Why do classical, quantum, or hybrid trajectories satisfy linear master equations?

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Abstract

It is shown that linearity of classical/quantum/hybrid ensemble dynamics follows from bases of statistics. Hybrid classical--quantum trajectories and their hybrid master equations are discussed. We stress that the interaction between a classical and a quantum subsystem requires monitoring the quantum subsystem because its action on the classical subsystem can only be realized by the emerging classical signal.

Random trajectories: classical, quantum, or hybrid

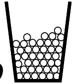
Stochastic processes (trajectories) in different spaces:

classical	$X_t \in R^n$	– – – – –
quantum	$\Psi_t \in \mathcal{H}$	————————
hybrid	$(X_t, \Psi_t) \in (R^n \times \mathcal{H})$	– – – – – ————————

Ensemble statistical interpretation

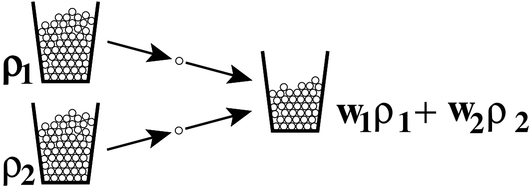
ρ: probability distribution/density matrix/hybrid density

Urn model (visualization):

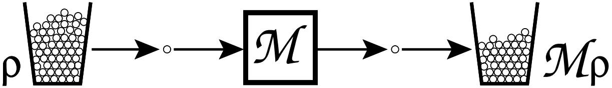


$$\rho \quad \{\mathbf{o}\} \;=\; \{X\} \text{ or } \{\Psi\} \text{ or } \{(X,\Psi)\}$$

Mixing, with weights w_1, w_2:



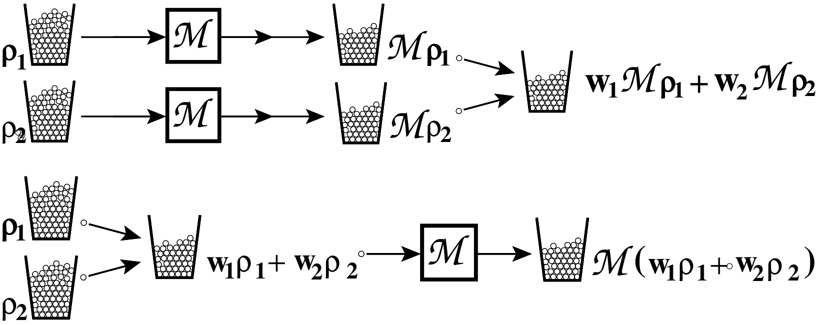
Norm & positivity conserving map $\mathcal{M}$, not necessarily linear:



But really, could it be nonlinear?

Interchangeability of operation & mixing

Mandatory feature of statistical interpretation:



$$w_1\mathcal{M}\rho_1 + w_2\mathcal{M}\rho_2 = \mathcal{M}\left(w_1\rho_1 + w_2\rho_2\right)$$

Nonlinearity invalidates statistical interpretation. Applying the (invalid!) statistical interpretation is illegitimate. If you still apply it, get strange things (e.g.: superluminality, most famously).

Classical examples and counterexample

Liouville/Hamilton:

$$
\begin{aligned}
\dot\rho(q,p) &= \{H(q,p),\rho(q,p)\}_{\text{Poisson bracket}} \\
\dot q &= \partial_p H(q,p), \quad \dot p = -\partial_q H(q,p)
\end{aligned}
$$

Fokker–Planck/Langevin:

$$
\begin{aligned}
\dot\rho(q,p) &= \left(D\partial_p^2 - (p/m)\partial_q + \eta\partial_p p\right)\rho(q,p) \\
\dot q &= p/m, \quad \dot p = -\eta p + \sqrt{2D} w_t
\end{aligned}
$$

Boltzmann/???:

$$
\begin{aligned}
\dot\rho(q,p) &= -(p/m)\partial_q\rho(q,p) + \dot\rho(q,p)|_{\text{collision}} \\
\dot q &= ??? \quad \dot p = ???
\end{aligned}
$$

Quantum examples and counterexample

Lindblad/QSD trajectory):

$$\begin{aligned}\dot{\hat\rho} &= -i[\hat H,\hat\rho] + \hat L\hat\rho\hat L^\dagger - \mathrm{Herm}\hat L^\dagger\hat L\hat\rho\\ \dot\Psi &= -i\hat H'\Psi - \tfrac{1}{2}(\hat L - \langle\hat L\rangle)^\dagger(\hat L-\langle\hat L\rangle)\Psi + (\hat L-\langle\hat L\rangle)\xi_t\Psi\end{aligned}$$

