Duality in higher categories 2

Last time: Simplicial localization

Theorem (Dwyer-Kan)

$(\mathcal{C}, W)$ relative category

$\rightsquigarrow \quad L_W\mathcal{C} \in \mathrm{Cat}_{sSet}$ $\left(\begin{array}{l}\text{sSet-enriched} \\ \text{categories}\end{array}\right)$

$\pi : \mathcal{C} \to L_W\mathcal{C}$

with universal property

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] \arrow[dr, "F"'] & L_W\mathcal{C} \arrow[d, dashed, "\exists !"] \\ & \mathcal{X} \end{tikzcd}$$

$\forall \; F$ s.t. $F(w)$ weakly invertible $\forall w \in W$.

Homotopy category:

$\mathcal{X} \in \mathrm{Cat}_{sSet} \mapsto h\mathcal{X} \in \mathrm{Cat}$

$\mathrm{Hom}_{h\mathcal{X}}(X, Y) \simeq \pi_0 \mathrm{Map}_{\mathcal{X}}(X, Y)$.

Fact: $\boxed{\mathrm{Hom}_{hL_W\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X, Y)}$

Simplicial abelian groups vs. chain complexes.

$\mathcal{C}$ category

$\rightsquigarrow \quad s\mathcal{C} := \mathrm{Fun}(\Delta^{op}, \mathcal{C})$

simplicial objects in $\mathcal{C}$.

Examples: sSet $= \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$

sAb $= \mathrm{Fun}(\Delta^{op}, \mathrm{Ab})$

abelian groups,

etc.

Rekha's talks:

$$\mathrm{sSet} \underset{\mathrm{Sing}}{\overset{|\,|}{\rightleftarrows}} \mathrm{Top}$$

Quillen adjunction, which is in fact a Quillen equivalence of model categories.

Analogous statement for $\mathcal{C} = \mathrm{Ab}$?

$$\mathrm{sAb} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathbb{Z})$$

{simplicial abelian groups} $\longrightarrow$ {chain complexes of abelian groups concentrated in non-negative degrees}

$$(A_\cdot : \Delta^{op} \to \mathrm{Ab}) \longmapsto (C(A_\cdot), d)$$

$$C(A_\cdot)_n = A_n$$

$$d_n : C(A_\cdot)_n \longrightarrow C(A_\cdot)_{n-1}$$

$$d_n = \sum (-1)^n \partial_n$$

Normalized version

$$N : sAb \to Ch_{\geq 0}(\mathbb{Z})$$

$$N(A_\bullet)_n = \bigcap_{i=0}^{n-1} \ker(\partial_i) \subseteq A_n .$$

$$d : N(A_\cdot)_n \to N(A_\cdot)_{n-1}$$

$$d := (-1)^n \partial_n$$

$d^2 = 0$ because we have the simplicial identity $\partial_{n-1}\partial_n = \partial_{n-1}\partial_{n-1}$.

Prop: $N(A_\cdot) \hookrightarrow C(A_\cdot)$ is a homotopy equiv and $N(A_\cdot)$ is isomorphic to the quotient of $C(A_\cdot)$ by degenerate simplices.

Def: A morphism $f : A_\cdot \to B_\cdot$ in sAb is a weak equivalence if the underlying morphism in $sSet$ is a weak homotopy equivalence.

Theorem (Dold-Kan correspondence).

There is an adjunction

$$\begin{tikzcd} sAb \arrow[r, shift left, "N"] & Ch_{\geq 0}(\mathbb{Z}) \arrow[l, shift left, "\Gamma"] \end{tikzcd}$$

(with N the left adjoint) that is part of a Quillen equivalence of model categories

$$\boxed{\pi_i(A_\cdot) = H_i(NA_\cdot)}$$

Remark: We can replace $Ab = Mod_{\mathbb{Z}}$ by any abelian category $\mathcal{A}$ to get

$$s\mathcal{A} \rightleftarrows Ch_{\geq 0}(\mathcal{A})$$

Example: $\mathcal{A}$ = k-modules for any commutative ring k

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Compatibility with monoidal structure

