Level Rank duality in Quantum Mechanics

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References

- Talk based on: ArXiv 2409.????? J. Djororovic, I. Haldar, L. Janagal, S.M. , C. Patel and N. Prabhakar

Introduction I

- It has been clear for 30 years that strong weak coupling dualities are ubiquitious in QFT. While there is strong circumstantial evidence for many such dualities, not one of them has been derived in $d \geq 3$. There is no real understanding of why they are correct.
- The situation with topological field theories is better. Pure Chern Simons (CS) theory has been shown to enjoy invariance under the intensely interesting level rank duality that relates the $U(N)_{k,k}$ and $SU(k)_{-N}$ Chern Simons theories. As this duality flips the t'Hooft coupling (rank divided by level), it is a strong weak coupling duality that is extremely well understood.
- However topological theories lack propagating degrees of freedom, and so dualities here may (at first) seem disconnectedy different from dualities in genuine QFTs, which host propagating degrees of freedom. Not quite the case.

Introduction II

- Over the last 12 years, considerable evidence has built up in support of the following conjecture. $U(N)_{k,k}$ CS theory coupled to one fundamental multiplet mass deformed Wilson Fisher theory of one multiplet of fundamental bosons, is dual to $SU(k)_{-N}$ CS theory coupled to (an otherwise free) fundamental multiplet of massive fermions.
- In the deep IR this QFT duality reduces to the rigorously proved duality between topological CS theories described above.
- Thus this example connects a poorly understood QFT duality to well understood topological dualities in a limit. Can we learn something about the QFT dualities by starting with the topological dualities and 'flowing upwards'?

Non Relativistic QM coupled to CS Theory

- In order to to flow a little bit up from the deep IR, we study a modified - but still tractable - low energy limit that retains some propagating degrees of freedom.
- We focus on configurations with a fixed collection of particles, and zoom in to energies that are just larger than the sum of masses of these particles. At such energies, the particles all move slowly compared to the speed of light and are well described by non relativistic quantum mechanics.
- In this talk I construct the non relativistic system (Hilbert space, Schrodinger equation) on both sides of the duality, and study the relationship between them.

Review of Quantization of CS with fixed sources

- It is useful to first recall Witten's classic quantization of Chern Simons theory in the presence of fixed sources.
- Witten identified the Hilbert Space of CS theory interacting with a collection of fixed sources, in representations R_1, $R_2 \ldots R_n$ of the gauge group with WZW conformal blocks with primary fields at the same location and in the same representations.
- Here we use the term'conformal blocks' in an ultralocal manner. With this notion, the space of conformal blocks to be that subspace $\text{Inv}(\bigotimes_i R_i)_{q,\mathbf{z}}$ of the (Lie Algebra) invariant subspace of the tensor product of the representations R_i, $\text{Inv}(\bigotimes_i R_i)$ that obeys the Gepner Witten constraints of CFT at insertion locations $\mathbf{z}$.
- The Gepner Witten constraints are a function of the insertion locations $\mathbf{z}$ of the primaries, and so the embedding of the space of conformal blocks - within the space of Lie algebra invariants - rotates as $\mathbf{z}$ varies.

Berry's connection

- As Witten worked with fixed sources, he obtained a $2n$ parameter set of Hilbert Spaces, parameterized by the insertion locations $\mathbf{z}$. If we would like to compare states in the distinct Hilbert Spaces, we need a connection - a Berry's connection - on this collection of Hilbert Spaces.
- A userul Berry's connection is obtained from the construction of states, as a path integral on the solid ball in the presence of a tangle of Wilson Lines, with end points in the given representations and at the given locations.
- We define the Berry's connection by the condition that a state defined by a tangle with given end points - and a second state defined by the same tangle, with infinitesimally separated insertion locations (and framing vector rotated by the spin connection with spin h) - are parallel transports of each other.

Local flatness of the Berry's connection

- A little thought will convince you that the local curvature of this Berry's connection lies in the $U(1)$: infact it is the same as the curvature of the $U(1)$ connection. This connection is given by $\mathcal{A}^{s}_{\mu_i} = (h_i)\omega_{\mu_i})$. It follows that, locally,

$$\mathcal{A} = \mathcal{A}^{\mathrm{Berry}} - \mathcal{A}^{s}_{\mu_i} = -V^{-1}(\partial_{z_i} V) \tag{1}$$

- From the path integral, it is easy to convince oneself that the connection $\mathcal{A}^{\mathrm{Berry}}$, hence $\mathcal{A}$, has δ function curvatures at the locations where two insertions collide. It follows that the function V is multivalued as one point is taken around another.

