Lecture 1. Coulomb branches — Examples

I will follow notations in Bullimore's lectures
(maybe slight changes).

But I do not assume good or ugly conditions
unlike his or Hanany's lectures.

G : complex reductive group $\supset G_c$: maximal cpt subgrp
$\mathbf{N}$: representation of G $\quad \rho : G \longrightarrow GL(\mathbf{N})$
$\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$: symplectic representation of G
$\quad \parallel \mathbf{N}' \oplus \mathbf{N}'^*$ (cotangent type)
$\rightsquigarrow$ 3d $N=4$ SUSY gauge theory

$\tilde{G} \triangleright G$ (normal) $\qquad \tilde{G} \curvearrowright \mathbf{N} \qquad G_F = \tilde{G}/G$
$\cup$ flavor symmetry
T_F maximal torus

mass parameter $m \in \operatorname{Lie} T_F$
We also consider $m \in \operatorname{Hom}(\mathbb{C}^\times, T_F)$

T_C = Pontryagin dual to $\pi_1(G)$
$= \operatorname{Hom}_{grp}(\pi_1(G), \mathbb{C}^\times)$

Our examples $\pi_1(G) = \mathbb{Z}^n$

FI parameter $\zeta \in \operatorname{Lie} T_C$
may be $\operatorname{Hom}(\mathbb{C}^\times, T_C)$

1°. Higgs branch $M_H = \mathbb{M} /\!/\!/ G$

$\llcorner$ symplectic reduction

($\mathbb{M} /\!/\!/ G_{\mathbb{C}}$ hyperkähler quotient)

T_F (over G_F) $\curvearrowright M_H$

mass parameter m : equivariant variable

FI parameter $\zeta \in \mathrm{Hom}(\mathbb{C}^\times, T_{\mathbb{C}})$ ☆

$= \mathrm{Hom}(\pi_1(G), \pi_1(\mathbb{C}^\times))$

$\underset{\pi_1}{\cong} \mathrm{Hom}_{grp}(G, \mathbb{C}^\times)$

Lie alg elem $\zeta \in \mathrm{Lie}\, T_{\mathbb{C}} \cong \mathrm{Hom}_{\text{Lie alg.}}(\mathfrak{g}, \mathbb{C})$

One can deform the moment map equation

($\mu : \mathbb{M} \longrightarrow \mathfrak{g}^*$ $\quad \mathbb{M} /\!/\!/ G = \mu^{-1}(0) /\!/ G$)

$\mu^{-1}(\zeta) /\!/ G = M_H^{\zeta}$

Or using ☆, one can consider GIT quotient

$\mu^{-1}(0) /\!/_{\zeta}\, G$

FI parameter ζ : deformation / Kähler parameter

$\mathcal{M}_C$: Coulomb branch

(Braverman-Finkelberg-N)

Gr_G : affine Grassmannian

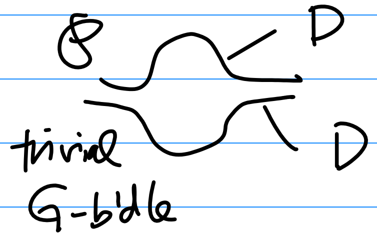
= moduli space of pairs

— $\mathcal{E}$: principal G-b'dle over D = formal disk $\mathrm{Spec}\,\mathbb{C}[[z]]$

Hecke correspondence

— $\varphi : \mathcal{E}|_{D^\times} \longrightarrow D^\times \times G$

formal punctual disk $D^\times$ = $\mathrm{Spec}\,\mathbb{C}((z))$



glued along $D^\times$

$\mathcal{R}$ = moduli space of triples

— $\mathcal{E}$ same

— φ

— s : section of $\mathcal{E} \times^G \mathbf{N}$

s.t. $\varphi(s) \in \mathbf{N}[[z]]$

s is a section of this b'dle over $D \cup_{D^\times} D$

(raviolo)

$G(\mathcal{O})$ = gauge transformation for triv.

