

Model Categories.

Topological spaces \xrightarrow{y} study upto homotopy.

Naive Homotopy Category of Topological spaces whose objects are top. spaces

$$\text{mor}(X, Y) = [X, Y] = \frac{\mathcal{F}(X, Y)}{\sim}$$

Want to study top. space upto up to w. homotopy equivalences

$$X \xrightarrow{f} Y \text{ is w. eq iff } \pi_*(X) \xrightarrow{\cong} \pi_*(Y)$$

Ch_R : Chain complexes of R -modules
(R assoc. ring)

$$\text{mor } \{M_k\} \xrightarrow{f_k} \{N_k\}$$

$$\begin{array}{ccc} M_k & \xrightarrow{f_k} & N_k \\ \partial \downarrow & & \downarrow \partial \\ M_{k-1} & \xrightarrow{f_{k-1}} & N_{k-1} \end{array}$$

We can study $Ch_{\mathbb{R}}$ upto chain homotopy equivalence

Homotopy Catg of $Ch_{\mathbb{R}}$ obj Chain complex
 $\text{mor}(M, N) = \frac{Ch_{\mathbb{R}}(M, N)}{\sim}$

often we want study $Ch_{\mathbb{R}}$ up to quasi isomorphism.

ie maps which induce isomorphisms on Homology of the chain complexes

- 1) Want to localize the category of Top spaces w/ot weak h-eq.
- 2) Want to localize the Categ. $Ch_{\mathbb{R}}$ up to quasi isomorphism. (non-negatively graded)

One solution to this problem is

offered by Model Categories

Defn: A model category is a category

M with three distinguished classes of morphisms, called weak eq., cofibrations, and fibrations, satisfying the following the axioms.

1) Finite limits and finite colimits exist.

2) If 2 out of $f, g, fg \in W \Rightarrow$ the third one belongs to W .

3) All the classes are preserved under retracts.

4)
$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow i & \dashrightarrow & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$
 a) The dotted arrow exists if $i \in C$ and $p \in F \cap W$.

b) The dotted arrow exists if $i \in C \cap W$ and $p \in F$.

5) Given any map $f: X \rightarrow Y$ in M

it can be factored as a map
 $i \in \mathcal{E} \cap \mathcal{W}$ $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \searrow^{i'} & & \nearrow^p \\ & X' & \end{array}$ and $p \in \mathcal{F}$

and also as $\mathcal{E} \begin{array}{ccc} X & \rightarrow & Y \\ \searrow^{i'} & & \nearrow^p \\ & X' & \end{array} \in \mathcal{F} \cap \mathcal{W}.$

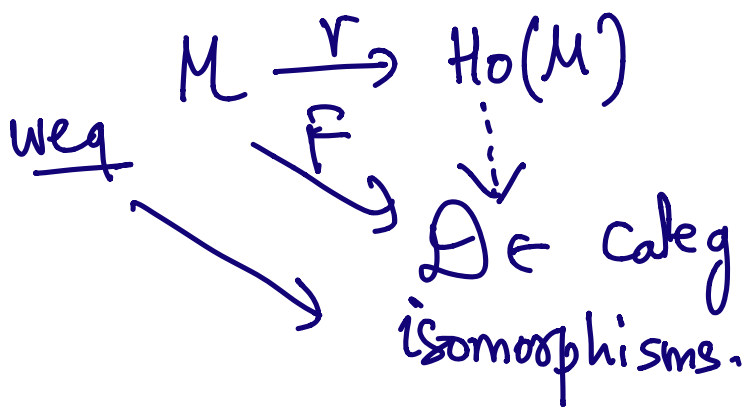
Conseq.!) A morphism in \mathcal{M} is a fibration
 iff it has Right lifting property
 with respect to all acyclic
 cofibrations (cofib + w.eq)

2) A morphism is a cofibration iff
 it LLP w/ all acyclic fibrations
 (fibr + w.eq)

This set up allows us to define
 a notion of homotopy.

$\text{Ho}(\mathcal{M})$ - obj objects of \mathcal{M}

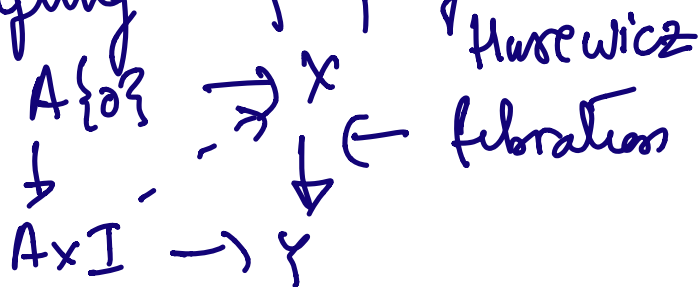
$\underline{\underline{=}}$ morphism $(X, Y) := [X_{\text{cof}}, Y_{\text{cof}}]$



Examples: Topological spaces

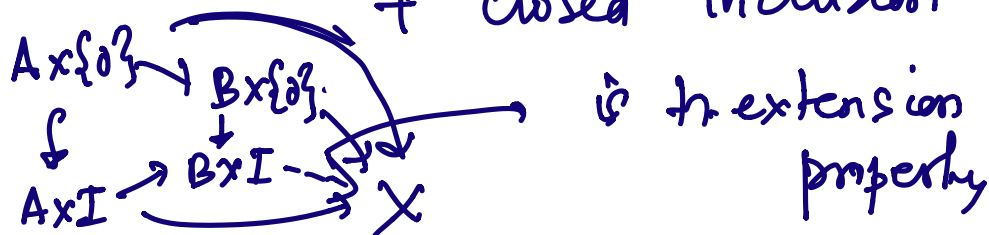
$W =$ homotopy equivalence

$F = f \in F$ if it has homotopy lifting property w/ all space.