The Berry's connection is the KZ connection

- A connection with all these properties is familar from the study of CFT. Recall that in the study of correlators of WZW theory, we make use the term 'conformal blocks' in a stronger sense than so far. While these CFT conformal blocks do obey all the Gepner Witten constraints, in addition, their variation with **z** is determined by the KZ equations

$$D_{\mu_i}\mathcal{B}_{\beta'}(\mathbf{z}) = 0,$$

$$D_{\bar{z}_i} = \partial_{\bar{z}_i}\ , \quad D_{z_i} = \partial_{z_i} + \mathcal{A}^T_{z_i KZ}\ , \quad \text{with} \quad \mathcal{A}^T_{z_i KZ} = \frac{1}{\kappa}\sum_{j\neq i}\frac{T^a_{R_i}T^a_{R_j}}{z_i - z_j}\ ,$$

- The KZ connection has been demonstrated to equal the Berry's connection in a particular choice of gauge. We thus have an algebraic construction of this Berry's connection.
- Note that, by definition, $\mathcal{B}_{\beta'}(\mathbf{z})$ are locally covariantly constant.

The Inner Product

- From the path integral viewpoint one computes $\langle\phi|\psi\rangle$ as follows. One reflects the path integral that defines $|\phi\rangle$ about the boundary of the ball, and complex conjugates all representations. One then glues this path interal to the one that defines $|\psi\rangle$.
- Topological invariance guarantees that simultaneous variation of the end points **z** (for the states $\langle\phi|$ and $|\psi\rangle$) , leave the inner product $\langle\phi|\psi\rangle$ unchanged. But we have defined the connection to ensure that $\langle\phi|$ and $|\psi\rangle$ are covariantly constant under such variations. Consquently, the same must be true of the inner products.
- Concretely, let $\mathcal{I}_\alpha(\mathbf{z})$, $\alpha = 1 \ldots d$ represent a basis of blocks. Then the matrix

$$Q_{\alpha^*\beta}(\mathbf{z}) = (\mathcal{I}_\alpha(\mathbf{z}), \mathcal{I}_\beta(\mathbf{z}))$$

is covariantly constant.

Dynamical Particles: Bundle of Blocks

- For dynamical particles, the Hilbert space is clearly is the union of $\text{Inv}(\bigotimes_i R_i)_{q,\mathbf{z}}$ over all $\mathbf{z}$. In mathematical language, this is a Hilbert Space of sectons, with base space $\mathbf{z}$, and fibre space is the space of blocks at location $\mathbf{z}$. A state in the Hilbert Space takes the form

$$\psi^{\alpha}(\mathbf{z})\mathcal{I}_{\alpha}(\mathbf{z}). \tag{2}$$

- We are free to perform a local change of basis

$$\mathcal{I}' = W\mathcal{I}$$

under which

$$\psi' = \psi W^{-1}$$

Consequently our fibres are subject to $Gl(d, C)$ gauge transformations.

Dynamical particles: covariant derivative

- The covariant derivative on our space uses a linear combination of of the KZ and spin connections, and takes the explicit form

$$D_{\mu_i} = \partial_{\mu_i} + \mathcal{A}^T_{\mu_i} + s^i \omega_{\mu_i}$$

where

$$s_i = s_i^{\rm int} + h_+ i$$

- s_i^{int}, the intrinsic spin of the i^{th} particle, is external data we need to specify. s_i^{int} is zero for scalar particles, $\frac{m_i}{2}$ for Dirac particles, etc. In contrast, the 'statistical' spin h_i is the spin the particle picks up by virtue of its interation with the Chern Simons gauge field. Value of stat spin clear from 2d viewpoint. Can also be seen directly in 3d from a Noether charge analysis on a spherical lump of charge (contribution somewhat analogous to $\vec{E} \times \vec{B}$ in 4 dimensions).