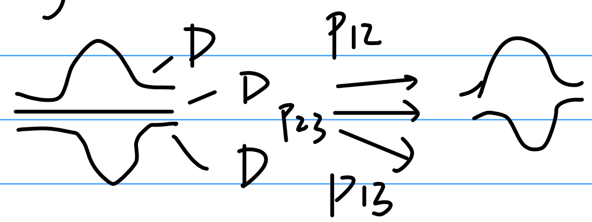
$\circlearrowright$ $\mathcal{R}$

$H_*^{G(\mathcal{O})}(\mathcal{R})$ = equivariant Borel-Moore homology.

↖ ring of <u>monopole operators</u>

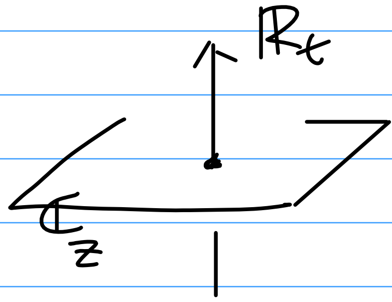
<u>Prop</u>. One can define an associative product on $H_*^{G(\mathcal{O})}(\mathcal{R})$ by the convolution:

(rough idea



Recall Hurtubise's 1st lecture

monopoles on $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} = \underset{z}{\mathbb{C}} \times \underset{t}{\mathbb{R}}$



$*F_A = d_A \varphi$ $\qquad \nabla = d + A$

$\Longrightarrow [\nabla_t + i\varphi, \nabla_{\partial/\partial \bar z}] = 0$

Allow singularity at 0 — Kronheimer 1985

S^1- equivariant instantons on $\mathbb{R}^4 \circlearrowleft S^1$

$\swarrow$

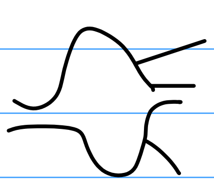
$\mathbb{R}^3$

singularity type $\longleftrightarrow$ S^1-action on fiber at $0 \in \mathbb{R}^4$ (up to gauge)

$\longleftrightarrow$ $\mathrm{Hom}(S^1, G_c)/\mathrm{conj}.$

(i.e. cocharacter)

$\nabla_{\partial/\partial \bar{z}}$ changes when we cross $t=0$



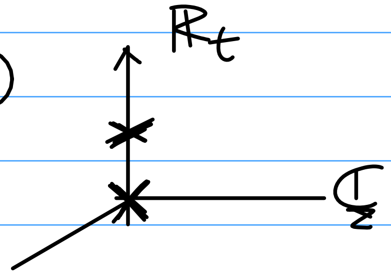
$D \Leftarrow t>0$

$D \Leftarrow t<0$

$H_*^{G(\mathcal{O})}(\mathcal{R})$ = ring of monopole operators

<u>Th.</u> The product is commutative.

☹



$\mathcal{O} \in D$ ↖ in fact, has a positive radius and then ~~you~~ one can move $\mathcal{O} \in D$ freely //

<u>Def.</u> $\mathcal{M}_C \equiv \mathcal{M}_C(G, \mathbf{N}) = \operatorname{Spec} H_*^{G(\mathcal{O})}(\mathcal{R})$

irreducible algebraic variety

$\left(H_*^{G(\mathcal{O})}(\mathcal{R}) = \text{ring of polynomial functions on } \mathcal{M}_C . \right)$

Poincaré polynomial of $\mathcal{R}$ is given by monopole formula when good or ugly theories

– FI parameter

$$\pi_0(\mathcal{R}) \cong \pi_0(\underbrace{Gr_G}_{\substack{\cong \\ \Omega G_c}}) = \pi_1(G)$$

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \bigoplus_{\gamma \in \pi_1(G)} H_*^{G(\mathcal{O})}(\mathcal{R}^\gamma)$$

$$\rightsquigarrow \quad T_C = \pi_1(G)^\wedge \curvearrowright \mathcal{M}_C$$

— mass parameter

$\tilde{G} \triangleright G$ $\quad$ Assume $\tilde{G}/G = T_F$ for simplicity

$\circlearrowleft \mathbf{N}$

We replace G by $\tilde{G}$ in the above constr.

