Thom space of a vector bundle

V is a vector bundle $/X$

fibrewise one-point compactification $\rightsquigarrow$ sphere bundle S^V

∞-section : $s : X \longrightarrow S^V$

Define: $Th(V) :=$ cofibre(s).

Other definitions

1) X compact $\rightsquigarrow$ inner product on V

$D(V) =$ all elements of norm ≤ 1
"Disk bundle"

$S(V) = \ldots$ norm 1 "sphere bundle"

$Th(V) = D(V)/S(V)$.

2) 0-section : $X \to V$

$x \longmapsto (x, 0)$

$\in$ fibre at x

$(V - \text{0-section}) \longrightarrow V$

homotopy cofibre : $Th(V)$

3) X cpt. $\rightsquigarrow$ $Th(V)$ is the 1-pt compactification of V.

<u>Example</u>: $X \times \mathbb{R}^n \longrightarrow X$

Thom space : $X \times S^n / X \times \infty$

is

$\Sigma^n X_+$.

V is orientable if $\exists\ \alpha \in H^n(Th(V))$

s.t for each fibre,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] \arrow[d] & V \arrow[d] \\ pt \arrow[r] & X \end{tikzcd} \qquad S^n \longrightarrow Th(V)$$

α pulls back to a generator of $H^n(S^n)$.

<u>Thom isomorphism</u>

V orientable $\Rightarrow$ $H\mathbb{Z} \wedge Th(V)$ is $H\mathbb{Z} \wedge \Sigma^n X_+$.

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \pi_2^*(V) \arrow[d] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

$$\rightsquigarrow \quad Th(V) \to Th(\pi_2^*(V))$$

$$\begin{tikzcd} V \arrow[d] & W \arrow[d] & & V \boxtimes W \arrow[d] \\ X & Y & \rightsquigarrow & X \times Y \end{tikzcd}$$

$$Th(V \boxtimes W) \simeq Th(V) \wedge Th(W)$$

$$\pi_2^*(V) = \bullet \boxtimes V \qquad \begin{array}{c} Th(\pi_2^*(V)) \\ \| \\ X_+ \wedge Th(V) \end{array}$$

$$Th(V) \longrightarrow X_+ \wedge Th(V)$$

$$
\begin{tikzcd}
Th(V)\wedge H\mathbb{Z} \arrow[r] \arrow[ddr, bend right] & X_+\wedge Th(V)\wedge H\mathbb{Z} \arrow[d, "id\wedge\alpha\wedge id"] \\
& X_+\wedge\Sigma^n \underbrace{H\mathbb{Z}\wedge H\mathbb{Z}}_{} \arrow[d] \\
& \Sigma^n X_+\wedge H\mathbb{Z}
\end{tikzcd}
$$

$\alpha \in H^n(Th(V))$

$\rightsquigarrow$

$Th(V) \to \Sigma^n H\mathbb{Z}$

α Thom class implies this is a weak equiv.

Thom spectrum

$Th(V \otimes W) = Th(V) \wedge Th(W)$

$Th(V \oplus \varepsilon) \simeq \Sigma Th(V)$.

$$\begin{array}{c} \gamma_n \\ \downarrow \\ BO(n) \end{array}$$

universal n-dim'l bundle

Define: $MO_n = Th(\gamma_n)$

$$\begin{tikzcd} \gamma_n \oplus \varepsilon \arrow[r] \arrow[d] & \gamma_{n+1} \arrow[d] \\ BO(n) \arrow[r, "i_n"'] & BO(n+1) \end{tikzcd}$$

$\rightsquigarrow \quad Th(\gamma_n \oplus \varepsilon) \longrightarrow Th(\gamma_{n+1})$

$$\Sigma MO_n \longrightarrow MO_{n+1}$$

$\rightsquigarrow$ str. maps of a spectrum MO.

Theorem (Thom)

$$\pi_k MO \cong \left\{ \begin{matrix} k\text{-dim'l closed} \\ \text{mflds} \end{matrix} \right\} \Big/ \text{cobordism}$$

$\longleftarrow$

$$M^k \underset{\nu}{\hookrightarrow} \mathbb{R}^{N+k} \qquad N \gg 0$$

Tubular nbd thm

Pontryagin-Thom const^{n}

$$M^k \subseteq \underset{\underset{D(N(\nu))}{\text{is}}}{U} \subseteq \mathbb{R}^{N+k}$$

$\downarrow$

$$\left[S^{N+k} \to \underset{\underset{D(N(\nu))/S(N(\nu))}{\shortparallel}}{Th(N(\nu))} \right.$$

pts in $U \longmapsto$ corr pt.