Dynamical particles: an ‘equivalence principle’

- Consider a bunch a particles interacting via Chern Simons exchange. The Chern Simons equation of motion $2\pi kF = -*J$ tells us that the $U(N)$ field strength is delta function localized on particle world lines. It follows that particles are locally free. They ‘see’ the Chern Simons interaction only when they wind around each other. Simple example: motion of a particle in 2d around a point like solenoid at the origin.
- In our dynamical problem, consequently, it must locally be possible to move to a $GL(d, C)$ gauge in which motion reduces to free motion on the space of particle positions time unchanged motion in the space of conformal blocks.
- Such a gauge does exist. It is obtained by choosing our basis vectors $\mathcal{I}_\alpha$ to be covariantly constant, i.e. to obey the KZ equation. Equation can be solved beause $\mathcal{A}$ is locally flat. $\mathcal{A}$ vanishes in new gauge. Gauge an only be chosen locally because relevant $\mathcal{I}_\alpha$ multivalued.

Dynamical particles: inner product for sections

- We need an inner product on sections. Our inner product must be positive definite and gauge covariant. Moreover our ‘equivalence principle’ tells us that it must locally reduce to the product of the free particle inner product and Witten’s inner product in the ‘irregular gauge’ of the prevous slide.
- The unique inner product that meets all these constraints is

$$\langle\psi|\chi\rangle = \int \prod_i \sqrt{g_i} dz^i d\bar{z}^i \psi^\dagger Q \chi$$

Dynamical particles: Hamiltonian

- The Hamiltionian of our system is constrained by gauge invariance and the equivalence principle, to take the form

$$H = \sum_i -\frac{1}{2|m_i|}\left(D_i^2\psi + \frac{a_i R}{2}\psi\right) \; ,$$

- We have allowed for each particle to couple to the curvature in an arbitrary manner because free particles have such couplings. Infact if one takes the non relativistic limit of the Dirac, or massive spin one equation, one finds $a_i = |s_i|$. The chioice $a_i = |s_i|$ may, therefore have some special properties. To be general, however, we leave the couplings a_i arbitrary.

Hermiticity and Boundary Conditions

- Using the convarint constancy of Q, it is not difficult to demonstrate that the Hamiltonian of the previous slide is Hermitian. However there is a subtlety. The proof works if one can integrate by parts and ignore boundary terms. This is potentially problematic when two particles approach each other. We pause to study this point.
- It is convenient to work in irregular gauge and then later transform back to a regular gauge. To study the approach of i and j to each other, we choose our basis of blocks to diagonalize i, j fusion. Consider a basis element in which i and j fuse to m. For this element, the monodromy (on taking R_i arond R_j) is given by

$$e^{2\pi i \nu_{ij}^m}, \quad \nu_{ij}^m = h_i + h_j - h_m - [h_i + h_j - h_m] \tag{3}$$

Boundary conditions

- In the limit that r is small, it is easy to check that the most general solution to the wave equation takes the form

$$\psi = \sum_n \psi_n(r) e^{i(n+\nu)\theta}, \qquad \psi_n(r) = a_n r^{n+\nu} + \frac{b_n}{r^{n+\nu}} \qquad (4)$$

- For $n > 1$ the condition of square integrability (well definedness of the norm) sets $b_n = 0$. b_0 is allowed to be zero. However it turns out that our proof of Hermiticity of the Hamiltonian goes through if and only if the ratio $\frac{b_0}{a_0}$ is the same for all wave functions. This ratio - for every choice of i, j, m - thus has to be specified once and for all, and is part of the definition of our Hilbert Space.
- Physically, this ratio contains information about non Chern Simons 'contact type' interactions between particles. Note this ratio has dimension mass$^{-2\nu}$. When our QM is obtained as the low energy limit of a UV QFT, we expect that this ratio vanishes generically. 'RG flow Univerality'.

Level Rank Duality

- Our construction of our multi particle quantum mechanics is now complete. We can construct a quantum mechanics with particles in representations R_i in the $U(N)_{k,k}$ theory. We can separately construct the quantum mechanics of particles in representations $\tilde{R}_i$ (level rank dual reps) in the $SU(k)_{-N}$ theory. Is there a relationship between these distinct quantum systems?
- In order to address this question, we review a standard construction in the study of level rank duality for topological CS theories. Consider a 2d theory of Nk complex chiral fermions. This theory enjoys invariance under $U(Nk)_1$. The $U(Nk)_1$ primaries of this theory are product of (at most Nk) ψ or the product of at most Nk $\bar{\psi}$.

