# Stable homotopy category $\mathcal{S}$

- $\mathrm{Top}_* \xrightarrow[\Sigma^\infty]{} \mathcal{S}$

  suspension spectrum

- $\mathcal{S} \xrightarrow[\Omega^\infty]{} \mathrm{Top}_*$

- Triangulated category

- cofibre sequence

  $X \xrightarrow{f} Y \to Cf \to \Sigma X$

- homotopy pushouts are also homotopy pullbacks.

- $\wedge$ $\qquad$ $F(-,-)$

make $\mathcal{S}$ into a symmetric monoidal category & unit $= S^0$

# Homology & cohomology theories defined by spectra

$E$ spectrum

Define: $E^n(X) = \{X, \Sigma^n E\} = \pi_{-n}(F(X,E))$

$n \in \mathbb{Z}$

is the cohomology theory induced by $E$.

$$E_n(X) = \pi_n(E \wedge X)$$

is the homology theory induced by $E$.

$$\pi_n(H\mathbb{Z} \wedge X) \cong \tilde{H}_n(X;\mathbb{Z})$$

$$\pi_{-n}(F(X, H\mathbb{Z})) \cong \tilde{H}^n(X;\mathbb{Z}).$$

$(\mathcal{C}, \otimes, \mathbb{1})$ is a symm. mon. cat.

<u>Def</u>: $X$ is said to be <u>dualizable</u> if $\exists$ another object $Y$ + maps

$$X \otimes Y \xrightarrow{\lambda} \mathbb{1}, \quad \mathbb{1} \xrightarrow{\mu} Y \otimes X$$

s.t.

$$\begin{tikzcd} X \cong X \otimes \mathbb{1} \arrow[r, "id \otimes \mu"'] \arrow[ddr, "id"'] & X \otimes Y \otimes X \arrow[d, "\lambda \otimes id"] \\ & \mathbb{1} \otimes X \\ & X \arrow[u, phantom, "\cong" description] \end{tikzcd}$$

$$\begin{tikzcd} Y \cong \mathbb{1} \otimes Y \arrow[r, "\mu \otimes id"'] \arrow[ddr, "id"'] & Y \otimes X \otimes Y \arrow[d, "id \otimes \lambda"] \\ & Y \otimes \mathbb{1} \\ & Y \arrow[u, phantom, "\cong" description] \end{tikzcd}$$

& in this case $Y$ is said to be the dual of $X$.

Example: $\mathcal{C}$ = Vector spaces$/_k$.

Dualizable objects = Finite dimensional vector spaces

Question: What are the dualizable objects in $\mathcal{S}$?

- $S^0$, $S^k \wedge S^{-k} \simeq S^0$

  $S^k$ is dualizable with dual $S^{-k}$.

Exercise: $X$ is dualizable & $Y$ is its dual. Then $\forall\, W, Z$

$$\mathrm{Hom}_{\mathcal{C}}(W \otimes X, Z) \cong \mathrm{Hom}_{\mathcal{C}}(W, Y \otimes Z)$$

....

In $\mathcal{S}$, suppose $X$ is dualizable, then

$$\{W \wedge X, Z\} \cong \{W, DX \wedge Z\}$$

$DX$ = dual of $X$.

$\Rightarrow$ $X$ is compact.

$$\left( \{X, \operatorname{colim} -\} \cong \operatorname{colim}\{X, -\} \right)$$

$X$ dualizable $\Rightarrow$

$$X \wedge DX \longrightarrow S^0$$

$$\rightsquigarrow \quad DX \longrightarrow F(X, S^0)$$

<u>Exercise</u>: This is a weak equiv. i.e. $X$ is dualizable then $DX \simeq F(X, S^0)$ & $X \simeq F(F(X, S^0), S^0)$

**Exercise** $X, Y$ is dualizable

$f : X \to Y$ then $Cf$ is dualizable.

$$X \to Y \to Cf \qquad \text{cofibre}$$

$$\downarrow$$

$$F(Cf, S^0) \to F(Y, S^0) \to F(X, S^0)$$

cofibre

$$\downarrow$$

$$F(F(X,S^0),S^0) \to F(F(Y,S^0),S^0) \to F(F(Cf,S^0),S^0)$$

cofibre

$\Rightarrow$ All finite spectra are dualizable.

