Basic homotopy theory

Study topological spaces up to homotopy equivalence.

Abstract Homotopy theory

Realize that this is applicable in other categories

Study categories with special kinds of equivalences.

Homotopy

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y \end{tikzcd}$$

A homotopy

$$\begin{tikzcd} X \times 0 \arrow[d] \arrow[dr, "f"] & \\ H : X \times [0,1] \arrow[r] & Y \\ X \times 1 \arrow[u] \arrow[ur, "g"'] & \end{tikzcd}$$

$[X, Y]$ = set of homotopy classes of maps

Top : category of top. spaces

Top_* : category of based spaces

$[\ ,\ ]_*$: Based htpy classes

Homotopy equiv.

$X \xrightarrow{f} Y$ is a htpy equiv.

if $\exists$ $g : Y \to X$ s.t.

$$g \circ f \simeq id_X$$

$$f \circ g \simeq id_Y .$$

Homotopy gps $\quad X \in ob(Top_*)$

$$\pi_n(X) = [S^n, X]_* \quad n \geq 0.$$

$\pi_0(X)$ = set of path components of X

$\pi_1(X)$ = fundamental gp

$\pi_n(X)$: Abelian gp if $n \geq 2$.

CW complex : Space X

$+$

filtration $X = \bigcup_{k \geq 0} X^{(k)}$

$X^{(k)} \subseteq X^{(k+1)}$ & $X^{(k+1)}$ is obtained out of $X^{(k)}$ by attaching $(k+1)$-cells.

$$\begin{tikzcd} \coprod_{\alpha \in I} \partial D^{k+1}_\alpha \arrow[r] \arrow[d] & \coprod_{\alpha \in I} D^{k+1}_\alpha \arrow[d] \\ X^{(k)} \arrow[r, hook] & X^{(k+1)} \end{tikzcd}$$

pushout

Back to htpy gps

- X CW cx & $Y \xrightarrow[p]{} X$ covering space then,

$$\pi_k(Y) \cong \pi_k(X) \text{ if } k \geq 2.$$

- $A \subseteq X$

$$\pi_n(X,A) = \left[(S^n, S^n_+, *), (X, A, *) \right]$$

$$\pi_n(X,A) \begin{cases} \text{set} & n \leq 1 \\ \text{gp} & n = 2 \\ \text{Abelian} & n \geq 3. \end{cases}$$

- Long exact seq

$$\cdots \pi_k(A) \to \pi_k(X) \to \pi_k(X,A) \to \pi_{k-1}(A) \to \cdots$$

$$\pi_k(S^n) = 0 \qquad k<n$$

$$\pi_n(S^n) \cong \mathbb{Z}$$

$$f \longmapsto \deg(f)$$

$\pi_{>n}(S^n)$: difficult to calculate

<u>Def</u>: A map $f: X \to Y$ is called a weak equivalence if it induces an iso_ on all htpy gps.

$$X \simeq Y$$

Theorem (Whitehead's thm)

A weak equiv between CW complexes is a homotopy equiv.

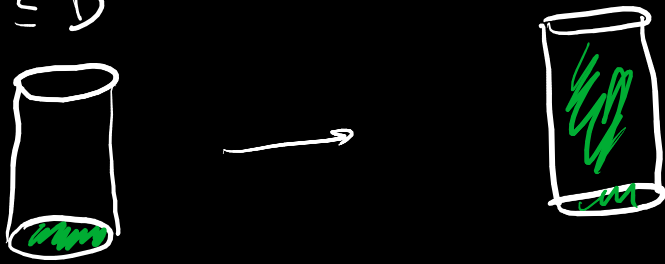
Cofibrations

$f: A \to X$ is called a cofibration if it has the homotopy ext$^{\underline{n}}$ property.

$$\begin{tikzcd}
A \times [0,1] \cup_f X \times 0 \arrow[d] \arrow[drr] & & \\
X \times [0,1] \arrow[rr, dashed, "\exists"'] & & Z
\end{tikzcd}$$

Example

$$S^n \subseteq D^{n+1}$$



$$S^n \times [0,1] \cup D^{n+1} \times 0 \longrightarrow D^{n+1} \times [0,1]$$

$X \subseteq Y$ subcomplex of CW cx

then the incl. $X \to Y$ is a cofibration.