Get

$$\mathcal{M}_C(\tilde{G}, \mathbf{N}) \curvearrowleft \pi_1(\tilde{G})^\wedge$$

$$\uparrow$$

$$\pi_1(T_F)^\wedge = T_F^\vee$$

dual torus

<u>Lemma</u> $\quad \mathcal{M}_C(G, \mathbf{N}) = \mathcal{M}_C(\tilde{G}, \mathbf{N}) /\!/\!/ T_F^\vee$

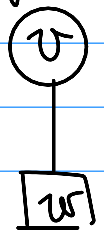
symplectic reduction

$(\operatorname{Lie} T_F^\vee)^* \cong \operatorname{Lie} T_F \ni m$

$\rightsquigarrow$ deformation/Kähler parameter for $\mathcal{M}_C$

Examples

quiver gauge theory for A_1



$$G = GL(V), \quad \mathbb{N} = \mathrm{Hom}(\mathbb{C}^W, \mathbb{C}^V)$$

$$\left(\begin{array}{l} \lambda = W \\ \mu = W - 2V \end{array} \right. \qquad \text{weights of } \mathfrak{sl}_2 = \mathfrak{g}_{A_1\text{-quiver}}$$

Higgs branch

$\mathcal{M}_H(\lambda, \mu)$ (non perturbed one)

$$= \{ X \in \mathfrak{gl}(W) \mid X^2 = 0, \ \mathrm{rk}\, X \leq V \}$$

cpx part $= 0$ $\curvearrowright$

$$\mathcal{M}_H^{\xi}(\lambda, \mu) = T^*\mathrm{Gr}(V, W)$$

$\llcorner$ Grassmannian of V-planes in $\mathbb{C}^W$

$$\begin{tikzcd} \mathbb{C}^V \arrow[d, shift right, "b"'] \\ \mathbb{C}^W \arrow[u, shift right, "a"'] \end{tikzcd} \qquad X = ba$$

$\mathcal{M}_H^{\xi}(\lambda, \mu)$ $\quad b$: inj. $\quad \mathrm{Im}\, b \subset \mathbb{C}^W$

- $\pi : \mathcal{M}_H^{\xi}(\lambda, \mu) \longrightarrow \mathcal{M}_H(\lambda, \mu)$

$$(X, \mathrm{Im}\, b) \longmapsto X$$

surjective

$\iff 2V \leq W$

$\iff \mu \geq 0$

- $\mathcal{M}_H^{\xi} \neq \phi \iff V \leq W$

$$\mathcal{M}_H(\lambda, \mu) = \underset{X:\ \mathrm{rk}\,0}{\mathcal{M}_H^{\mathrm{reg}}(\lambda, \lambda)} \cup \underset{X:\ \mathrm{rk}\,1}{\mathcal{M}_H^{\mathrm{reg}}(\lambda, \lambda - 2)} \cup \cdots$$

$\mathcal{M}_C(\lambda,\mu)$ $\qquad$ $\mathcal{M}_C^{m}(\lambda,\mu) \longrightarrow \mathcal{M}_C(\lambda,\mu)$

birational (always) $\qquad$ interested in this

Bow variety (N-Takayama)

use Hanany-Witten Transition

$$\mathcal{M}_C(\lambda,\mu) \cong \left\{ X = \left[\begin{array}{ccc|ccc} 1 & & * & & & * \\ & \ddots & \vdots & & 0 & * \\ & & 1\,* & & & * \\ \hline * & \cdots & * & 1 & & \vdots \\ & 0 & & & \ddots & \\ & & & & & 1\,* \end{array}\right] \;\middle|\; X : \text{nilpotent} \right\}$$

$(v < w)$ $\qquad \underbrace{}_{v} \quad \underbrace{}_{w-v}$

If $w - 2v \geq 0$, $\quad \uparrow \cong$ Slodowy slice $\cap \mathcal{N}$ (nilpotent cone)

If $w - 2v < 0$, $\quad \uparrow$ is larger than Slodowy slice $\cap \mathcal{N}$

One can study stratification (symplectic leaves) of $\mathcal{M}_H$ and $\mathcal{M}_C$

$$\mathcal{M}_C(\lambda,\mu) = \mathcal{M}_C^{reg}(\lambda,\mu) \cup \mathcal{M}_C^{reg}(\lambda-2,\mu) \cup \cdots$$

regular locus open

$\exists$ order reversing bijection between strata