M is model category three distinguished classes of maps $\underset{\text{cofib}}{\mathcal{C}}$, $\underset{\text{fibr}}{\mathcal{F}}$, $\underset{\text{weak eq.}}{\mathcal{W}}$

$M[\mathcal{W}^{-1}] = Ho(M)$ obj same as obj M

$\mathrm{mor}(X,Y) = [X_{cf}, Y_{cf}]$

M has an initial obj ϕ & a terminal obj $*$.

An object A in M is said to be cofibrant if $\phi \to A$ is a cofibrant.

Every object A in M has a cofibrant replacement

$$\begin{tikzcd} \phi \arrow[r] \arrow[dr, tail, "\text{cofib}"'] & A \\ & A_c \arrow[u, "\sim"', "\text{fibr}"] \end{tikzcd}$$

A_c.

An object is fibrant if $X \to *$ is a fibration.

Every object has a fibrant replacement

$$\begin{tikzcd} X \arrow[r] \arrow[dr, hook, "\sim"'] & * \\ & X_f \arrow[u] \end{tikzcd}$$

$X_f \longrightarrow$ fibrant replacement

Recall in Top spaces the $[0,1]=I$ is the object which track homotopy

$$\begin{tikzcd} X\times\{0\} \arrow[d] \arrow[dr, "f"] & \\ X\times I \arrow[r, "\text{homotopy}"] & Y \\ X\times\{1\} \arrow[u] \arrow[ur, "g"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \amalg X \arrow[rr, "(id_X,\, id_X)"] \arrow[dr] & & X \\ & X\times I \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[rr, bend left, "(f,g)"] & Y^I \arrow[r, "(\alpha(0),\alpha(1))"] & Y\times Y \end{tikzcd}$$

$\alpha \in \mathcal{J}(I, Y)$

In a model category we can talk about upto weak equivalence an object like $X\times I$ or Y^I. (may not be arising from another object I).

It allows us to define two different notions of homotopy (left hmpty, right homotopy)

For a object which is both fibrant & cofibrant these notions coincide and we can show that this notion of homotopy is an equivalence relation.

$Ho(M) :=$ category obj same as M

& $\mathrm{mor}(X,Y) = [X_{cf}, Y_{cf}]$

→ We can understand the category in terms of cofibrant fibrant.

Typically we can control what the fibrations are or the cofibrations.

Model categories allow us to change our models upto homotopy theory

Simplicial Sets: $\leftarrow$ alternate defn to what Somnath did

Δ: Category of finite ordered sets

$\underline{n} = \{0, 1, 2, \ldots, n\}$

morphisms order preserving maps.

A simplicial set X is a contravariant functor from Δ to Sets.

$X : \Delta^{op} \to \text{Sets}$.

$\{X_n\}$

$\underline{n-1} \xrightarrow{d^i} \underline{n}$ $\hat{i}$ skips one element

$\{0, 1, 2, \ldots, n-1\} \to \{0, 1, 2, \ldots, n\}$

$d_i : Y_n \to Y_{n-1}$

face maps.

degeneracies.

$s^j : Y_{n-1} \to Y_n$

$\underline{n} \xrightarrow{s_j} n-1$

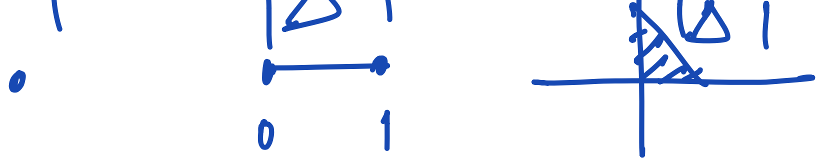
$\{0, 1, 2, \ldots, j, j+1, \ldots, n\}$

$$\{0,1,2,\dots,j,\dots,n\}$$

$$|\Delta^n| = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \,\Big/\, \sum_{i=0}^{n} t_i = 1 \right\}$$

$$0 \le t_i \le 1 \}$$

$|\Delta^0|$ $\qquad$ $|\Delta^1|$ $\qquad$ $|\Delta^2|$



0 $\qquad$ 1

We have a covariant functor

$$\Delta \longrightarrow \text{Top}$$

$$\underline{n} \longrightarrow |\Delta^n|$$

It is easy to check that this is a functor

<u>Examples</u>:

1) X is a top. space. We can

define a simplicial set

$$(\mathrm{Sing}(X))_\bullet = \mathcal{T}op(|\Delta^n|, X)$$

Note given any simplicial set Y we can define simplicial group by

$$\mathbb{Z}Y : \Delta^{op} \longrightarrow \text{Ab group}$$

$$\underline{n} \longrightarrow \text{free ab group on } Y_n$$

$$\mathbb{Z}[Y_n]$$

Note that we can define chain cplx using this & if $Y = \mathrm{Sing}(X)$ ~~this~~ then $\mathbb{Z}Y$ allows us to define the singular chain complex.