pts outside $U \longmapsto \infty$

$$S^{N+k} \longrightarrow Th(N(\nu))$$

$$\rightsquigarrow N(\nu) \qquad (\dim : N+k-k) = N$$

$$\begin{tikzcd} N(\nu) \arrow[d, squiggly] & \\ N(\nu) \arrow[r] \arrow[d] & \gamma_N \arrow[d] \\ M \arrow[r] & BO(N) \end{tikzcd}$$

$$\rightsquigarrow Th(N(\nu)) \longrightarrow Th(\gamma_N)$$
$$\cap$$
$$MO_N .$$

$$\Longrightarrow S^{N+k} \longrightarrow MO_N$$

$$\rightsquigarrow \pi_k MO.$$

General construction

$$X \xrightarrow{f} BO$$

X_n = pullback

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\ BO(n) \arrow[r] & BO \end{tikzcd}$$

$$Th(f)_n = Th(f_n^* \gamma_n)$$

$\rightsquigarrow$ yields a spectrum

Thom spectrum of f.

$$BSO \to BO \rightsquigarrow MSO$$

$$\big\downarrow$$

$\pi_k MSO =$ oriented k-mflds / cob.

$$BU \to BO \rightsquigarrow MU$$

$$\big\downarrow$$

$\pi_k MU$: complex cob gps.

$$* \to BO \rightsquigarrow S^0$$

$$\big\downarrow$$

framed cobordism gps.

Restrict to finite complexes

X.

$X \xrightarrow{f} BO$ classifies

a virtual bundle of dim 0

$V - W$ V, W vector bundles

$\dim(V) - \dim(W) = 0$.

$$\begin{array}{c} V \\ \downarrow \\ X \end{array}$$

X finite cx.

$\exists\, W$ s.t.

$V \oplus W$ = trivial bundle.

Every virtual bundle (of dim 0) may be expressed as $V - \dim V$ for some vector bundle V.

<u>Observation</u>: In this case, the Thom spectrum is

$$\simeq \Sigma^{-\dim V} Th(V).$$

$\rightsquigarrow$ define Thom spectra for any virtual bundle.

<u>Orientations</u> α virtual bundle

Thom class for $Th(\alpha) \in H^{\dim\alpha}(Th(\alpha))$

$\dim\alpha = 0$:

$$\begin{tikzcd} X_n \arrow[r, "i_n"] \arrow[d, "\alpha_n"'] & X \arrow[d, "\alpha"] \\ BO(n) \arrow[r] & BO \end{tikzcd}$$

$$\rightsquigarrow \quad \Sigma^{-n} \underbrace{Th(\alpha_n^* \gamma_n)} \to Th(\alpha)$$

Θ is a Thom class for α

if $i_n^*(\Theta) \in H^0(\Sigma^{-n} Th(\alpha_n^* \gamma_n))$
$\qquad\qquad\qquad\qquad \cong$
$\qquad\qquad\qquad H^n(Th(\alpha_n^* \gamma_n))$

is a Thom class for $\alpha_n^* \gamma_n$.

<u>Thom iso</u> : A Thom class

Θ induces an equiv of spectra

$$Th(\alpha) \wedge H\mathbb{Z}$$

is

$$\Sigma^{\dim \alpha} X_+ \wedge H\mathbb{Z}.$$

Atiyah Duality

Spanier - Whitehead duality

$$\rightsquigarrow \quad X \subseteq S^n$$

$$\Rightarrow \quad \Sigma^{1-n}(S^n - X) \simeq D(X)$$

$$X_+ \subseteq S^n$$

$$+ \longleftrightarrow \infty$$

$$D(X_+) \simeq \Sigma^{1-n}(S^n - X_+)$$

$$\| $$

$$\Sigma^{1-n}(\mathbb{R}^n - X)$$

$$\mathbb{R}^n - X \longrightarrow \underset{\substack{\text{IS} \\ *}}{\mathbb{R}^n} \longrightarrow \mathbb{R}^n / \mathbb{R}^n - X$$

$$\mathbb{R}^n / \mathbb{R}^n - X \simeq \Sigma(\mathbb{R}^n - X).$$

$$\Rightarrow \quad D(X_+) \simeq \Sigma^{-n}\left(\mathbb{R}^n / \mathbb{R}^n - X\right)$$

X closed manifold. $\qquad N$: normal bundle

Tubular nbd : $U \simeq D(N)$

$$\mathbb{R}^n / \mathbb{R}^n - X \simeq U / \partial U = D(N) / S(N) =: \mathrm{Th}(N)$$

$$D(X_+) \simeq \Sigma^{-n}\left(\mathbb{R}^n / \mathbb{R}^n - X\right)$$

is

$$\Sigma^{-n}\left(Th(N)\right)$$

$N = n - TX.$

$$Th(n - TX) \simeq \Sigma^n Th(-TX)$$

$$\Rightarrow \quad D(X_+) \simeq Th(-TX)$$

Atiyah duality.

If X is orientable, & dim $= m$

$$H\mathbb{Z} \wedge Th(-TX)$$

$$\simeq \Sigma^{-m} H\mathbb{Z} \wedge X_+$$

$$\Rightarrow H\mathbb{Z} \wedge D(X_+) \cong \Sigma^{-m} H\mathbb{Z} \wedge X_+$$

$$\Big\downarrow \pi_{-k}$$

$$H^k(X) \cong H_{m-k}(X)$$

yields Poincaré duality.