$N_\cdot$ is lax monoidal:

there are natural maps

$$N(A_\cdot) \otimes N(B_\cdot) \to N(A_\cdot \otimes B_\cdot)$$

define levelwise

Eilenberg-MacLane shuffle map

Consequence: We get a functor

$$\left\{\begin{array}{c}\text{associative}\\ \text{simplicial}\\ \text{rings}\end{array}\right\} \xrightarrow{N} \left\{\begin{array}{c}\text{connective}\\ \text{differential}\\ \text{graded}\\ \text{rings}\end{array}\right\}$$

$\cup$

$Alg(sAb)$

and this is an equivalence of homotopy theories.

There is an op-lax monoidal structure $N(A_\cdot \otimes B_\cdot) \to N(A_\cdot) \otimes N(B_\cdot)$ given by the Alexander-Whitney map, but this is not symmetric in $A_\cdot, B_\cdot$.

Warning: The homotopy theory of commutative simplicial rings is equivalent to the homotopy theory of connective commutative dg-rings if char $= 0$, but this is NOT true if char $p \neq 0$.

Need to work with E_∞-rings.

$$\left\{ \begin{matrix} E_\infty\text{-simplicial} \\ \text{rings} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{connective} \\ E_\infty\text{-} \\ \text{dg-alg}/k \end{matrix} \right\}$$

over any field.

UPSHOT: We have equivalences of ∞-categories

$$\left\{ \begin{matrix} \text{connective} \\ Hk\text{-module} \\ \text{spectra} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{simplicial} \\ k\text{-modules} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{connective} \\ \text{chain complexes over } k \end{matrix} \right\}$$

for any commutative ring k.

"Stabilizing" both sides, we obtain a monoidal equivalence

$$\left\{\begin{array}{c}Hk\text{-module}\\ \text{spectra}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{chain}\\ \text{complexes}\\ \text{of } k\text{-modules}\end{array}\right\}$$

with weak equivalences defined as before.

$$\left\{\begin{array}{c}\text{connective}\\ \text{associative}\\ Hk\text{-algebras}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{simplicial}\\ k\text{-algebras}\end{array}\right\}$$

$$\simeq \left\{\begin{array}{c}\text{connective dg-algebras}\\ \text{over } k\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\text{associative}\\ Hk\text{-algebras}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{dg-algebras}\\ \text{over } k\end{array}\right\}$$

If char $k = 0$

$$\left\{\begin{array}{c}\text{connective}\\ \text{commutative}/E_\infty\\ Hk\text{-algebras}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{connective}\\ \text{commutative}\\ \text{dg-algebras}\\ \text{over } k\end{array}\right\}$$

$$\simeq \left\{\begin{array}{c}\text{commutative}\\ \text{simplicial } k\text{-algebras}\end{array}\right\}$$

and corresponding statements for non-connective versions.

Stable ∞-categories:

Fix your favorite model of ∞-categories. ((∞,1)-categories)

Let C be an ∞-category.

$* \in Ob(C)$ is initial (resp final) if

$Map(*, X)$ is contractible (resp. $Map(X, *)$ is contractible) for all $X \in Ob(\mathcal{C})$.

Defn: A zero object 0 in $\mathcal{C}$ is an object that is both initial and final.

Defn: An ∞-category $\mathcal{C}$ is stable if

(1) $\mathcal{C}$ has a 0 object

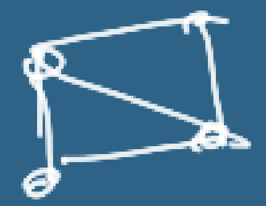
(2) finite (homotopy) colimits and limits exist in $\mathcal{C}$.

(3) pushout squares coincide with pullback squares.

Defn: The suspension Σ is defined (in any pointed ∞-category) by the pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

$\Delta^1 \times \Delta^1 \to \mathcal{C}$



and the looping functor Ω is defined by the pullback square

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

So (3) says that

$\Sigma\Omega Y \to Y$ is an equivalence in $\mathcal{C}$ (ie an isomorphism in $h\mathcal{C}$) (*)

and $X \to \Omega\Sigma X$ is an equivalence.

