Model Categories.

Topological spaces $\leftarrow$ study upto homotopy.

Naive Homotopy category of Topological spaces whose objects are top. spaces

$$\mathrm{mor}(X,Y) = [X,Y] = \mathcal{F}(X,Y)/\sim$$

Want to study top. space upto up to w. homotopy equivalences

$X \xrightarrow{f} Y$ is w.eq iff $\pi_*(X) \xrightarrow{\cong} \pi_*(Y)$

Ch_R : Chain complexes of R-modules (R assoc. ring)

$\{M_k, \partial\}$

mor $\{M_k\} \xrightarrow{f_k} \{N_k\}$

$$\begin{tikzcd} M_k \arrow[r, "f_k"] \arrow[d, "\partial"'] & N_k \arrow[d, "\partial"] \\ M_{k-1} \arrow[r, "f_{k-1}"'] & N_{k-1} \end{tikzcd}$$

We can study Ch_R upto chain homotopy equivalence

Homotopy categ of Ch_R obj Chain cplx

$mor(M,N) = Ch_R(M,N) / \sim$

often we want study Ch_R up to quasi isomorphism.

i.e maps which induce isomorphisms on Homology of the chain complexes

1) Want to localize the category of Top spaces wrt weak h-eq

2) Want to localize the Categ. Ch_R (non-negatively graded) up to quasi isomorphism.

One solution to this problem is offered by Model Categories

Defn: A model category is category M with three distinguished classes of morphisms, called weak eq. W, cofibrations C, and fibrations F. satisfying the following the axioms.

1) Finite limits and finit. colimits exist.

2). If 2 out of $f, g, fg \in W \Rightarrow$ the third one belongs to W.

3) All the classes are preserved under retracted.

4)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

a) The dotted arrow exists if $i \in C$ and $p \in F \cap W$.

b) The dotted arrow exists if $i \in C \cap W$ and $p \in F$.

5). Given any map $f: X \to Y$ in M

it can be factored as a map $i \in \mathcal{C} \cap W$ and $p \in \mathcal{F}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "i"'] & & Y \\ & X' \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

and also as

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\mathcal{C}"'] & & Y \\ & Y' \arrow[ur, two heads, "\in \mathcal{F} \cap W"'] & \end{tikzcd}$$

Conseq. 1) A morphism in M is a fibration iff it has Right lifting property with respect to all acyclic cofibrations (cofib + w.eq)

2) A morphism is a cofibration iff it LLP wrt all acyclic fibrations (fibr + w.eq)

This set up allows us to define a notion of homotopy.

$Ho(M)$ – obj objects of M

morphism $\underline{(X, Y) := [X_{cf}, Y_{cf}]}$

$$\begin{tikzcd} M \arrow[r, "r"] \arrow[dr, "F"'] & Ho(M) \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd}$$

$\mathcal{D} \in$ categ

weq $\longrightarrow$ isomorphisms.

<u>Examples</u>: Topological spaces

W = homotopy equivalence

F = $f \in F$ if it has Homotopy lifting property wrt all space.

$$\begin{tikzcd} A\{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ A\times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$\leftarrow$ Hurewicz fibration

C = $f \in C$ if it has HEP + closed inclusion

$$\begin{tikzcd} A\times\{0\} \arrow[r] \arrow[d] & B\times\{0\} \arrow[d] \arrow[dr] & \\ A\times I \arrow[r] \arrow[rr, bend right] & B\times I \arrow[r, dashed] & X \end{tikzcd}$$

is the extension property

Strom proved this defines a model category and the homotopy category is same as what we called the naive homotopy categ.

Alternate Top spaces $\mathcal{W} :=$ weak h. equiv

$\mathcal{F} :=$ Serre fibration

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & X \arrow[d] \\ D^n \times I \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \quad \forall n.$$

$$D^n = \{ x \in \mathbb{R}^n \,/\, \|x\| \leq 1 \}$$

$$S^{n-1} = \{ x \in \mathbb{R}^n \,/\, \|x\| = 1 \}$$

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "i"] & Y' \arrow[r, "r"] & X' \end{tikzcd}$$

g is retract of f

$\mathcal{C} :=$ retracts of generalized CW complexes

This forms a Model Category.

$$Ho(\mathcal{T})(A, X) \simeq [A, X]$$

A is CW complex

(3) Ch_R (non negatively graded) W: quasi iso.

Cofibr: $0 \to M \to N \to$ cokernel

$\uparrow$ proj R-module

fibr: epimorphisms

This is a model category and $Ho(Ch_R)$ equivalent to the category of chain complexes of proj R-modules / chain homotopy equivalence

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$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \quad \xrightarrow{\text{h-eq diagram}} \quad \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

$$* \not\xrightarrow{h} S^n$$

Pushouts do not preserve w-eq of diagrams

In a model categ set up we can formally define what we call homotopy colimits and holimits which are preserved under weq. diagrams.

$\longrightarrow$ Let M be a model categ with ϕ as initial obj & $*$ as the terminal obj

$$\begin{tikzcd} \phi \arrow[r] \arrow[dr, "\text{cofib}"'] & X \\ & X_c \arrow[u, two heads, "\sim"'] \end{tikzcd}$$

X_c is called cofibrant replacement

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\sim"'] & * \\ & X_f \arrow[u, two heads, "\text{fibration}"'] \end{tikzcd}$$

X_f is called fibrant replacement

$$\mathrm{Ho}\mathscr{C}(X, Y) := [X_{cf}, Y_{cf}]$$

In $\mathcal{M}$ if cofibrations

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & \sqcup \end{tikzcd}$$

cofibrations

cofibrant

pushout

will be preserved under weakly eq. diagrams.

$$Z \leftarrow X \rightarrow Y$$

homotopy pushout

Quillen pair of Model categories.

Adjoint pair of functors

$$\mathcal{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{N}$$

X obj of $\mathcal{M}$ & Y is a obj of $\mathcal{N}$.

$$\mathcal{M}(X, RY) \cong \mathcal{N}(LX, Y)$$

if L preserves cofibr & acyclic cofibr.

or if R preserves fibr & acyclic fibr

This induces a adjoint pair.

$$Ho(M) \rightleftarrows Ho(N).$$

If (L, R) give a Quillen equivalence. $X \xrightarrow{\sim} RLX$ is a w.eq for all cofibrant object

or equivalently $LRY \xrightarrow{\sim} Y$ is true for all fibrant weq.

induces a equivalences of Homotopy category.