Level Rank Branching Rules

- Now $U(Nk)_1$ has a $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ subgroup. This subgroup is complete, in the sense that the sum of the Sugawra central charges of the subgroup equals the Sugawara central charge of $U(Nk)_1$
- Any primary of $U(Nk)_1$ can be decomposed into a finite sum over $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ representations. In some cases the $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ primaries are also $U(Nk)_1$ primaries. In other cases, the $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ are descendents at level n_i. Note that $H_i + n_i = h_i + \hat{h}_i$
- For this reason the $U(Nk)_1$ blocks - simply free fermion correlators - can be decomposed into sums of products of $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ blocks. In equations

$$\mathcal{P}_i \left(\Psi_{\vec{p}}(\mathbf{z}) \right) = \Phi_{\vec{p}}(\mathbf{z}) \sum_{\alpha,\beta} C^{\alpha\hat{\beta}} \mathcal{B}_{\alpha}^{R_i}(\mathbf{z}) \hat{\mathcal{B}}_{\hat{\beta}}^{\tilde{R}_i}(\mathbf{z}). \qquad (5)$$

Here $C^{\alpha\beta}$ are constants.

The covariantly constant section $C^{i\hat{j}}$

- The equation on the previous slide may be rewritten as follows. First, the $U(1)_{Nk}$ block Φ be absorbed into the $SU(N)_k$ block, turning it into a $U(N)_{k,k}$ block. Second, we insert $\mathcal{B}_\alpha = \mathcal{B}^i_\alpha \mathcal{I}_i$ (and a similar equation for the dual blocks) into this equaton to obtain

$$\mathcal{P}_i\left(\Psi(\mathbf{z})\right) = \sum_{i,\hat{j}} C^{i\hat{j}}(\mathbf{z})\mathcal{I}_i\,\hat{\mathcal{I}}_{\hat{j}}$$

$$C^{i\hat{j}}(\mathbf{z}) = C^{\alpha\hat{\beta}}\mathcal{B}^i_\alpha(\mathbf{z})\hat{\mathcal{B}}^{\hat{j}}{}_{Y^T\hat{\beta}}(\mathbf{z})$$

- $C^{i\hat{j}}$, defined above, is a section in the product of original and level rank dual bundles. It is also covariantly constant. When we work in irregular gauge in each of the two bundles, the covariantly constant section $C^{i\hat{j}}$ reduces to the constant matrix $C^{\alpha\hat{\beta}}$

Map between Sections

- The formula
$$\phi^{\alpha^*} = C^{*\alpha^* \hat{\beta}^*} \hat{Q}_{\hat{\beta}^* \hat{\gamma}} \hat{\phi}^{\hat{\gamma}} \tag{6}$$
maps dual sections to (the conjugates of) usual sections. As Q and C are both covariantly constant, it follows immediately that if $\hat{\phi}^{\hat{\gamma}}$ obeys the dual Schrodinger equation, the complex conjugate of ϕ^{α^*} obeys the regular Schrodinger equation.
- The spin of the RHS of this map is $\hat{s}_i^{int} + \hat{h}_i$. The spin of the LHS equals $-\hat{s}_i^{int} - h_i$ (recall complex conjugation interchanges z and $\bar{z}$ and so flips spin). Consequently this map only works if $s_i^{int} + \hat{s}_i^{int} = -(h_i + \hat{h}_i) = -H_i - n_i$.

Invertibility of the map

- In matrix notation, the map above is

$$\phi^* = C^* \hat{Q} \hat{\phi} \tag{7}$$

Upon complex conjugating

$$\phi = C \hat{Q}^* \hat{\phi}^* \tag{8}$$

- Since original and level rank dual sections are on equal footing, we have a similar map from regular sections to the complex conjugate of dual sections.

$$\hat{\phi}^* = C^\dagger Q \phi \tag{9}$$

It is natural to expect (9) to be the complex conjugate of (8). This is the case provided

$$C \hat{Q}^* C^\dagger Q = I \tag{10}$$

Invertibility and the inner product

- One can also view (10) in the following terms. Assuming this equation is true we can use it to solve for $\hat{Q}$ in terms of C, $C^\dagger$ and Q. The RHS of this equation is clearly a covariantly constant matrix, in the product of dual sections and their complex conjugates. We independently konw that $\hat{Q}$ also has this property. A proof that sections with this property are unique would amount to a proof of (10)
- While we suspect that a proof of (10) exists somewhere in the literature, we have not been able to find it. For now we proceed conjecturing (10) holds, As some evidence for this conjecture, we have proved covariantly constant sections $\hat{Q}$ are indeed unique in the $SU(2)_k$ WZW theory, as well as for the special case of two fundamental and two antifundamental insertions in the general $SU(N)_k$ theory.
- Using (10), it is a simple matter to show that our map between Hilbert Spaces preserves the inner product, and therefore matrix elements of the Hamiltonian.

