Stable homotopy theory

Poincaré duality: M closed, oriented n-manifold

$$\Rightarrow \quad H_k(M) \cong H^{n-k}(M).$$

Alexander duality: X closed subset of S^n

$$\tilde{H}_k(S^n - X) \cong \tilde{H}^{n-k-1}(X)$$

Objective: These theorem are part of more general results about "dual objects" in "stable homotopy theory"

Last time: defined homotopy groups.

S^1 $$\pi_k(S^1) = \begin{cases} \mathbb{Z} & k=1 \\ 0 & k \geq 2 \end{cases}$$

$$\begin{tikzcd} \mathbb{R} \arrow[d, "\text{cov space}"] \\ S^1 \end{tikzcd}$$

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d, "\eta"] \\ & S^2 \end{tikzcd} \Rightarrow \pi_k(S^2) \cong \pi_k(S^3) \text{ if } k \geq 3$$

$$\pi_2 S^2 \cong \mathbb{Z}.$$

η: Hopf map

Homology : ⇝ excision

Suspension iso -

$$\tilde{H}_{k+1}(\Sigma X) \cong \tilde{H}_k(X).$$

But it is not true that

$$\pi_{k+1}(\Sigma X) \cong \pi_k(X).$$

<u>Def</u> : X is said to be k-conn if $\pi_i(X) = 0 \quad \forall \ i \leq k$

<u>Freudenthal suspension thm</u>

X (k-1)-conn.

$$\pi_r(X) \xrightarrow{\Sigma} \pi_{r+1}(\Sigma X) \text{ is } \begin{cases} \cong \text{ if } r \leq 2k-2 \\ \text{surj } r = 2k-1 \end{cases}$$

Stable homotopy groups

$$\pi_k^s(X) = \varinjlim_n \pi_{k+n}(\Sigma^n X)$$

$$S^{k+n} \to \Sigma^n X \quad \overset{}{\underset{\Sigma}{\rightsquigarrow}} \quad S^{k+n+1} \to \Sigma^{n+1} X$$

$k \geq 0$

Fr. susp. thm $\Rightarrow$ the maps in $\varinjlim$ are isos after a finite stage.

$\pi_k^s(S^0)$

$k=0$ $\quad \pi_n S^n \quad n \gg 0$

$\cong \mathbb{Z}$

$k=1$ $\quad \pi_{n+1} S^n \quad n \gg 0$

$\cong \mathbb{Z}/2$

gen by $\Sigma^{n-2}\eta$.

Stable htpy classes of maps.

$$\{X, Y\} = \varinjlim_{n} [\Sigma^n X, \Sigma^n Y]_*$$

Spanier Whitehead category

$\mathcal{SW}$: objects : CW complexes

Maps : $\{-,-\}$

Example:

$\eta : S^3 \to S^2$

$\Sigma^2 \eta \in \{S^3, S^2\}$

$$\boxed{\eta \in \{S^1, S^0\}}$$

$C\eta = ?$

$$\begin{array}{c} C\Sigma^2\eta \\ \parallel \\ \mathbb{CP}^2 \end{array} \simeq \Sigma^2 C\eta$$

$C\eta$ does not exist in $\mathcal{SW}$.

Def: A spectrum X is a sequence of spaces $\{X_n\}$ + structure maps $\Sigma X_n \longrightarrow X_{n+1}$

Maps between spectra:

A map $X \to Y$ is a sequence of maps $X_n \to Y_n$ which commutes with the str. maps.

LATER: $\{-,-\}$.

Def: X space

$\Sigma^\infty X$: suspension spectrum of X

$(\Sigma^\infty X)_n = \Sigma^n X$ & the str. maps are identity.

$$\begin{tikzcd} \mathrm{Top}_* \arrow[r, shift left, "\Sigma^\infty"] & \text{Spectra} \arrow[l, shift left, "\Omega^\infty"] \end{tikzcd}$$

$\Omega^\infty : Y \longmapsto Y_0$.

Σ^∞ is left adjoint to Ω^∞.

Defn: Homotopy gps.

X spectrum

$$\pi_k(X) = \varinjlim_n \pi_{k+n}(X_n)$$

- Are defined for $\forall\ k \in \mathbb{Z}$.
- $\pi_k(\Sigma^\infty X) \cong \pi_k^s(X)$.

  (0 if $k<0$.)

Def: A weak equivalence of spectra is a map which induces an iso- on all π_k.

Example Eilenberg MacLane spectra.

A abelian

$K(A,n)$ satisfies

$$\pi_r(K(A,n)) = \begin{cases} 0 & r \neq n \\ A & r = n \end{cases}$$

$$K(A,n) \simeq \Omega K(A,n+1)$$

$$\rightsquigarrow \quad \Sigma K(A,n) \to K(A,n+1)$$

$\rightsquigarrow$ HA s.t. $HA_n = K(A,n)$

$$\pi_k HA = \begin{cases} A & k = 0 \\ 0 & k \neq 0 . \end{cases}$$

X spectrum: ΣX by

$$(\Sigma X)_n = \Sigma X_n$$

$\Sigma^{-1} X$: $(\Sigma^{-1} X)_n = X_{n-1}$

$$(\Sigma^{-1} X)_0 = *$$

$$\Sigma \Sigma^{-1} = \Sigma^{-1} \Sigma$$

$$\Sigma \Sigma^{-1} X \longrightarrow X$$

$$\left[n : \Sigma X_{n-1} \xrightarrow{\qquad} X_n \right]$$

$\nwarrow$ str map

is a weak equiv. of spectra

Mappingcones : defined levelwise

Def: X is said to be an Ω-spectrum if the adjoint of the structure maps

$$X_n \longrightarrow \Omega X_{n+1}$$

are weak equiv of spaces.

Examples: HA

$$\Omega^2 \mathbb{Z} \times BU \simeq \mathbb{Z} \times BU \text{ [Bott periodicity]}$$

$$\mathbb{Z} \times BU,\ \Omega(\mathbb{Z} \times BU),\ \mathbb{Z} \times BU, \dots .$$

KU : Complex K-thy

Homotopy classes of maps

X, Y spectra $\quad \underset{\text{space}}{\mathrm{Map}(X,Y)}$

Y is an Ω-spectrum:

$$\{X, Y\} := \pi_0 \mathrm{Map}(X,Y).$$

Thm: For every spectrum X, $\exists$ Ω-spectrum LX + map $X \longrightarrow LX$ which is a weak equiv.

Cofibre seq

$$X \xrightarrow{f} Y \to Cf \to \Sigma X \to \cdots$$

$\{-, W\}$ ⇝ long exact sequences.

$\boxed{\{W, -\} \to \text{long exact sequences.}}$

π_k^s : homology theory

Smash product of spectra

X, Y spectra

$$(X \wedge Y)_n \sim \bigvee_{0 \leq k \leq n} X_k \wedge Y_{n-k}$$

Prodblem : defining str maps

Function spectrum

X, Y spectra

$$F(X, Y)_n \sim \mathrm{Maps}(X, \Sigma^{-n} Y)$$

similar problem

THEOREM: There is a category Quillen equiv to the category of spectra which is closed symmetric monoidal recovering the homotopical smash product & function spectrum described here.

The unit object is $\mathbb{S} = \Sigma^\infty S^0$.

symmetric spectra / orthogonal spectra

- Construction using operads.