# Duality in higher categories 1

Pranav Pandit

ICTS Program : "Dualities in algebra and topology".

February 6, 2021

## Outline :

Introductory lecture

1. Preliminaries : conventions, "homotopical language".

2. Examples of dualities. (What *is* duality?)

3. Formulate our goals for the mini-course : a (higher) categorical framework for duality.

## 1. Conventions / Language.

By default, we work

(a) Homotopy theoretically
(everything is derived)

(b) Model independently
(don't specify set-theoretical models).

(c) "Algebra $\subseteq$ higher algebra"
(viewing ordinary rings / modules as discrete ring spectra).

### (a) Homotopy theory.

A "homotopy theory" can mean

(i) $(\mathcal{C}, W)$, $\mathcal{C}$ category (relative cat)
$W \subseteq \mathcal{C}$ sub-cat (weak eq.)

(ii) A category $\mathcal{C}$ enriched in Top / sSet
(a topological / simplicial category).
$X, Y \in Ob(\mathcal{C})$, $Map_{\mathcal{C}}(X,Y) \in$ Spaces

(iii) A quasi-category $\mathcal{C}$ (Rekha's talk)

(iv) A complete Segal space : (later)

etc. (other models for $(\infty,1)$-cat)

All these are "equivalent" notions.

(i) $\longleftrightarrow$ (ii)

Dwyer-Kan simplicial localization.

$(\mathcal{C}, W) \rightsquigarrow L_W\mathcal{C} \in$ simplicial categories

$\mathcal{C}$ rel-cat $\qquad + \ \pi : \mathcal{C} \to L_W\mathcal{C}$

s.t.
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] \arrow[dr, "F"'] & L_W\mathcal{C} \arrow[d, dotted, "\exists !"] \\ & \mathcal{X} \end{tikzcd}$$

Dwyer Kan simplicial localization.