$\hat x$-decoherence—Brownian motion/wf localization

$$\begin{aligned}\dot{\hat\rho} &= -i[\hat H,\hat\rho] - \tfrac{1}{8}\gamma[\hat x,[\hat x,\hat\rho]]\\ \dot\Psi &= -i\hat H\Psi - \tfrac{1}{8}\gamma(\hat x - <\hat x>)^2\Psi - \tfrac{i}{2}\sqrt{\gamma}\hat x w_t\Psi\\ \dot\Psi &= -i\hat H\Psi - \tfrac{1}{8}\gamma(\hat x - <\hat x>)^2\Psi + \tfrac{1}{2}\sqrt{\gamma}(\hat x - <\hat x>)w_t\Psi\end{aligned}$$

Nonlinear Schrödinger–Newton equation:

$$\begin{aligned}\dot{\hat\rho} &= ???\\ \dot\Psi &= -i\left(\hat H + V^\Psi(\hat x)\right)\Psi\end{aligned}$$

Hybrid prototype

SME of $\hat{x}$-monitoring with outcome signal $x_t = \dot{X}_t$

$$\begin{aligned}\dot{\hat{P}} &= -i[\hat{H},\hat{P}] + \tfrac{1}{8}[\hat{x},[\hat{x},\hat{P}]] + \sqrt{\gamma}\mathrm{Herm}(\hat{x}-\langle\hat{x}\rangle)\hat{\rho}w_t \\ \dot{X} &= \langle\hat{x}\rangle + w_t/\sqrt{\gamma}\end{aligned}$$

Monitoring yields hybrid trajectory $(X_t, \hat{P}_t)$ — the prototype!
Hybrid density:

$$\hat{\rho}_t(X) := \mathrm{Mean}\hat{P}_t\delta(X - X_t)$$

$\rho(X) = \mathrm{tr}\hat{\rho}(X)$ and $\hat{\rho} = \int\hat{\rho}(X)dX$ and $\hat{\rho}|_X = \hat{\rho}(X)/\rho(X)$
HME follows from SME:

$$\dot{\hat{\rho}}(X) = \tfrac{1}{8}\gamma[\hat{x},[\hat{x},\hat{\rho}(X)]] - \hat{x}\partial_X\hat{\rho}(X) + \tfrac{1}{2}\gamma^{-1}\partial_X^2\hat{\rho}(X)$$

General hybrid dynamics

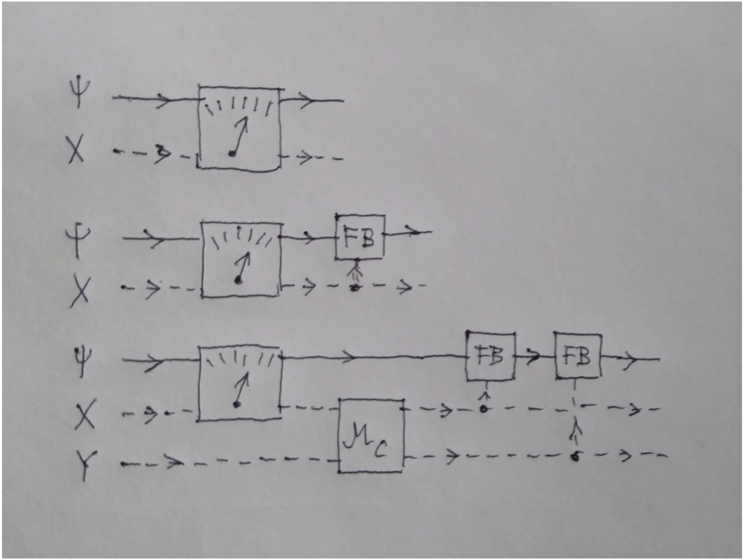
=coupled dynamics of classical (not necessarily dynamical) and quantum subsystems, resp.

About an individual quantum system the random measurement outcomes are the only true classical variables, legitimate to couple to other classical variables.

Accordingly:

- Dynamical action of the quantized subsystem on the classical subsystem is only possible via the stochastic classical signal of quantum monitoring of some quantum observable(s)
- Monitoring implies decoherence of the quantum subsystem. Signal noise implies diffusion in the classical subsystem. Hybrid dynamics is irreversible.

Hybrid dynamics flowcharts



Summary

1) Linearity of classical/quantum/hybrid dynamics follows from bases of statistics

2) SME and HME formalisms of quantum monitoring are equivalent.

3) Hybrid dynamics contains quantum monitoring, may contain self-dynamics of the signal, signal's coupling to further classical subsystems, feedback controlled by the classical variables.

4) Main works: D, Tilloy, Oppenheim group