"Finite spectra": All spectra obtained by attaching finitely many cells. That is,

$$\begin{array}{l} X^{(0)} \to X^{(1)} \to \cdots \quad X^{(n-1)} \to X^{(n)} = X \\ \parallel \\ * \end{array}$$

$$\mathrm{Cone}(X^{(i-1)} \to X^{(i)}) = S^{n_i}$$

$$n_i \in \mathbb{Z}$$

$$\Rightarrow \mathrm{Cone}(S^{n_i - 1} \to X^{(i-1)}) = X^{(i)}.$$

- $\Sigma^\infty$(finite CW cx.) is dualizable.

$X$ finite CW cx $\qquad DX = D(\Sigma^\infty X)$

$$= F(X, S^0)$$

$$E_k DX = \pi_k(E \wedge DX) \cong \{S^k, E \wedge DX\} \cong \{S^k \wedge X, E\} \cong E^{-k}(X).$$

$$E_k DX = E^{-k}(X)$$

$$E^k(DX) = E_{-k}(X).$$

$X$ finite spectrum the so is

$F(X, S^0)$.

$$\begin{tikzcd} X^{(0)} \arrow[r] \arrow[d, equal] & X^{(1)} \arrow[r] & \cdots \arrow[r] & X^{(n)} \simeq X \\ * & & & \end{tikzcd}$$

$$\begin{tikzcd} F(X/X^{(n)}, S^0) \arrow[r] \arrow[d, equal] & F(X/X^{(n-1)}, S^0) \arrow[r] & \cdots \arrow[r] & F(X/X^{(0)}, S^0) \arrow[d, equal] \\ * & & & F(X, S^0) \end{tikzcd}$$

<u>Example</u> $D(\mathbb{C}P^2)$

$$\mathbb{C}P^2 = \mathrm{Cone}\,(S^3 \xrightarrow[\eta]{} S^2)$$

$$D(\mathbb{C}P^2) = \mathrm{Cone}\,(S^{-2} \xrightarrow[\eta]{} S^{-3})$$

$X$ finite spectrum

1) We may rearrange the attachments so that the cells are attached in increasing dimension.

2) $\exists\, N$ s.t. $\Sigma^N X \simeq \Sigma^\infty Y$ where $Y$ is a CW complex.

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$X \subseteq S^n$

finite complex

$A \subseteq S^n - X$

finite CW cx.

s.t. $A \simeq S^n - X$.

$p \in S^n$ outside both $X$ & $A$.

st : Stereographic proj$^n$: $S^n - p \xrightarrow{\cong} \mathbb{R}^n$

$$X \times A \xrightarrow{\hat{\mu}} S^{n-1}$$

$$(x, a) \longmapsto \frac{st(x) - st(a)}{\|st(x) - st(a)\|}$$

Hopf const$^n$:

$$X * A \xrightarrow{\mu} \Sigma S^{n-1} = S^n$$

$$tx + (1-t)a \longmapsto (\hat{\mu}(x,a), t)$$

$$X * A \simeq \Sigma(X \wedge A)$$

$$\Sigma(X \wedge A) \xrightarrow[\mu]{} S^n$$

$\rightsquigarrow$ stable map

$$X \wedge A \longrightarrow S^{n-1}$$

$$\rightsquigarrow \quad A \longrightarrow F(X, S^{n-1})$$

is

$$\Sigma^{n-1} F(X, S^0)$$

$$\|$$

$$DX$$

$$\rightsquigarrow \Sigma^{1-n} A \longrightarrow DX.$$

Take homology on both sides,

$$\tilde{H}_k(\Sigma^{1-n} A) \longrightarrow \tilde{H}_k(DX)$$

$$\tilde{H}_k(\Sigma^{1-n} A) \cong \tilde{H}_{k+n-1}(S^n - X) \qquad \tilde{H}_k(DX) \cong \tilde{H}^{-k}(X)$$

$\tilde{H}^{n-k'-1}(X)$

$k' = k+n-1$

$$\tilde{H}_{k'}(S^n - X)$$

iso by Alexander duality.

Consequence:

$$\Sigma^{1-n} A \longrightarrow DX$$

iso- on $H_*$

Suspend both sides

$$: \Sigma^M \Sigma^{1-n} A \to \Sigma^M DX$$

map of suspension spectra

$\Rightarrow$ weak equiv by Whitehead's thm.

$$\Rightarrow \Sigma^{1-n} A \xrightarrow[\sim]{} DX$$ of spectra.