$\forall$ $\mathcal{X}$ simplicial category

$\forall$ $F$ s.t $F(w)$ is invertible in $h\mathcal{X}$. $\forall$ $w \in W$

$hL_W\mathcal{C} \simeq \mathcal{C}[W^{-1}]$

$h\mathcal{X}$ ordinary category

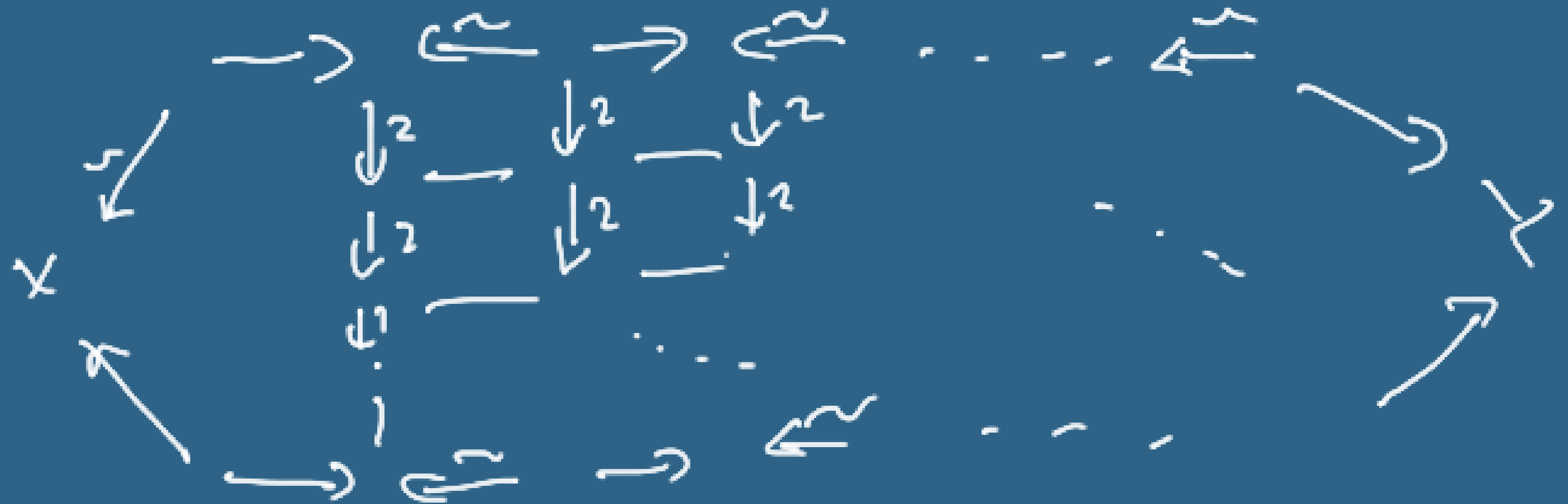
$Ob(h\mathcal{X}) = Ob(\mathcal{X})$

$Hom_{h\mathcal{X}}(X, Y) = \pi_0 Map_{\mathcal{X}}(X, Y)$.

Rmk: Can also formulate this using Top-enriched categories.

Rmk: Explicit model: Hammock Localization $L^H_W\mathcal{C}$

k-simplices in $Map_{L^H_W\mathcal{C}}(X, Y)$ rep. by equiv classes of hammocks



Columns have length k, $\xleftarrow{\sim} \in W$

"Working homotopy theoretically"

**Slogan**: "Equality is bad",
remember isomorphism / path
witnessing the equivalence of two
~~entities~~ objects.

- Replace Sets by Spaces everywhere.

- Pass to $\pi_0$ only in the "final step".

**Example**: $(X, x) \in \mathrm{Spaces}_*$

$$* \xrightarrow{x} X$$

**Naive pullback**:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

**Problem**: Does not preserve
weak equivalences

$\{\text{paths } \gamma \mid \gamma(0) = x\}$
$\|$

$$\begin{tikzcd} & * \arrow[d, "x"] \\ * \arrow[r, "x"'] & X \end{tikzcd} \quad \xrightarrow{\text{w.e.}} \quad \begin{tikzcd} & P_x X \arrow[d] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

$P_x X \in$ path space

$$\gamma \mapsto \gamma(1)$$

but pullbacks not equiv:

$$* \not\xrightarrow{\sim} \Omega X.$$

## Homotopy pullback:

$$\begin{tikzcd} Z \times^h_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

$$Z \times^h_X Y = \{ (y, z, \gamma) \mid f(y) \xrightarrow{\gamma} g(z) \text{ path} \}$$

Example:

$$* \times^h_X * \simeq \Omega_x X = \{ \gamma : I \to X \mid \gamma(0) = x = \gamma(1) \}$$

Exercise: Convince yourself that

$$\begin{tikzcd} * \amalg * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd}$$

is a homotopy pushout.

## Model categories:

rel. cat $(\mathcal{C}, W)$ + extra structure

Extra str. = cofibrations + fibrations

is not intrinsic to the homotopy thy

Analogy (Toën?): Choosing model str. with given underlying $(\mathcal{C}, W)$ is like choosing coordinates on a fixed underlying manifold $M$; namely a tool for computations;

**Example:** Mapping spaces

- $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(QX, RY)/\sim_{\text{htpy}}$

  $\mathrm{Hom}_{\mathcal{C}}(QX, \ldots)$: cofib replacement

  $RY$: fibrant replacement.

  $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X,Y) = \pi_0 \mathrm{Map}_{LwC}(X,Y)$

- Can in fact use (co)-simplicial resolutions in model cat to compute $\mathrm{Map}_{\mathcal{C}}(X,Y)$.

**Example:** Computing homotopy limits & colimits.

**Example:** Computing other derived functors.

**Sub-example:** Given

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ Z & \to & X \end{array}.$$

to compute homotopy pullback replace $Y \to X$ by any weakly equiv fibration $RY \to X$

$$Y \times^h_X Z \simeq RY \times_X Z.$$

$P_x X \to X$ is a fibration. weakly equivalent to $* \to X$

$$\rightsquigarrow \; * \times^h_X * \simeq * \times_X P_x X \simeq \Omega_x X.$$

$\infty$-category = homotopy theory.