Interplay with exchange statistics

- Recall that any level rank pair of representations, R_i and $\hat{R}_i$, descends from the branching rule of some primary operator of $U(Nk)_1$ theory. Every such primary is a product of fermion fields. If the $U(N)_k$ primary is made up of an odd number of fermions, correlators of this primary are odd under interchange of two identical insertions. On the other hand, if the primary is built out of an even number of fermions, correlators of this primary are even under interchange of identical insertions.
- This can be used to show that our map between sections preserves statistics it the representations descend from an even number of fermions, but interchanges symmetry with antisymmetry if the representations descend from primaries with an odd number of Fermions. Restated, our map between Hilbert spaces is either of the Bose-Bose, Fermi-Fermi sort, or of the Bose-Fermi, Fermi-Bose sort, depending on gauge reps.

Parity

- One irritation in the construction above, is that our map takes sections to complex conjugate sections. Conjugate sections carry the opposite fluxes as compared to ordinary sections (can be seen from the KZ connection). This can be undone by performing a parity transformation, which flips the sign of the KZ connection. This parity tranfromation flips all spins. After performing this flip, the map between spins becomes

$$s_i^{int} = \hat{s}_i^{int} - H_i - n_i \tag{11}$$

  where the hatted side is assumed to have negative level.
- In summary, duality works when the participating particles have equal masses, transoform in level rank dual representations, and when
  1) The intrinsic spins of the particles are related as above
  2) The duality flips Bose/Fermi statistics if and only if H_i is a half (as opposed to full integer).

Comparison with UV dualities

- Consider $SU(N)$ theory coupled to fundamental Bosons/Fermions. These conjectured dualities have two massive phases. In the first, the level of the fermionic side has the same sign as its mass. In this phase the intrinsic spin of the bosons is zero, and the intrinsic spin of the fermions $-\frac{m_F}{2}$.
- The second phase occurs when the level of the fermions has the opposite sign as its mass. In this phase the Higgs mechansm turns the Bosons into vectors with spin $-\mathrm{sgn}(\mathrm{k_B}) = \mathrm{sgn}(\mathrm{k_F})$. The fermion continues to have spin equal to $-\frac{m_F}{2}$.
- In this exaple $H_i = \frac{1}{2}$ and $n_i = 0$. Easy to check that the relations beween spins does obey (11) in both phases. Also, of course, the relation between statistics. Similar agreement in the case of every conjectured matter Chern Simons duality. Our analysis can be thought of as first flow upwards to these UV dualities.

No UV dualities with large representations

- When going through the list of conjectured UV dualities involving matter CS theories, one is struck by the following fact.
- Each of these dualities have matter in small representations of the gauge group. One finds the fundamental, adjoint, two box symmetric .... . However there are no conjectured dualities invovling matter in, e.g. n boxes the first row of the Young Tableuax, with $n > 4$. One might wonder why this is the case
- (11) offers an explanation. If $H_i + n_i > 2$ then atleast one of $s_i^{\rm int}$ and $\hat{s}_i^{int}$ would have to have a spin of modulus greater than one. Field theories of this sort do not exist (or atleast have not yet been understood). This gives an explanation for this observation.

Summary and Future directions

- Modulo establishing (10) in generality, we have presented a complete proof of duality between level rank dual pairs of Q Mech. Perhaps first step in flowing 'upwards' from the well understood level rank duality of topological theories to a proper understanding of dualities of UV QFTs
- To complete our work, it would be nice to find a clear proof of (10). Its possible that such a proof exists in the literature and just needs to be located.
- Would be interesting to try to solve our Schrodinger equations in simple contexts (two particles, four particles etc). Strong weak coupling duality described above may prove useful here, as it relates naively strongly coupled quantum mechanics to a nearly free theory.
- It would be interesting to rederive the duality of quantum mechanics using a path integral approach. Peraps such a derivation could generalize to field theory, for instance, in world line formalism?