Fibration

A map $p: Y \to Z$ is called a fibration if $\exists$ homotopy lifting in diagrams

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ S^n \times [0,1] \arrow[r] \arrow[ur, dashed, "\exists"] & Z \end{tikzcd}$$

[Serre fibration]

$\Rightarrow$ can replace S^ns by any CW cxs.

Example: Fibre bandle.

$$\begin{array}{c} Y \\ \downarrow p \\ Z \end{array}$$

fibration

Z path connected. $\Rightarrow$ All the fibres are htpy equiv.

$p^{-1}(*) = F$

$$\begin{array}{ccc} F & \to & Y \\ & & \downarrow \\ & & Z \end{array}$$

$$\boxed{\begin{array}{c} \pi_k(Y, F) \\ \cong \\ \pi_k(Z) \end{array}}$$

$\rightsquigarrow$ Long exact seq

$$\cdots \pi_k F \to \pi_k Y \to \pi_k Z \to \pi_{k-1} F \to \cdots$$

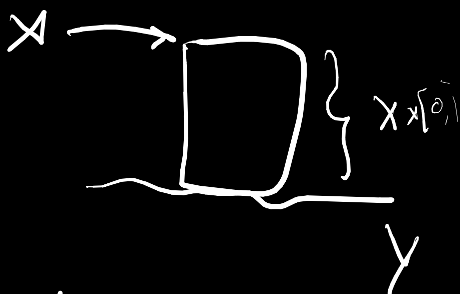
$$X \xrightarrow{f} Y$$

Mapping cylinder Mf $\qquad X \times [0,1] \sqcup Y \,/\, (x,1) \sim f(x)$

$$\begin{aligned} Mf &\longrightarrow Y \\ (x,t) &\longmapsto f(x) \\ y &\longmapsto y \end{aligned}$$

fibration + htpy equiv

$$\begin{aligned} X &\longrightarrow Mf \\ x &\longmapsto (x,0) \end{aligned}$$



$$X \longrightarrow Mf \longrightarrow Y$$

cofibration

fibration + $\simeq$

$$X \xrightarrow{f} Y$$

$$E(f) = \{ (x,\gamma) \in X \times \mathrm{Map}([0,1],Y) \mid \gamma(0) = f(x) \}$$

$$\begin{aligned} E(f) &\longrightarrow Y \\ (x,\gamma) &\longmapsto \gamma(1) \end{aligned}$$

fibration

$$\begin{aligned} X &\longrightarrow E(f) \\ x &\longmapsto (x, \mathrm{const}_{f(x)}) \end{aligned}$$

cofibration + $\simeq$.

fibration

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "f"'] & E(f) \arrow[r] & Y \end{tikzcd}$$

CW approximation

For every space X, $\exists$ CW cx. Y + map $Y \longrightarrow X$ which is a weak equivalence.

Consequence of

$$\left(\begin{array}{c}\text{CW approx} \\ + \\ \text{Whitehead}\end{array}\right) : \quad \mathrm{Top}/\simeq$$

is equiv to the category

$\mathcal{H}$: $\mathrm{ob}(\mathcal{H})$ = CW cxs.

$\mathrm{mor}(\mathcal{H})$ = htpy classes of maps

Cofibre sequence   Baratt - Puppe seq

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \longrightarrow \Sigma Y \to \cdots$$

$$C_f = Y \cup_f CX$$

The functor $[-, W]_*$

$$\begin{tikzcd} \cdots \arrow[r] & {[C_f, W]_*} \arrow[r] \arrow[rr, bend left, "*"] & {[Y, W]_*} \arrow[r] & {[X, W]_*} \end{tikzcd}$$

$$\exists \longmapsto \alpha \longmapsto *, \quad \alpha \in [Y,W]_*$$

Fibre sequences

htpy fibre = fibre of $E(f) \to Y$

$$\cdots \to \Omega X \to \Omega Y \to F \to X \xrightarrow{f} Y$$

$[W, -]_*$: gives a long exact seq of pointed sets.