## (b) Model independence.

$(Top, W) \simeq (sSet, W') \simeq (Fun(\Theta^{op}, set), W')$

as homotopy theories

$sSet = Fun(\Delta^{op}, Set)$

$\simeq \cdots$

- we suppress the model for the homotopy theory of spaces that we use

$(relcat, W_{relcat}) \simeq (Cat_{Top}, W'_{DK})$

$\simeq (Cat_{sSet}, W_{DK}) \simeq$

$(quasi\text{-}cat, W_{Joyal})$

$\simeq$ (Complete Segal Spaces, $W_{Rezk}$)

$\simeq \cdots$

- We suppress the model for homotopy theories / $\infty$-categories that we use.

**Defn:** Let $F : \mathcal{C} \to \mathcal{D}$ be a Top-enriched functor of topological categories.

$F$ is a **Dwyer-Kan equivalence** if

1) $\forall\, X, Y \in Ob(\mathcal{C})$

$F_* : Map_{\mathcal{C}}(X,Y) \to Map_{\mathcal{D}}(FX, FY)$

is a weak equivalence of spaces.

2) $h\mathcal{C} \xrightarrow{hF} h\mathcal{D}$ is an equivalence of categories.

[ 1) + 2) $\Leftrightarrow$ 1) + $hF$ essentially surjective ]

## (C) Algebra vs higher algebra

$A$ abelian group $\rightsquigarrow$ $HA$ Eilenberg-Maclane spectrum.

$Ab = \mathbb{Z}\text{-Mod}$

Spectra $= Mod_{\mathbb{S}}$

$\mathbb{S}$ = sphere spectrum.

$\mathbb{S} \to H\mathbb{Z}$

$\mathbb{S} \wedge \mathbb{S} \to \mathbb{S}$

$H$ - lax monoidal.

$A$ ring $\Rightarrow$ $HA$ ring spectrum.

**Convention:** we will often suppress $H$ from the notation & view a ring $A$ as a discrete ring spectrum

## Notation:

- $R$ ring spectrum

$Mod_R$ - $R$-module spectra

$(Mod_R, W)$ homotopy theory.

equivalences of underlying spectra.

$M$ spectrum

$R \wedge M \longrightarrow M$ satisfying all the usual constraints.

$Mod_R^{\omega}$ - compact spectra

- $R$ ordinary ring or dg-ring

$Mod_R$ $\infty$-category of $R$-modules

$Ob(Mod_R) =$ Chain complexes of $R$-modules

$W = \{\text{quasi-isos}\}$

$hMod_R := D(R)$

$Mod_R^{\omega} =: Thick(R) =: Perf(R)$

compact/small objects

Ordinary alg:

Assoc algebras $A$

Commutative algebras $A$

$xy = yx$

Higher algebra

Assoc alg ($\mathbb{E}_1$-alg or $A_\infty$-alg)

more commutative

$\mathbb{E}_2$-alg

$\vdots$

$\mathbb{E}_n$-alg

$\vdots$

$E_\infty$-alg

homotopy commutative algebras

## Examples of duality

1) Poincaré duality

2) Serre duality

3) Local duality

4) Koszul duality

## Some references:

Simplicial localization: apart from the three seminal papers of Dwyer-Kan from 1980, there is the following book:

1. Dwyer, Hirschhorn, Kan and Smith, "Homotopy limit functors on model categories and homotopical categories".

2. Barwick-Kan, "Relative categories: another model for the homotopy thy of homotopy theories." Indag. Math (2012)

3. Toën, "Towards an axiomatization of the theory of higher categories." K-theory, 2005.

$A_\infty$-alg : $A$

(Associative in homotopy sense)

$\mathrm{Disc}_1$ Top enriched cat

Objects = disjoint unions of 1-dim discs

$$A^{\otimes 2} \longrightarrow A$$

$\mathrm{Hom}(M,N) :=$ smooth embeddings of 1-mflds

$$A : \mathrm{Disc}_1^{\sqcup} \longrightarrow \mathrm{Spaces}^{\times}$$

$$A(\,\text{—}\;\;\text{—}\,) \simeq A \times A$$

$$A : \mathrm{Disc}_1^{\sqcup} \longrightarrow \mathcal{C}^{\otimes}$$

$X$ space $\quad$ $\Omega_x X$ is an $A_\infty$-algebra