2). Let G be any group.

$$(Y_G)_0 = \{e\}$$

$$(Y_G)_1 = G$$

$$(Y_G)_2 = \underline{G \times G}$$

$$\vdots$$

$$(Y_G)_n = \underbrace{G \times \cdots \times G}_{n \text{ times}}$$

3) Given any (small) categ $\mathcal{C}$ we can define its nerve as simplical set

$$(N\mathcal{C})_0 = \operatorname{obj} \mathcal{C}$$

$$(N\mathcal{C})_1 = \operatorname{mor} \mathcal{C}$$

$$(N\mathcal{C})_2 = (\text{composable morp})$$

$$= \coprod_{\substack{(X,Y,Z) \\ \text{obj of } \mathcal{C}}} \operatorname{Hom}(X,Y) \times \operatorname{Hom}(Y,Z)$$

$$(N\mathcal{C})_n \longrightarrow n \text{ composable morphism}$$

Notice the previous example is special case of nerve of the category obj $*$
mor G.

We defined a functor.

$$\mathrm{Top} \longrightarrow \text{Simplicial sets}$$
$$X \longrightarrow \mathrm{Sing}(X)$$

There is a functor adjoint to this called the geometric realization

$$\text{Simplicial set} \longrightarrow \mathrm{Top}.$$
$$X \longrightarrow \frac{\coprod_{n \geq 0} X_n \times \Delta^n}{\sim}$$

$$(\theta^*(x), \sigma) \sim (x, \theta_*(\sigma))$$

$$\forall \ \theta \in \Delta([m],[n])$$

$$\theta^* : X_n \to X_m$$

$$\theta_* : |\Delta^m| \to |\Delta^n|$$

$$\coprod_{\substack{\theta \in ([m],[n]) \\ m,n \geq 0}} X_n \times |\Delta^m| \underset{(id,\theta^*)}{\overset{\theta^*, id}{\rightrightarrows}} \coprod_{n \geq 0} X_n \times |\Delta^n| \xrightarrow{\text{coeq.}} |X|.$$

$$|X| = \operatorname*{colim}_{\substack{n \\ \Delta^n \to X}} |\Delta^n| \longrightarrow \text{is always a CW complex}$$

$$\hom_{Top}(|X|, Y) \cong \hom_{Top}\left(\operatorname*{colim}_{\substack{n, \\ \Delta^n \to X}} |\Delta^n|, Y\right)$$

$$= \operatorname*{colim}_{n, \Delta^n \to X} \hom_{Top}(|\Delta^n|, Y)$$

Let $\begin{cases} \Delta^n : \Delta^{op} \longrightarrow \text{Sets} \\ \quad \underline{k} \longrightarrow \text{Hom}_\Delta(\underline{k}, \underline{n}) \end{cases}$

$$= \operatorname*{colim}_{n,\, \Delta^n \to X} \hom_{S.Set}(\Delta^n, S(Y))$$

$S(Y) := \text{Sing}(Y)$

$$= \hom_{S.Set}\Big(\operatorname*{colim}_{\Delta^n \to X} \Delta^n, S(Y)\Big)$$

$$= \hom_{S.Sets}(X, S(Y))$$

We have a adjoint pair of functors

$$|\;| : S.Sets \rightleftarrows Top : S$$

w.eq. $X \to Y$ if $|X| \to |Y|$ is a weak eq.

w.eq.

weak h.eq

Serre fibration

cofibration are

retracts of generalized CW cplxes

Δ^n is simplicial

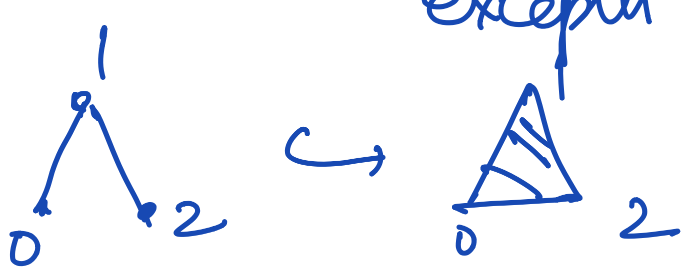
$(\Delta^n)_k = \hom(\underline{k}, \underline{n})$

Let $i_n = id : \underline{n} \to \underline{n}$

$(\partial\Delta^n) \subseteq \Delta^n$ an simplicial set generated $d_j(i_n)$. $\forall\ 0 \le j \le n$

$\Lambda^n_k \subseteq (\partial\Delta^n) \subseteq \Delta^n$.

generated by $d_j(i_n)$ except the k^{th} face



Then in simplicial sets a map $X \to Y$ is said to

be a Kan fibration if
$\forall\, k, n$ the dotted arrow exist.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d] & \leftarrow \text{fibration} \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S &
\end{tikzcd}$$

If X has the property

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[r] \arrow[ur, dashed] & * \arrow[from=1-2]
\end{tikzcd}$$

that the dotted arrows exist $\forall\, n, k$

then X is said to be Kan complex.

<u>Thm</u>: S.Sets form a model categ with:

w. equivalences defined as $X \to Y$

$|X| \to |Y|$ is w.eq.

fibrations as Kan fibrations

& cofibrations as inclusions

$$\text{s.Sets} \underset{S}{\overset{|\ |}{\rightleftarrows}} \text{Top}.$$

is a Quillen equivalence.