Rmk: In the defn of stable ∞-cats enough to ask for

(2)' pushouts / pullbacks to exist

(3)' $\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$ is pullback iff it is a pushout.

Non-examples: (Exercise)

- Spaces_* (based spaces) is pointed but not stable $\Sigma\Omega Y \not\simeq Y$.

- Mod_k^{cn} - ∞-category of connective chain complexes $/k$ is pointed, but not stable.

Indeed $\Sigma M_\bullet = M_\bullet[1]$ (Exercise)

with $(M[1])_n = M_{n-1}$

$d_{M[1]} = (-1)^n d^n_M$

Examples

- Spectra (or more generally Mod_R for some $R \in \mathrm{Alg}(\mathrm{Mod}_{\mathbb{S}})$) is stable.

- If R is a dg. alg $/k$

$\mathrm{Mod}_R \leftarrow$ { the ∞-category presented by the relative category $(Ch(R), \text{quasi-isos})$

is stable.

$\Sigma M_\bullet = M[1]$
$\Omega M_\bullet = M[-1]$
} Exercise $\Sigma\Omega \simeq id$.

Theorem (Lurie)

If $\mathcal{C}$ is a stable ∞-category then $h\mathcal{C}$ is a triangulated category.

Exact triangles: images in $h\mathcal{C}$ of pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Cf \end{tikzcd} \quad \leftarrow \text{cone of } f$$

Exercise: Show that if $\mathcal{C}$ is stable then $\mathcal{C}$ admits a natural enrichment in $\mathrm{Spectra} = \mathrm{Mod}_{\mathbb{S}}$

$\mathbb{S}$ = sphere spectrum.

(ie, $\forall X, Y \in \mathrm{ob}(\mathcal{C})$ $\exists\ \underline{\mathrm{Mor}}(X,Y) \in \mathrm{Mod}_{\mathbb{S}}$ st $\Omega^\infty \underline{\mathrm{Mor}}(X,Y) \simeq \mathrm{Map}(X,Y)$)

Defn (Cone) Let $f : X \to Y$ be a morphism in a pointed ∞-category $\mathcal{C}$, and let

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

be a pushout square in $\mathcal{C}$.

Then we call Z the cone of f and denote it by $\mathrm{Cone}(f)$ or $C(f)$.

Long exact sequences.

$$\begin{tikzcd} \Omega Z \arrow[r] \arrow[d] & 0 \arrow[d] & \\ X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

$$\to \Omega Z \to X \to Y \to Z \to \Sigma X \to \cdots$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ 0 \arrow[r] \arrow[drr, bend right] & C(f) \arrow[dr, dashed, "\exists !"] & \\ & & Z \end{tikzcd}$$

The Dold-Kan correspondence induces

$$\left\{\begin{matrix}\text{categories}\\ \text{enriched in } sMod_k\end{matrix}\right\} \xrightarrow{\tilde{N}} \left\{\begin{matrix}\text{categories}\\ \text{enriched in } Ch_{\geq 0}(k)\end{matrix}\right\}$$

and this induces an equivalence of homotopy theories

$$\left\{\begin{matrix}\text{k-linear}\\ \text{stable } \infty\text{-categories}\end{matrix}\right\} \simeq \left(\left\{\begin{matrix}\text{triangulated}\\ \text{categories}\\ \text{enriched in } Ch(k)\end{matrix}\right\}, \text{Dwyer-Kan equivalences}\right)$$

$$\simeq$$

$$\left\{\text{triangulated dg-categories}/k\right\}$$

(differential graded categories)

Some references

1. Lurie, "Higher algebra", available on author's web-page.

2. Schwede-Shipley, "Algebras and modules in monoidal model categories" Proc. Lond. Math. Soc., 2000.

3. Schwede-Shipley, "Stable categories are categories of modules", Topology 2003

4. Toën, "Lectures on dg-categories" Lecture Notes in Math, 2008.