$C = f \in C$ if it has HEP

+ closed inclusion



Strom proved this defines a model category and the homotopy category is same as what we called the naive homotopy caty.

Alternate Top spaces $W :=$ weak h. equiv

$\mathcal{F} :=$ Serre fibration

$$\begin{array}{ccc} D^n & \rightarrow & X \\ \downarrow & \twoheadrightarrow & \downarrow \\ D^n \times I & \rightarrow & Y \end{array} \quad \neq n.$$

$$D^n = \{ x \in \mathbb{R}^n / \|x\| \leq 1 \} \quad \neq \downarrow \circlearrowleft \downarrow \circlearrowleft \downarrow$$

$$S^{n-1} = \{ x \in \mathbb{R}^n / \|x\| = 1 \} \quad \neq \downarrow \circlearrowleft \downarrow \circlearrowleft \downarrow$$

$\mathcal{C} :=$ Retracts of generalized CW complexes

This forms a Model Category.

$$\text{Ho}(\mathcal{F})(A, X) \cong [A, X]$$

\uparrow
A is CW complex

(3) Ch_R W : quasi iso.
 (non negatively graded) $\text{Cofiber} : 0 \rightarrow M \rightarrow N \rightarrow \text{cokernel}$
 \uparrow
 $\text{proj } R\text{-mod}$

$\text{fibr} : \text{epimorphisms}$

This is a model category and
 $\text{Ho}(\text{Ch}_R)$ equivalent to the
 category of chain complexes of
 $\text{proj } R\text{-modules}$

chain homotopy
equivalence.

$$\begin{array}{ccc}
 S^{n-1} & \longrightarrow & * \\
 \downarrow & \searrow \cong & \downarrow \\
 * & \longrightarrow & *
 \end{array}$$

h-eq
diagram

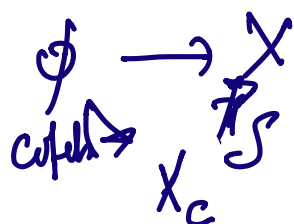
$$\begin{array}{ccc}
 S^{n-1} & \longrightarrow & D^n \\
 \downarrow & & \downarrow \\
 D^n & \longrightarrow & S^n
 \end{array}$$

$$* \xrightarrow{w} S^n$$

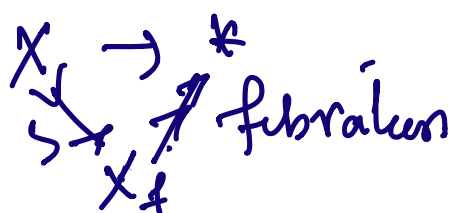
pushouts do not
 w-eq of preserve
 diagrams

In a model category set up we can formally define what we call homotopy colimits and homotopy limits which are preserved under weak-equivalences.

→ Let M be a model category with \emptyset as initial object and $*$ as the final object



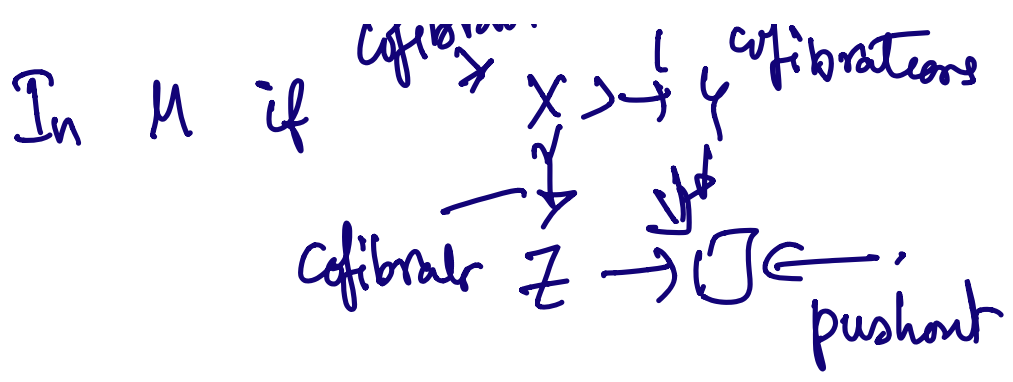
X_c is called cofibrant replacement



X_f is called fibrant replacement

$$\text{Ho} \mathcal{C}(X, Y) := [X_{cf}, Y_{cf}]$$

... ..



$Z \leftarrow X \rightarrow Y$
 \downarrow
 homotopy pushout

will be preserved under weakly eq. diagrams.

Quillen pair of Model categories.

Adjoint pair of functors

$$M \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} N$$

X obj of M & Y is a obj

$$M(X, RY) \cong N(LX, Y)$$

if L preserves cofibr & acyclic cofibr.

or if R preserves fibr & acyclic fibr

This induces a adjoint pair $\text{Ho}(\mathcal{M}) \rightleftarrows \text{Ho}(\mathcal{N})$.

If (L, R) give a Quillen

equivalence $X \xrightarrow{L} RLX$

is a w.eq for all cofibrant objects

or equivalently $LR Y \xrightarrow{R} Y$

is true for all fibrant objects

induces a equivalence

Homotopy category.