

$$\leq \frac{L}{\pi^2 h^2} |\mathbf{q}|_{L^2} \left[\sum_{|k_1|\geq \frac{\kappa_N}{2}, |k_2|\geq 1} \frac{1}{k_1^2 k_2^2} + \sum_{|k_2|\geq \frac{\kappa_N}{2}, |k_1|\geq 1} \frac{1}{k_1^2 k_2^2} \right]^{1/2}$$

$$\leq \frac{L}{\pi^2 h^2} |\mathbf{q}|_{L^2} \left[\left(\sum_{|k_1|\geq \frac{\kappa_N}{2}} \frac{1}{k_1^2} \right) \left(\sum_{|k_2|\geq 1} \frac{1}{k_2^2} \right) + \left(\sum_{|k_2|\geq \frac{\kappa_N}{2}} \frac{1}{k_2^2} \right) \left(\sum_{|k_1|\geq 1} \frac{1}{k_1^2} \right) \right]^{1/2}$$

$$\leq c \frac{L}{h^2} |\mathbf{q}|_{L^2} \frac{1}{\kappa_N^{1/2}} \leq c \frac{L^{1/2}}{h^2} |\mathbf{q}|_{L^2} \frac{1}{\lambda_{N+1}^{1/4}}.$$

From (A.10)–(A.13), we obtain that

$$|I_h(\mathbf{q})(x)| \leq c \frac{L^{1/2}}{h^2} |\mathbf{q}|_{L^2} \frac{1}{\lambda_{N+1}^{1/4}}, \quad \forall \mathbf{x} \in \Omega, \tag{A.14}$$

which proves (A.4). ∎

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